

Algorithms and combinatorics of geometric graphs

Third homework, due November 18

This homework is due November 18, via e-mail at arnaud.de-mesmay@univ-eiffel.fr. Also do not hesitate to ask me if you have questions. The language of your homework can be either English or French. The exercises are independent and can be treated in any order.

Exercise 1:

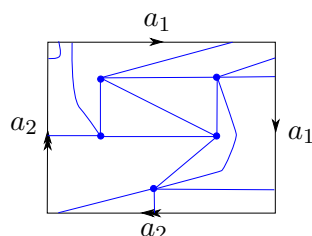
For $G = (V, E)$ a simple graph (without loops nor multiple edges), the **complement** of G is the graph with the same vertex set V , and where two vertices u and v are connected if and only if they are *not* connected in G .

1. Let G be a simple planar graph with 11 vertices. Prove that the complement of G is not planar.
2. Let G be a simple graph embeddable on an orientable surface of genus g with n vertices. For which values of n (depending on g) can we prove that the complement of G is not embeddable on an orientable surface of genus g ? Bonus points depending on how good your bound is.

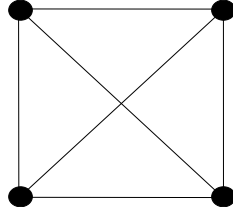
Addendum: The bound in the first item is not tight (sorry, I said the opposite in class), actually even for $n=9$, there are no planar graphs whose complements are also planar (I am not linking a reference as it would trivialize the exercise). So no need to try to prove that your bound is tight for the second item either, just solve the equation that naturally appears and I will be happy.

Exercise 2:

A convenient way to represent a graph on a non-orientable surface is to draw it on top of its canonical polygonal scheme $a_1 a_1 \dots a_g a_g$. For example, here is a cellular embedding of K_5 on a non-orientable surface of genus two.



1. Provide an explicit cellular embedding of the graph pictured below on a non-orientable surface of genus 3.
2. Let G be a simple graph with v vertices, e edges cellularly embedded on a non-orientable surface of genus g . Prove that $g \leq e - v + 1$.



3. Let G be a simple graph with v vertices and e edges, and let g_1 be the smallest genus of a non-orientable surface on which G embeds. Prove that for any g such that $g_1 \leq g \leq e - v + 1$, G can be cellularly embedded on a non-orientable surface of genus g .
4. In particular, G can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.

Exercise 3:

1. Let G be a graph embedded on an orientable surface of genus g , not necessarily cellularly. Prove that $v - e + f \geq 2 - 2g$, where v , e and f denote respectively the number of vertices, edges and faces of the graph embedding.
2. Let G be a simple graph cellularly embedded on an orientable surface of genus g , with the properties that (1) all the faces have degree three (i.e., are incident to three edges), and (2) each cycle¹ of length 3 in the graph bounds a face. The set of such (triangular) faces is denoted by T . Use the previous question to show that in any embedding of G on an orientable surface of genus g , the number of faces is $|T|$. Deduce that the embedding of G on an orientable surface of genus g is unique up to homeomorphism.

¹In the graph-theoretical sense: a walk in the graph without repeated edges or faces