Some exercises on embedded graphs

Exercise 1:

For $G = (V, E)$ a simple graph, the **complement** of G is the graph with the same vertex set V, and where two vertices u and v are connected if and only if they are *not* connected in G.

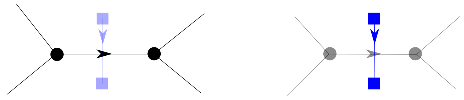
1. Let G be a simple planar graph with 11 vertices. Prove that the complement of G is not planar.
2. Let G be a simple graph embedded on an orientable surface of genus g with n vertices. For which values of n (depending on g) can we prove that the complement of G is not embeddable on an orientable surface of genus g?

Exercise 2:

In this exercise, when we refer to cycles, we mean cycles in the graph-theoretical sense: walks in a graph without repeated vertices and edges.

Let G be a **directed** planar graph, i.e., where each edge is endowed with a direction from one vertex to the other one. A directed graph is **strongly connected** if any vertex can be connected to any other vertex using a directed path. A directed graph is **acyclic** if it contains no non-trivial directed cycle. A **source**, respectively a **sink**, is a vertex whose incident edges are all outgoing, respectively incoming. A **regular** vertex in a directed plane graph is a vertex whose cycle of incident edges consists of a single interval of incoming edges and a single interval of outgoing edges.

Graph duality extends to directed plane graphs as pictured in the figure below: if a primal edge goes from left to right, the dual edge goes from top to bottom.



1. Prove that a directed planar graph is strongly connected if and only if its dual graph G^* (with respect to *any* embedding) is acyclic.
2. Is this true for a non-planar graph? If not, provide an example of a strongly connected graph with an embedding on a surface whose dual is not acyclic.
3. Let G be a directed planar graph with a unique source s and a unique sink t. Prove that in every planar embedding of G, every vertex except s and t is regular.

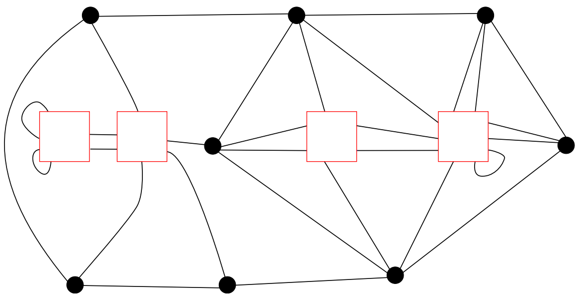
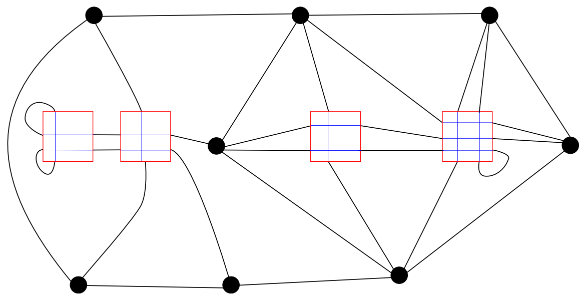
Exercise 3:

1. Let G be a graph embedded on an orientable surface of genus g, not necessarily cellularly. Prove that $v - e + f \geq 2 - 2g$, where v, e and f denote respectively the number of vertices, edges and faces of the graph embedding.

2. Let G be a simple graph cellularly embedded on an orientable surface of genus g, with the properties that (1) all the faces have degree three (i.e., are incident to three edges), and (2) each cycle of length 3 in the graph bounds a face. The set of such (triangular) faces is denoted by T. Use the previous question to show that in any embedding of G, the number of faces is $|T|$. Deduce that the embedding of G on an orientable surface of genus g is unique up to homeomorphism.

Exercise 4:

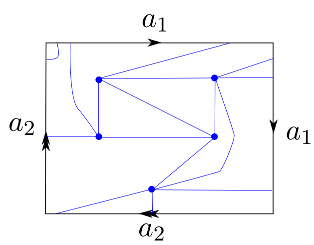
We consider the following way of representing non-planar graphs with boxes. There are k disjoint squares called **boxes** drawn in the plane, and each side acts as a teleporter to the same point on the opposite side. A graph is embedded in the plane with k boxes if it is drawn without crossings in the plane when the edges are allowed to use these teleporters: when an edge intersects a point on the box, it continues on the same point on the opposite side. Note that each edge is allowed to use the same box any number of times. For example, here is a picture of a graph embedded in the plane with four boxes (left picture). Equivalently, a box is a way to hide a grid of crossings (see the right picture).

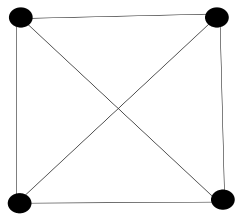
1. Provide an embedding of $K_{3,3}$ in the plane with a single box.
2. Prove that a graph can be embedded in the plane with g boxes if and only if it can be embedded on a surface of genus g.
3. Let G be a graph embedded on a surface of genus g. By the previous question, G can be embedded in the plane with g boxes. Find a function $f(g)$ so that the following strengthening holds (and prove it): G can be embedded in the plane with g boxes so that each edge of G crosses at most $f(g)$ boxes (counted with multiplicity). Any function (even non-polynomial) will do, but the smaller ones are better!

Exercise 5:

Recall that a cellular embedding is an embedding where all the faces are disks, and that a non-orientable surface of genus g is a surface with polygonal scheme $a_1a_1a_2a_2\dots a_ga_g$. A convenient way to represent a graph on a non-orientable surface is to draw it on top of this polygonal scheme. For example, here is a cellular embedding of K_5 on a non-orientable surface of genus two.



1. Provide an explicit cellular embedding of K_4 (pictured below) on a non-orientable surface of genus 3.



2. Let G be a simple graph with v vertices, e edges cellularly embedded on a non-orientable surface of genus g. Prove that $g \leq e - v + 1$.

3. Let G be a simple graph with v vertices and e edges, and let g_1 be the smallest genus of a non-orientable surface on which G embeds. Prove that for any g such that $g_1 \leq g \leq e - v + 1$, G can be cellularly embedded on a non-orientable surface of genus g.

4. In particular, G can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.