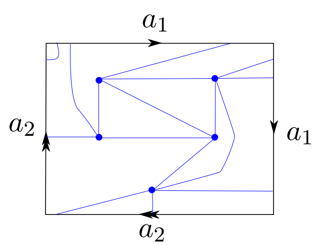
Some exercises on embedded graphs.

Exercise 1:

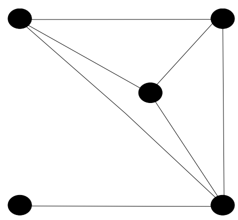
1. A simple bipartite cellularly embedded planar graph is called **bibipartite** if its dual graph is simple and also bipartite. Give a complete list of all bibipartite planar graphs and prove that it is complete. *Hint: it is non-empty!*.
2. Find universal constants α, β and γ (not depending on n or g) such that the following holds: For all integers n and g such that $n \geq \gamma g$, every simple n-vertex graph embedded on a surface of genus g has an independent set[1] of size n/α, in which every vertex has degree at most β.
3. Describe an algorithm to find such an independent set in $O(n)$ time.

Exercise 2:

Recall that a cellular embedding is an embedding where all the faces are disks, and that a non-orientable surface of genus g is a surface with polygonal scheme $a_1a_1a_2a_2 \ldots a_ga_g$. A convenient way to represent a graph on a non-orientable surface is to draw it on top of this polygonal scheme. For example, here is a cellular embedding of K_5 on a non-orientable surface of genus two.



1. Provide an explicit cellular embedding of the graph pictured below on a non-orientable surface of genus 3.



2. Let G be a simple graph with v vertices, e edges cellularly embedded on a non-orientable surface of genus g. Prove that $g \leq e - v + 1$.

[1]An independent set in a graph G is a subset of the vertices of G, no two of which are connected by an edge in G.

3. Let G be a simple graph with v vertices and e edges, and let g_1 be the smallest genus of a non-orientable surface on which G embeds. Prove that for any g such that $g_1 \leq g \leq e - v + 1$, G can be cellularly embedded on a non-orientable surface of genus g.

4. In particular, G can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.

Exercise 3:

In this exercise, we look at a graph G cellularly embedded on an orientable surface of genus g, where each edge uv has an orientation from its **start** to its **end**: either $u \to v$ or $v \to u$[2]. This induces an orientation on the edges of the dual graph, as pictured in Figure 1. We will only be considering oriented cycles in this exercise: a cycle in G is a sequence of oriented edges $(v_1 \to v_2), (v_2 \to v_3) \dots (v_k \to v_1)$ with no repeated vertices. Recall that a cycle is **contractible** if it can be continuously deformed to a point on the surface, or equivalently if it bounds a closed disk.

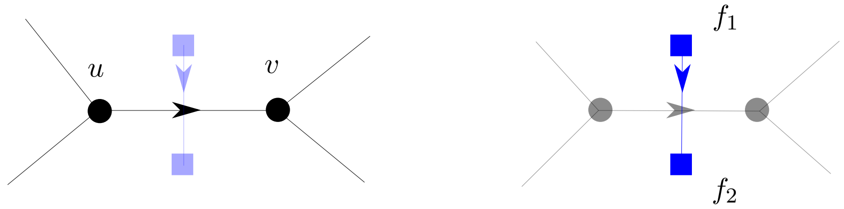


Figure 1: The start of the dual edge (here f_1) lies to the left of the primal edge.

1. List all the cycles of the toroidal graph pictured below, as well as all the cycles of its dual. To make this easier, edges are named. Which of these cycles are contractible?

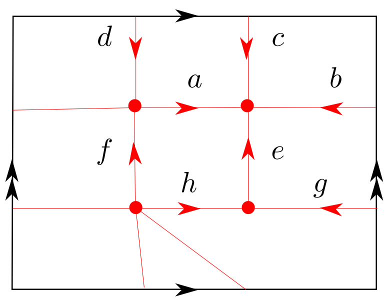


Figure 2: An oriented graph (in red) embedded on a torus. To be clear: the black edges are not part of the graph and are just here to represent the torus (top identified to bottom and left identified to right).

2. Let e be an edge so that the dual edge e^* belongs to a cycle. Prove that e can not belong to a contractible cycle.

3. Suppose that the dual graph G^* has no cycle. Prove that if G has a contractible cycle, G^* has a **sink**, i.e., a vertex v where all the incident edges are oriented towards v, or a **source**, i.e., a vertex v where all the incident edges are oriented away from v.

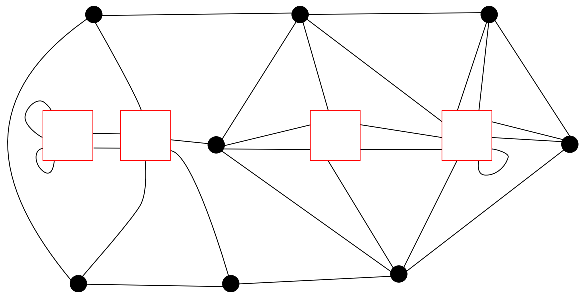
4. Still supposing that the dual graph G^* has no cycle, deduce that if G has a contractible cycle, then it has a face whose boundary is a cycle. *Hint: Be careful that a facial walk might have repeated vertices.*

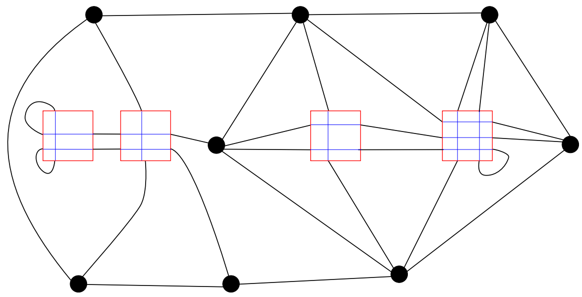
[2]Orientations of loops are a bit annoying to define. To keep things simple, for the purpose of this exercise, we use the same definition: a loop based at u is oriented $u \to u$. In particular it has a unique possible orientation.

5. Provide a linear-time algorithm to test whether an oriented graph G embedded on an orientable surface contains a contractible cycle.

Exercise 4:

We consider the following way of representing non-planar graphs with boxes. There are k disjoint squares called **boxes** drawn in the plane, and each side acts as a teleporter to the same point on the opposite side. A graph is embedded in the plane with k boxes if it is drawn without crossings in the plane when the edges are allowed to use these teleporters: when an edge intersects a point on the box, it continues on the same point on the opposite side. Note that each edge is allowed to use the same box any number of times. For example, here is a picture of a graph embedded in the plane with four boxes (left picture). Equivalently, a box is a way to hide a grid of crossings (see the right picture).





1. Provide an embedding of K_5 in the plane with a single box.

2. Prove that a graph can be embedded in the plane with g boxes if and only if it can be embedded on a surface of genus g.

3. Let G be a graph embedded on a surface of genus g. By the previous question, G can be embedded in the plane with g boxes. Find a function $f(g)$ so that the following strengthening holds (and prove it): G can be embedded in the plane with g boxes so that each edge of G crosses at most $f(g)$ boxes (counted with multiplicity). Any function (even non-polynomial) will do, but the smaller ones are better!