An analysis of the equational properties of the well-founded fixed point

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Abstract

Well-founded fixed points have been used in several areas of knowledge representation and reasoning and in particular to give semantics to logic programs involving negation. They are an important ingredient of approximation fixed point theory. We study the logical properties of the (parametric) well-founded fixed point operation. We show that the operation satisfies several, but not all of the standard equational properties of fixed point operations described by the axioms of iteration theories.

1 Introduction

Fixed points and fixed point operations have been used in just about all areas of computer science. There has been a tremendous amount of work on the existence, construction and logic of fixed point operations. It has been shown that most fixed point operations, including the least (or greatest) fixed point operation on monotonic functions over complete lattices, satisfy the same equational properties. These equational properties are captured by the notion of iteration theories, or iteration categories, cf. [2] or [14] for a recent survey.

For an account of fixed point approaches to logic programming containing original references, we refer to [21]. These approaches, and in particular the stable and well-founded fixed point semantics of logic programs with negation,

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based on the notion of bilattices, have led to the development of an elegant abstract 'approximation fixed point theory', cf. [9, 10, 28].

In this paper, we study the equational properties of the well-founded fixed point operation as defined in [9, 10, 28] with the aim of relating well-founded fixed points to iteration categories. We extend the well-founded fixed point operation to a parametric operation giving rise to an external fixed point (or dagger) operation [2, 3] over the cartesian category of approximation function pairs between complete bilattices. We offer an initial analysis of the equational properties of the well-founded fixed point operation. Our main results show that several identities of iteration theories hold for the well-founded fixed point operation, but some others fail.

2 Complete lattices and bilattices

Recall that a *complete lattice* [6] is a partially ordered set L, ordered by a relation $\leq$, such that each $X \subseteq L$ has a supremum $\bigvee X$ and hence also an infimum $\bigwedge X$. In particular, each complete lattice has a least and a greatest element, respectively denoted either $\bot$ and $\top$, or 0 and 1. We say that a function $f : L \to L$ over a complete lattice L is monotonic (anti-monotonic, resp.) if for all $x, y \in L$, if $x \leq y$ then $f(x) \leq f(y)$ ($f(x) \geq f(y)$, resp.).

A *complete bilattice*[1] [21, 20, 22] $(B, \leq_p, \leq_t)$ is equipped with two partial orders, $\leq_p$ and $\leq_t$, both giving rise to a complete lattice. We will denote the $\leq_p$-least and greatest elements of a complete bilattice by $\bot$ and $\top$, and the $\leq_t$-least and greatest elements by 0 and 1, respectively.

An example, depicted in Figure 1, of a complete bilattice is $\mathcal{FOUR}$, which has 4 elements, $\bot, \top, 0, 1$. The nontrivial order relations are given by $\bot \leq_p 0, 1 \leq_p \top$ and $0 \leq_t \bot, \top \leq_t 1$.

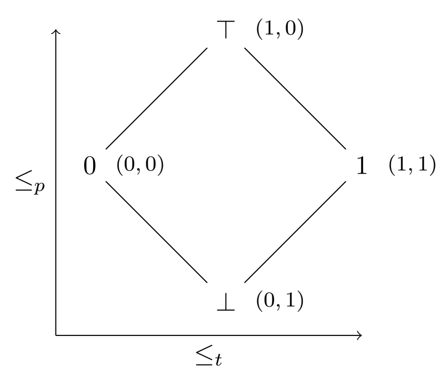

Figure 1: A representation of $\mathcal{FOUR} \approx \mathbf{2} \times \mathbf{2}$ taken from [21].

[1] Sometimes bilattices are equipped with a negation operation and the bilattices as defined here are called pre-bilattices.

Two closely related constructions of a complete bilattice from a complete lattice are described in [9] and [20], see also [5] and [22] for the origins of the constructions. Here we recall one of them. Suppose that $L = (L, \leq)$ is a complete lattice with extremal (i.e., least and greatest) elements 0 and 1. Then define the partial orders $\leq_p$ and $\leq_t$ on $L \times L$ as follows:

$$\begin{aligned} (x, x') \leq_p (y, y') \quad &\Leftrightarrow \quad x \leq y \ \wedge \ x' \geq y' \\ (x, x') \leq_t (y, y') \quad &\Leftrightarrow \quad x \leq y \ \wedge \ x' \leq y'. \end{aligned}$$

Then $L \times L$ is a complete bilattice with $\leq_p$-extremal elements $\perp = (0, 1)$ and $\top = (1, 0)$, and $\leq_t$-extremal elements $0 = (0, 0)$ and $1 = (1, 1)$. Note that when L is the 2-element lattice $\mathbf{2} = \{0 \leq 1\}$, then $L \times L$ is isomorphic to $\mathcal{FOUR}$ as depicted in Figure 1. In this paper, we will mainly be concerned with the ordering $\leq_p$.

In any category, we usually denote the composition of morphisms $f : A \to B$ and $g : B \to C$ by $g \circ f$ and the identity morphisms by id_A. We let SET denote the category of sets and functions and we denote by CL the category of complete lattices and monotonic functions. Both SET and CL have all products and hence are *cartesian categories*. The usual direct product, equipped with the pointwise order in CL, serves as categorical product. In CL, a terminal object is a 1-element lattice T. In both categories, for any sequence $A_1, \ldots, A_n$ of objects, the categorical projection morphisms $\pi_i^{A_1 \times \cdots \times A_n} : A_1 \times \cdots \times A_n \to A_i$, $i \in [n] = \{1, \ldots, n\}$, are the usual projection functions.

Products give rise to a *tupling* operation. Suppose that $f_i : C \to A_i$, $i \in [n]$ in SET or CL, or in any cartesian category. Then there is a unique $f : C \to A_1 \times \cdots \times A_n$ with $\pi_i^{A_1 \times \cdots \times A_n} \circ f = f_i$ for all $i \in [n]$. We denote this unique morphism f by $\langle f_1, \ldots, f_n \rangle$ and call it the (target) tupling of the f_i (or pairing, when $n = 2$). Note that in SET and CL, we have $f(x) = (f_1(x), \ldots, f_n(x))$ for all $x \in C$.

And when $f : C \to A$ and $g : D \to B$, then we define $f \times g$ as the unique morphism $h : C \times D \to A \times B$ with $\pi_1^{A \times B} \circ h = f \circ \pi_1^{C \times D}$ and $\pi_2^{A \times B} \circ h = g \circ \pi_2^{C \times D}$. In SET and CL, $h(x, y) = (f(x), g(y))$ for all $x \in C$ and $y \in D$.

If $m, n \geq 0$, ρ is a function $[m] \to [n]$ and $A_1, \ldots, A_n$ is a sequence of objects in a cartesian category, we associate with ρ (and $A_1, \ldots, A_n$) the morphism

$$\rho^{A_1, \ldots, A_n} = \langle \pi_{\rho(1)}^{A_1 \times \cdots \times A_n}, \ldots, \pi_{\rho(m)}^{A_1 \times \cdots \times A_n} \rangle$$

from $A_1 \times \cdots \times A_n$ to $A_{\rho(1)} \times \cdots \times A_{\rho(m)}$ (Note that in SET and CL, $\rho^{A_1, \ldots, A_n}$ maps $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$ to $(x_{\rho(1)}, \ldots, x_{\rho(m)}) \in A_{\rho(1)} \times \cdots \times A_{\rho(m)}$.) With a slight abuse of notation, we usually let ρ denote this morphism as well. Morphisms of this form are sometimes called *base morphisms*. When $m = n$ and ρ is a bijection, then the associated morphism $A_1 \times \cdots \times A_n \to A_{\rho(1)} \times \cdots \times A_{\rho(n)}$ is an isomorphism. Its inverse is the morphism associated with the inverse ρ^{-1} of the function ρ. For each object A, the base morphism associated with the unique function $[m] \to [1]$ is the *diagonal morphism* $\Delta_m^A = \langle \mathrm{id}_A, \ldots, \mathrm{id}_A \rangle : A \to A^m$, usually denoted just Δ_m.

3 The category CL

The objects of **CL** are complete lattices. Suppose that A, B are complete lattices. A morphism from A to B in **CL**, denoted $f : A \stackrel{\bullet}{\to} B$, is a $\leq_p$-monotonic function $f : A \times A \to B \times B$, where $A \times A$ and $B \times B$ are the complete bilatices determined by A and B. Thus, $f = \langle f_1, f_2 \rangle$ such that $f_1 : A \times A \to B$ is monotonic in its first argument and anti-monotonic in the second argument, and $f_2 : A \times A \to B$ is anti-monotonic in its first argument and monotonic in its second argument. (Such functions f are called approximations in [28].) Composition is the ordinary function composition and for each complete lattice A, the identity morphism $\mathbf{id}_A : A \stackrel{\bullet}{\to} A$ is the identity function $\mathrm{id}_{A\times A} = \mathrm{id}_A \times \mathrm{id}_A = \langle \pi_1^{A\times A}, \pi_2^{A\times A} \rangle : A \times A \to A \times A$.

The category **CL** has finite products. (Actually it has all products). Indeed, a terminal object of **CL** is any 1-element lattice T. Suppose that $A_1, \ldots, A_n$ are complete lattices. Then consider the direct product $A_1 \times \cdots \times A_n$ as an object of **CL** together with the following morphisms $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n} : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_i$, $i \in [n]$. For each i, $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$ is the function

$$A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \to A_i \times A_i$$

defined by

$$\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}(x_1, \ldots, x_n, x_1', \ldots, x_n') \quad = \quad (x_i, x_i'),$$

so that in SET, $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$ can be written as

$$\langle \pi_i^{A_1\times\cdots\times A_n\times A_1\times\cdots\times A_n}, \pi_{n+i}^{A_1\times\cdots\times A_n\times A_1\times\cdots\times A_n} \rangle \quad = \quad \pi_i^{A_1\times\cdots\times A_n} \times \pi_i^{A_1\times\cdots\times A_n}.$$

It is easy to see that the morphisms $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$, $i \in [n]$, determine a product diagram in **CL**. To this end, let $f^i = \langle f_1^i, f_2^i \rangle : C \stackrel{\bullet}{\to} A_i$ in **CL**, for all $i \in [n]$, so that each f^i is a $\leq_p$-monotonic function $C \times C \to A_i \times A_i$. Then let $h = \langle h_1, h_2 \rangle$ be the function $C \times C \to A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n$, where $h_1 = \langle f_1^1, \ldots, f_1^n \rangle$ and $h_2 = \langle f_2^1, \ldots, f_2^n \rangle$. Thus, h_1 and h_2 are functions $C \times C \to A_1 \times \cdots \times A_n$.

We prove that h is the target tupling of $f^1, \ldots, f^n$ in **CL**. First, since each f_1^i is monotonic in its first argument and anti-monotonic in the second argument, the same holds for h_1. In the same way, h_2 is anti-monotonic in the first argument and monotonic in the second. Thus, h is $\leq_p$-monotonic. Next, writing just $\boldsymbol{\pi}_i$ for $\boldsymbol{\pi}_i^{A\times\cdots\times A_n}$ and π_i for $\pi_i^{A\times\cdots\times A_n}$, where $i \in [n]$, we have

$$\begin{aligned}
\boldsymbol{\pi}_i \circ h &= \boldsymbol{\pi}_i \circ \langle h_1, h_2 \rangle \\
&= (\pi_i \times \pi_i) \circ \langle \langle f_1^1, \ldots, f_1^n \rangle, \langle f_2^1, \ldots, f_2^n \rangle \rangle \\
&= \langle \pi_i \circ \langle f_1^1, \ldots, f_1^n \rangle, \pi_i \circ \langle f_2^1, \ldots, f_2^n \rangle \rangle \\
&= \langle f_1^i, f_2^i \rangle \\
&= f_i.
\end{aligned}$$

It is also clear that h is the unique morphism $C \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$ in **CL** with this property.

Proposition 1 **CL** *is a cartesian category in which the product of any objects* $A_1, \dots, A_n$ *agrees with their product in* CL.

By the above argument, the tupling of any sequence of morphisms $f^i = \langle f_1^i, f_2^i \rangle : C \stackrel{\bullet}{\to} A_i$ in **CL** is $h = \langle h_1, h_2 \rangle$, where h_1 is the tupling of the functions f_1^i and h_2 is the tupling of the functions f_2^i in SET. We will denote it by $\boldsymbol{\langle} f^1, \dots, f^n \boldsymbol{\rangle} : C \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$.

For further use, we note the following. Suppose that $\rho : [m] \to [n]$ and $A_1, \dots, A_n$ are complete lattices. Then the associated morphism $\boldsymbol{\rho}^{A_1,\dots,A_n} : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_{\rho(1)} \times \cdots \times A_{\rho(m)}$ in **CL** is the function

$$A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \quad \to \quad A_{\rho(1)} \times \cdots \times A_{\rho(m)} \times A_{\rho(1)} \times \cdots \times A_{\rho(m)}$$

given by

$$(x_1, \dots, x_n, x'_1, \dots, x'_n) \quad \mapsto \quad (x_{\rho(1)}, \dots, x_{\rho(m)}, x'_{\rho(1)}, \dots, x'_{\rho(m)}).$$

Thus,

$$\boldsymbol{\rho}^{A_1,\dots,A_n} = \rho^{A_1,\dots,A_n} \times \rho^{A_1,\dots,A_n},$$

where $\rho^{A_1,\dots,A_n}$ is the morphism associated with ρ and $A_1, \dots, A_n$ in SET (or CL). This is in accordance with $\mathbf{id}_A = \mathrm{id}_A \times \mathrm{id}_A$.

Suppose that $f : C \stackrel{\bullet}{\to} A$ and $g : D \stackrel{\bullet}{\to} B$ in **CL**, so that f is a function $C \times C \to A \times A$ and g is a function $D \times D \to B \times B$. Then $f \times g : C \times D \stackrel{\bullet}{\to} A \times B$ in the category **CL** is the function

$$(\mathrm{id}_A \times \langle \pi_2^{B \times A}, \pi_1^{B \times A} \rangle \times \mathrm{id}_B) \circ h \circ (\mathrm{id}_C \times \langle \pi_2^{D \times C}, \pi_1^{D \times C} \rangle \times \mathrm{id}_D)$$

from $C \times D \times C \times D$ to $A \times B \times A \times B$, where h is $f \times g : C \times C \times D \times D \to A \times A \times B \times B$ in SET. Hence, $h = \langle h_1, h_2 \rangle$ with

$$\begin{aligned} h_1(x, y, x', y') &= (f_1(x, x'), g_1(y, y')) \\ h_2(x, y, x', y') &= (f_2(x, x'), g_2(y, y')). \end{aligned}$$

3.1 Some subcategories

Motivated by [9, 10, 28], we define several subcategories of **CL**. Suppose that A, B are complete lattices. Following [9], we call an ordered pair $(x, x') \in A \times A$ *consistent* if $x \leq x'$. Moreover, we call $f : A \stackrel{\bullet}{\to} B$ in **CL** consistent if it maps consistent pairs to consistent pairs. It is clear that if $f : A \stackrel{\bullet}{\to} B$ and $g : B \stackrel{\bullet}{\to} C$ in **CL** are consistent, then so is $g \circ f : A \stackrel{\bullet}{\to} C$, moreover, $\mathbf{id}_A$ is always consistent. Also, for any sequence $A_1, \dots, A_n$ of complete lattices, the projections $\boldsymbol{\pi}_i^{A_1 \times \cdots \times A_n} : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_i$, $i \in [n]$ are consistent. And when $f_i : C \stackrel{\bullet}{\to} A_i$, for all $i \in [n]$, then $\boldsymbol{\langle} f_1, \dots, f_n \boldsymbol{\rangle} : C \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$ is consistent iff each f_i is. Hence, the consistent morphisms in **CL** determine a cartesian subcategory of **CL** with the same product diagrams. Let **CCL** denote this subcategory.

We define two subcategories of **CCL**. The first one, **ACL**, is the subcategory determined by those morphisms $f = \langle f_1, f_2 \rangle : A \stackrel{\bullet}{\to} B$ in **CL** such that $f_1(x,x) \leq f_2(x,x)$ for all $x \in A$. The second, **EACL**, is the subcategory determined by those $f : A \stackrel{\bullet}{\to} B$ with $f_1(x,x) = f_2(x,x)$. These are again cartesian subcategories with the same product diagrams.

As noted in [9], most applications of approximation fixed point theory use *symmetric* functions. We introduce the subcategory of **CL** having complete lattices as object but only symmetric $\leq_p$-preserving functions as morphisms.

Suppose that $f : A \stackrel{\bullet}{\to} B$ in **CL**, say $f = \langle f_1, f_2 \rangle$, We call f symmetric if $f_2(x, x') = f_1(x', x)$, i.e., when

$$f_2 \quad = \quad f_1 \circ \langle \pi_2^{A\times A}, \pi_1^{A\times A} \rangle : A \times A \to B.$$

We will express this condition in a concise way as $f_2 = f_1^{\mathrm{op}}$.

It is easy to prove that if $f : A \stackrel{\bullet}{\to} B$ and $g : B \stackrel{\bullet}{\to} C$ are symmetric, then so is $g \circ f$. Moreover, $\mathbf{id}_A$ is always symmetric. Thus, symmetric morphisms determine a subcategory of **CL**, denoted **SCL**. In fact, **SCL** is a subcategory of **EACL**, since when $f = \langle f_1, f_2 \rangle : A \stackrel{\bullet}{\to} B$ is symmetric, then necessarily $f_1(x,x) = f_2(x,x)$ for all $x \in A$. Moreover, it is again a cartesian subcategory with the same products.

Since the first component of a symmetric morphism uniquely determines the second component, **SCL** can be represented as the category whose objects are complete lattices having as morphisms $A \stackrel{\bullet}{\to} B$ (where A and B are complete lattices) those functions $f : A \times A \to B$ which are monotonic in the first and anti-monotonic in the second argument. Composition, denoted $\bullet$, is then defined as follows. Given $f : A \stackrel{\bullet}{\to} B$ and $g : B \stackrel{\bullet}{\to} C$, $g \bullet f : A \stackrel{\bullet}{\to} C$ is the function

$$g \circ \langle f, f^{\mathrm{op}} \rangle : A \times A \to C,$$

where f^{op} denotes $f \circ \langle \pi_2^{A\times A}, \pi_1^{A\times A} \rangle$, so that $g \bullet f(x, x') = g(f(x,x'), f(x',x))$. The identity morphism $A \stackrel{\bullet}{\to} A$ is the projection $\pi_1^{A\times A}$.

4 Fixed points

Suppose that A and B are complete lattices, ordered by $\leq$, and let $f : A \times B \to A$ be a monotonic function. The least fixed point operation on CL maps f to the monotonic function $f^\dagger : B \to A$ such that for all $y \in B$, $f^\dagger(y)$, sometimes also denoted $\mu x.f(x,y)$, is the least solution of the fixed point equation $x = f(x,y)$. The existence of $f^\dagger(y)$ is guaranteed by the Knaster-Tarski fixed point theorem. It is also known that $f^\dagger(y)$ is the least $z \in A$ such that $f(z,y) \leq z$ which implies the monotonicity of †.

Remark 2 Sometimes we will apply the least fixed point operation to functions $f : A \times B \to A$, where A, B are complete lattices, which are monotonic in the first argument but anti-monotonic in the second. Such a function may be viewed as a monotonic function $A \times B^d \to A$, where B^d is the order dual of B. Hence, in this case, $f^\dagger$ is a monotonic function $B^d \to A$, or –as we will consider it– an anti-monotonic function $B \to A$. More generally, we will also consider functions that are monotonic in some arguments and anti-monotonic in others, but always take the least fixed point w.r.t. an argument in which the function is monotonic.

In this section, we recall from [9] the construction of stable and well-founded fixed points. More precisely, only symmetric functions were considered in [9], but it was remarked that the construction also works for non-symmetric functions.

Suppose that $f = \langle f_1, f_2 \rangle : A \overset{\bullet}{\to} A$ in **CL**, so that f is a $\leq_p$-monotonic function $A \times A \to A \times A$. Then $f_1 : A \times A \to A$ is monotonic in its first argument and anti-monotonic in its second argument, and $f_2 : A \times A \to A$ is monotonic in its second argument and anti-monotonic in its first argument. Define the functions $s_1, s_2 : A \to A$ by

$$\begin{aligned} s_1(x') &= \mu x. f_1(x, x') \\ s_2(x) &= \mu x'. f_2(x, x') \end{aligned}$$

and let $S(f) : A \times A \to A \times A$ be the function $S(f)(x, x') = (s_1(x'), s_2(x))$. Since s_1 and s_2 are anti-monotonic, $S(f)$ is a morphism $A \overset{\bullet}{\to} A$ in **CL**. We call $S(f)$ the *stable function* for f. It is known that every fixed point of $S(f)$ is a fixed point of f, called a *stable fixed point* of f. Indeed, let (x, x') be a fixed point of $S(f)$, so that $x = s_1(x')$ and $x' = s_2(x)$. By the definition of s_1 and s_2, we have $s_1(x') = f_1(s_1(x'), x')$ and $s_2(x) = f_2(s_2(x), x)$. So

$$\begin{aligned} f(x, x') &= (f_1(x, x'), f_2(x, x')) \\ &= (f_1(s_1(x'), x'), f_2(s_2(x), x)) \\ &= (s_1(x'), s_2(x)) \\ &= (x, x'). \end{aligned}$$

We let $f^\triangle$ denote the set of all stable fixed points of f. Since $S(f)$ is $\leq_p$-monotonic, there is a $\leq_p$-least stable fixed point $f^\ddagger$, called the *well-founded fixed* point of f.

The above construction can slightly be extended. Suppose that $f = \langle f_1, f_2 \rangle : A \times B \overset{\bullet}{\to} A$ in **CL**, so that f is a function $A \times B \times A \times B \to A \times A$. Then $f_1 : A \times B \times A \times B \to A$ is monotonic in its first and second arguments and anti-monotonic in the third and fourth arguments, while $f_2 : A \times B \times A \times B \to A$ is monotonic in the third and fourth arguments and anti-monotonic in the first and second arguments. Now let $s_1, s_2 : A \times B \times B \to A$ be defined by

$$\begin{aligned} s_1(x', y, y') &= \mu x. f_1(x, y, x', y') \\ s_2(x, y, y') &= \mu x'. f_2(x, y, x', y'). \end{aligned}$$

We have that s_1 is monotonic in its second argument and anti-monotonic in the first and third arguments, and s_2 is monotonic in the third argument and anti-monotonic in the first and second arguments. Define $S(f) : A \times A \times B \times B \to A \times A$ by

$$S(f)(x, x', y, y') = (s_1(x', y, y'), s_2(x, y, y')).$$

Then $S(f)$, as a function $(A \times A) \times (B \times B) \to A \times A$, is $\leq_p$-monotonic in both of its arguments. We call $S(f)$ the stable function for f. (Note that $S(f)$ can be considered as a morphism $L \times L' \to L$ of the category CL, where L and L' are the complete bilattices $A \times A$ and $B \times B$ considered as complete lattices ordered by the relation $\leq_p$.) For each $y, y' \in B$, let $f^{\triangle}(y, y')$ denote the set of solutions of the fixed point equation $(x, x') = S(f)(x, x', y, y')$. Hence, $f^{\triangle}$ is a function from $B \times B$ to the power set of $A \times A$, that we call the stable fixed point function. In particular, for each $y, y' \in B$ there is a $\leq_p$-least element of $f^{\triangle}(y, y')$. We denote it by $f^{\ddagger}(y, y')$. Since $S(f)$ is $\leq_p$-monotonic, so is $f^{\ddagger} : B \times B \to A \times A$. Hence $f^{\ddagger} : B \stackrel{\bullet}{\to} A$ in **CL**.

We have thus defined a dagger operation ‡ on **CL**, called the (parametric) *well-founded fixed point operation.* In the next two sections, we investigate the equational properties of this operation.

Remark 3 The parametric well-founded fixed point operation ‡ is just the pointwise extension of the operation defined on morphisms $A \stackrel{\bullet}{\to} A$. Indeed, when $f : A \times B \stackrel{\bullet}{\to} A$ and $(y, y') \in B \times B$, then let $g : A \stackrel{\bullet}{\to} A$ be given by $g(x, x') = f(x, y, x', y')$. Then $f^{\ddagger}(y, y') = g^{\ddagger}$ and $f^{\triangle}(y, y') = g^{\triangle}$.

Remark 4 Suppose that $f : \mathbf{2} \stackrel{\bullet}{\to} \mathbf{2}$ is given by $f(x, x') = (\neg x', \neg x)$, where $\neg 0 = 1$ and $\neg 1 = 0$. Then f is symmetric but $f^{\ddagger}$ is not, since $f^{\ddagger} = (0, 1)$. Hence **SCL** is not closed w.r.t. the parametric well-founded fixed point operation. Since $f^{\dagger}$ is not in **EACL** but **SCL** is a subcategory of **EACL**, this example also shows that **EACL** is not closed under the parametric well-founded fixed point operation.

Remark 5 We provide an example showing that when $f : A \times B \stackrel{\bullet}{\to} A$ in **CL** is consistent, $f^{\ddagger}$ may not be consistent. Indeed, let $A = \mathbf{2}$ and $B = T$ (terminal object), and let $f : A \stackrel{\bullet}{\to} A$ be given by $f(x, x') = (1, \neg x \vee x')$. Then f is consistent, since $f(0, 0) = f(0, 1) = f(1, 1) = (1, 1)$, but $f^{\ddagger} = (1, 0)$, so that $f^{\ddagger}$ is not consistent. Since f is in fact in **EACL**, this example also shows that neither **ACL** nor **EACL** is closed with respect to the well founded fixed point operation.

Note that the above f is not symmetric. In fact, if $f : A \stackrel{\bullet}{\to} A$ is symmetric, then $f^{\ddagger} : T \stackrel{\bullet}{\to} A$ is consistent. This follows from Remark 3 and Theorem 23 in [9].

We summarize the results of this section.

Proposition 6 *The well-founded fixed point operation* ‡ *is an external dagger operation over* **CL**. *Neither of the subcategories* **CCL**, **ACL**, **EACL**, **SCL** *is closed under* ‡.

5 Some valid identities

Iteration categories capture the equational properties of several fixed point operations including the least fixed point operation over CL. Axiomatizations of iteration categories can be conveniently divided into two parts, axioms for Conway categories and the commutative [11, 2] or group identities [13], or the generalized power identities of [12]. Known axiomatizations of Conway categories include the group consisting of the parameter (1), composition (6) and double dagger (8) identities, and the group consisting of the parameter (1), fixed point (2), pairing (7) and permutation (3) identities. In this section we establish several of the above mentioned identities for the parametrized well-founded fixed point operation over **CL**. In the next section we will show that several others fail.

Proposition 7 *The parameter identity holds in* **CL***:*

$$(f \circ (\mathbf{id}_A \times g))^\ddagger \quad = \quad f^\ddagger \circ g, \tag{1}$$

for all $f : A \times B \stackrel{\bullet}{\to} A$ *and* $g : C \stackrel{\bullet}{\to} B$*.*

Proof. Let $h = f \circ (\mathbf{id}_A \times g) : A \times C \stackrel{\bullet}{\to} A$. Then $S(h) : A \times A \times C \times C \to A \times A$ is given by

$$\begin{aligned} S(h)(x, x', z, z') \quad &= \quad (\mu x.f_1(x, g_1(z, z'), x', g_2(z, z')), \\ &\qquad \mu x'.f_2(x, g_1(z, z'), x', g_2(z, z'))) \\ &= \quad S(f)((\mathrm{id}_{A \times A} \times g)(x, x', z, z')), \end{aligned}$$

where $f = \langle f_1, f_2 \rangle$ and $g = \langle g_1, g_2 \rangle$. Thus, $S(h) = S(f) \circ (\mathrm{id}_{A\times A} \times g)$ in SET (or CL), and therefore $h^\triangle = f^\triangle \circ (\mathrm{id}_{A\times A} \times g)$, using a suggestive notation. Moreover, $h^\ddagger = f^\ddagger \circ g$, since the parameter identity holds for the least fixed point operation over CL. □

Proposition 8 *The fixed point identity holds:*

$$f \circ \langle f^\ddagger, \mathrm{id}_B \rangle \quad = \quad f^\ddagger, \tag{2}$$

for all $f : A \times B \stackrel{\bullet}{\to} A$*.*

Proof. By Remark 3, it is sufficient to prove our claim only in the case when $f : A \stackrel{\bullet}{\to} A$, i.e., f is a $\leq_p$-monotonic function $A \times A \to A \times A$. But it is known, see e.g. Theorem 19 in [9], that if $f : A \stackrel{\bullet}{\to} A$, then each stable fixed point of f is a ($\leq_t$-minimal) fixed point, so $f \circ f^\ddagger = f^\ddagger$. (We also have $f \circ f^\triangle = f^\triangle$.) □

Proposition 9 *The permutation identity holds:*

$$(\boldsymbol{\rho} \circ f \circ (\boldsymbol{\rho}^{-1} \times \mathbf{id}_B))^\ddagger \quad = \quad \boldsymbol{\rho} \circ f^\ddagger, \tag{3}$$

for all $f : A_1 \times \cdots \times A_n \times B \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$ *and permutation* $\rho : [n] \to [n]$*.*

Proof. We prove this only when B is the terminal object (cf. Remark 3), so that f can be viewed as a morphism $f = \langle f_1, f_2 \rangle : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$, where f_1, f_2 are appropriate functions

$$A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \to A_1 \times \cdots \times A_n.$$

Let $g = \boldsymbol{\rho} \circ f \circ \boldsymbol{\rho}^{-1}$ in **CL**, so that $g = \langle g_1, g_2 \rangle$ where g_1, g_2 are functions

$$A_{\rho(1)} \times \cdots \times A_{\rho(n)} \times A_{\rho(1)} \times \cdots \times A_{\rho(n)} \to A_{\rho(1)} \times \cdots \times A_{\rho(n)}.$$

First we show that

$$S(g) \quad = \quad \boldsymbol{\rho} \circ S(f) \circ \boldsymbol{\rho}^{-1} \tag{4}$$

in **CL**, i.e.,

$$S(g) \quad = \quad (\rho \times \rho) \circ S(f) \circ (\rho^{-1} \times \rho^{-1})$$

in SET. Below we will denote by x, x' n-tuples in $A_1 \times \cdots \times A_n$. Similarly, let y, y' denote n-tuples in $A_{\rho(1)} \times \cdots \times A_{\rho(n)}$. Note that if $x = (x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$, then $\rho(x) = (x_{\rho(1)}, \ldots, x_{\rho(n)})$ in $A_{\rho(1)} \times \cdots \times A_{\rho(n)}$. And if $y = (y_1, \ldots, y_n) \in A_{\rho(1)} \times \cdots \times A_{\rho(n)}$, then $\rho^{-1}(y) = (y_{\rho^{-1}(1)}, \ldots, y_{\rho^{-1}(n)})$ in $A_1 \times \cdots \times A_n$. Let

$$\begin{aligned} s_1(x') &= \mu x . f_1(x, x') \\ s_2(x) &= \mu x' . f_2(x, x'). \end{aligned}$$

Then $S(f)(x, x') = (s_1(x'), s_2(x))$. Similarly, let

$$\begin{aligned} t_1(y') &= \mu y . \rho(f_1(\rho^{-1}(y), \rho^{-1}(y'))) \\ t_2(y) &= \mu y' . \rho(f_2(\rho^{-1}(y), \rho^{-1}(y'))). \end{aligned}$$

Then $S(g)(y, y') = (t_1(y'), t_2(y))$. Since the permutation and parameter identities hold for the least fixed point operation over CL, we obtain that

$$\begin{aligned} t_1(y') &= \rho(s_1(\rho^{-1}(y'))) \\ t_2(y) &= \rho(s_2(\rho^{-1}(y))), \end{aligned}$$

proving (4). Now from (4), since the permutation identity holds for the least fixed point operation over CL, it follows that $g^\ddagger = \boldsymbol{\rho} \circ f^\ddagger$ in **CL**. Moreover, it follows that the stable fixed points of g are of the form $(\rho(x), \rho(x'))$, where (x, x') is a stable fixed point of f. (A suggestive notation: $g^\triangle = \rho \circ f^\triangle$.) $\square$

We now establish a special case of the pairing identity (7). It will be shown later that the general form of the identity does not hold.

Proposition 10 *The following identity holds:*

$$\langle\!\langle f, g \circ (\boldsymbol{\pi}_2^{A \times B} \times \mathbf{id}_C) \rangle\!\rangle^\ddagger \quad = \quad \langle\!\langle f^\ddagger \circ \langle\!\langle g^\ddagger, \mathbf{id}_C \rangle\!\rangle, g^\ddagger \rangle\!\rangle, \tag{5}$$

where $f : A \times B \times C \stackrel{\bullet}{\to} A$ *and* $g : B \times C \stackrel{\bullet}{\to} B$.

Proof. It suffices to consider the case when there is no parameter (cf. Remark 3). So let $f = \langle f_1, f_2 \rangle : A \times B \stackrel{\bullet}{\to} A$ and $g = \langle g_1, g_2 \rangle : B \stackrel{\bullet}{\to} B$, so that $f_1, f_2 : A \times B \times A \times B \to A$ and $g_1, g_2 : B \times B \to B$. Let $h = \langle\!\langle f, g \circ \boldsymbol{\pi}_2^{A \times B} \rangle\!\rangle : A \times B \stackrel{\bullet}{\to} A \times B$ in **CL**. Then $h^\ddagger$ can be constructed as follows. First consider

$$\begin{array}{l} \mu(x,y).(f_1(x,y,x',y'), g_1(y,y')) \quad \text{and} \\ \mu(x',y').(f_2(x,y,x',y'), g_2(y,y')). \end{array}$$

Since (5) and the parameter identity hold for the least fixed point operation over CL, we know that these functions can respectively be written as

$$\begin{array}{l} (\mu x.f_1(x, \mu y.g_1(y,y'), x', y'), \mu y.g_1(y,y')) \quad \text{and} \\ (\mu x'.f_2(x,y,x', \mu y'.g_2(y,y')), \mu y'.g_2(y,y')). \end{array}$$

Now $h^\ddagger$ can be obtained by solving the system of equations

$$\begin{array}{rcl} (x,x') & = & (\mu x.f_1(x, \mu y.g_1(y,y'), x', y'), \\ & & \ \mu x'.f_2(x,y,x', \mu y'.g_2(y,y'))) \\ (y,y') & = & (\mu y.g_1(y,y'), \mu y'.g_2(y,y')) \end{array}$$

for its least solution w.r.t. $\leq_p$. However, this system of equations is equivalent to the system

$$\begin{array}{rcl} (x,x') & = & (\mu x.f_1(x, \mu y.g_1(y,y'), x', \mu y'.g_2(y,y')), \\ & & \ \mu x'.f_2(x, \mu y.g_1(y,y'), x', \mu y'.g_2(y,y'))) \\ (y,y') & = & (\mu y.g_1(y,y'), \mu y'.g_2(y,y')) \end{array}$$

in the sense that both systems have the same solutions. Now the second system of equations is just

$$\begin{array}{rcl} (x,x') & = & S(f)((x,x'), S(g)(y,y')) \\ (y,y') & = & S(g)(y,y'). \end{array}$$

It follows that $h^\triangle$ consists of all $((x,y),(x',y'))$ such that (y,y') is a stable fixed point of g and (x,x') is in $f^\triangle(y,y')$. In particular, since the least fixed point operation over CL satisfies (5), it holds that $h^\ddagger = \langle\!\langle f^\ddagger \circ g^\ddagger, g^\ddagger \rangle\!\rangle$ as claimed. □

Remark 11 The identity (5) has already been established in Theorem 3.11 of [28], see also the Splitting Set Theorem of [24].

We prove one more property that is not an identity, but a quasi-identity. It is stronger that the group or commutative identities [2, 13], yet most of the standard models satisfy it. (Actually the commutative identities were introduced in [11] in order to replace this quasi-identity by weaker identities, since when it comes to equational theories, the best way to present them is by providing equational bases.)

Proposition 12 *The weak functorial dagger implication holds: for all* $f : A^n \times B \overset{\bullet}{\to} A^n$ *and* $g : A \times B \overset{\bullet}{\to} A$ *in* **CL**, *if* $f \circ (\mathbf{\Delta}_n \times \mathbf{id}_B) = \mathbf{\Delta}_n \circ g$, *then* $f^\ddagger = \mathbf{\Delta}_n \circ g^\ddagger$.

Proof. First recall that $\mathbf{\Delta}_n^A$ (or just $\mathbf{\Delta}_n$ when A is understood) denotes the diagonal morphism $A \overset{\bullet}{\to} A^n$ in **CL** and Δ_n^A (or just Δ_n when A is understood) denotes the diagonal morphism $A \to A^n$.

We spell out the proof only in the case when B is a terminal object. So let $f : A^n \overset{\bullet}{\to} A^n$ and $g : A \overset{\bullet}{\to} A$ in **CL**, say $f = \langle f_1, f_2 \rangle$ and $g = \langle g_1, g_2 \rangle$, where $f_i : A^n \times A^n \to A^n$ and $g_i : A \times A \to A$ are appropriate functions for $i = 1, 2$.

The assumption $f \circ \mathbf{\Delta}_n = \mathbf{\Delta}_n \circ g$ can be rephrased as

$$f_i \circ (\Delta_n \times \Delta_n) = \Delta_n \circ g_i, \quad i = 1, 2,$$

i.e.,

$$\begin{aligned} f_1(x, \ldots, x, x', \ldots, x') &= (g_1(x, x'), \ldots, g_1(x, x')) \\ f_2(x, \ldots, x, x', \ldots, x') &= (g_2(x, x'), \ldots, g_2(x, x')) \end{aligned}$$

for all $x, x' \in A$. Since the weak functorial dagger implication and the parameter identity hold for the least fixed point operation over CL, it follows that

$$\begin{aligned} h_1(x', \ldots, x') &= (k_1(x'), \ldots, k_1(x')) \\ h_2(x, \ldots, x) &= (k_2(x), \ldots, k_2(x)) \end{aligned}$$

where $h_1(x'_1, \ldots, x'_n)$ and $h_2(x_1, \ldots, x_n)$ are respectively the components of the least solution of

$$\begin{aligned} (x_1, \ldots, x_n) &= f_1(x_1, \ldots, x_n, x'_1, \ldots, x'_n) \quad \text{and} \\ (x'_1, \ldots, x'_n) &= f_2(x_1, \ldots, x_n, x'_1, \ldots, x'_n) \end{aligned}$$

and $k_1(x')$ and $k_2(x)$ denote the components of the least solution of

$$\begin{aligned} x &= g_1(x, x') \quad \text{and} \\ x' &= g_2(x, x'). \end{aligned}$$

Hence

$$S(f)(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = (h_1(x'_1, \ldots, x'_n), h_2(x_1, \ldots, x_n)),$$

moreover, $S(g)(x, x') = (k_1(x'), k_2(x))$. Consider now the equations

$$(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = (h_1(x'_1, \ldots, x'_n), h_2(x_1, \ldots, x_n))$$

and

$$(x, x') = (k_1(x'), k_2(x)).$$

Since the weak functorial dagger implication and the parameter identity hold for the least fixed point operation over CL, the $\leq_p$-least solution of the first equation can be obtained as the $2n$-tuple whose first n components are equal to the first component of the $\leq_p$-least solution of the second equation, and whose second n components are equal to the second component of the $\leq_p$-least solution of the second equation. This means that $f^\ddagger = (\Delta_n \times \Delta_n) \circ g^\ddagger$ in SET, i.e., $f^\ddagger = \boldsymbol{\Delta}_n \circ g^\ddagger$ in **CL**. (It also holds that if (x, x') is a stable fixed point of g, then $(x, \ldots, x, x', \ldots, x')$ is a stable fixed point of f.) □

For the definition of the commutative and group identities, we refer to [2, 11].

Corollary 13 *The commutative identities and the identities associated with finite groups hold for the parametrized well-founded fixed point operator over* **CL**.

In fact, each identity associated with a finite automaton [13] holds.

6 Some identities that fail

Proposition 14 *The composition identity*

$$(f \circ \langle g, \boldsymbol{\pi}_2^{A \times C} \rangle)^\ddagger \quad = \quad f \circ \langle (g \circ \langle f, \boldsymbol{\pi}_2^{B \times C} \rangle)^\ddagger, \mathbf{id}_C \rangle, \tag{6}$$

$f : B \times C \overset{\bullet}{\to} A$, $g : A \times C \overset{\bullet}{\to} B$, *fails in* **CL**, *even in the following simple case:* $f \circ (f \circ f)^\ddagger = (f \circ f)^\ddagger$, *where* $f : A \overset{\bullet}{\to} A$.

Proof. Let $f : \mathbf{2} \overset{\bullet}{\to} \mathbf{2}$ be given by $f(x, x') = (\neg x', \neg x)$ (see also Remark 4). Then $f \circ f$ is the identity function on $\mathbf{2} \times \mathbf{2}$, hence $(f \circ f)^\ddagger = (0, 0)$. On the other hand, $f \circ (f \circ f)^\ddagger = (1, 1)$. □

Proposition 15 *The squaring identity* $(f \circ f)^\ddagger = f^\ddagger$ *fails, where* $f : A \overset{\bullet}{\to} A$.

Proof. Let f be as in the previous proof. Then $(f \circ f)^\ddagger = (0, 0)$ as shown above. But $f^\ddagger = (0, 1)$. □

Since the fixed point, parameter and permutation identities hold but the composition identity fails, the pairing identity (found in [1, 7]) also must fail, see [2]. We can give a direct proof.

Proposition 16 *The pairing identity*

$$\langle f, g \rangle^\ddagger \quad = \quad \langle f^\ddagger \circ \langle h^\ddagger, \mathbf{id}_C \rangle, h^\ddagger \rangle, \tag{7}$$

where $h = g \circ \langle f^\ddagger, \mathbf{id}_{B \times C} \rangle$ *fails, where* $f : A \times B \times C \overset{\bullet}{\to} A$ *and* $g : A \times B \times C \overset{\bullet}{\to} B$.

Proof. Let $f, g : \mathbf{2} \times \mathbf{2} \overset{\bullet}{\to} \mathbf{2}$ in **CL**, so that f and g are appropriate functions $\mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} \to \mathbf{2} \times \mathbf{2}$,

$$\begin{aligned} f(x, y, x', y') &= (\neg y', \neg y) \\ g(x, y, x', y') &= (\neg x', \neg x). \end{aligned}$$

Then

$$\langle f, g \rangle(x, y, x', y') \quad = \quad (\neg y', \neg x', \neg y, \neg x)$$

and thus $\langle f, g \rangle^\ddagger = (0, 0, 1, 1)$. On the other hand, $f^\ddagger(y, y') = (\neg y', \neg y)$, hence $h = g \circ \langle f^\ddagger, \mathbf{id_2} \rangle$ is the identity function on $\mathbf{2} \times \mathbf{2}$ and $h^\ddagger = (0, 0)$ and $f^\ddagger \circ h^\ddagger = (1, 1)$. It follows that $\langle f^\ddagger \circ h^\ddagger, h^\ddagger \rangle = (1, 0, 1, 0)$. □

Each of the above examples involved symmetric morphisms. We now refute the double dagger identity, but we use a non-symmetric morphism.

Proposition 17 *The double dagger identity*

$$f^{\ddagger\ddagger} \quad = \quad (f \circ (\langle \mathbf{id}_A, \mathbf{id}_A \rangle \times \mathbf{id}_B))^\ddagger, \tag{8}$$

$f : A \times A \times B \stackrel{\bullet}{\to} A$, *fails in* **CL**, *even in the particular case when* $B = T$ *(terminal object).*

Proof. Let $g : \mathbf{2} \times \mathbf{2} \stackrel{\bullet}{\to} \mathbf{2}$ be given by $g(x, y, x', y') = (\neg y', \neg x)$, and let $h = g \circ \langle \mathbf{id_2}, \mathbf{id_2} \rangle : \mathbf{2} \stackrel{\bullet}{\to} \mathbf{2}$, so that $h(x, x') = (\neg x', \neg x)$. We already know that $h^\ddagger = (0, 1)$. But $g^\ddagger(y, y') = (\neg y', y)$ and $g^{\ddagger\ddagger} = (1, 0)$. □

7 Some applications

The established identities can be seen as abstract versions of transformations over logic programs that preserve the well-founded semantics (in the bilattice setting). For one example, consider the simple propositional logic program

$$p :- \ q, \sim r \quad q :- \ r, \sim p \quad r :- \ p, \sim q$$

Identifying p, q, r, we obtain

$$p :- \ p, \sim p$$

By the weak functorial implication established above, the two programs are equivalent in the sense that each component of the well-founded semantics of the first program agrees with the well-founded semantics of the second. (For a treatment of the semantics of logic programs in approximation fixed point theory, see [9, 8].)

By formulating transformations as identities, one can use standard (many-sorted) equational logic to derive other identities that in turn give rise to new transformations. For example, the following identity is an equational consequence of those established in the paper:

$$\langle f, g \circ \boldsymbol{\pi}_2^{A \times B} \rangle^\ddagger = \langle f \circ (\mathbf{id}_A \times g), g \circ \boldsymbol{\pi}_2^{A \times B} \rangle^\ddagger$$

where $f : A \times B \stackrel{\bullet}{\to} A$ and $g : B \stackrel{\bullet}{\to} B$. Indeed,

$$\begin{aligned}
&\langle f \circ (\mathbf{id}_A \times g), g \circ \boldsymbol{\pi}_2^{A\times B}\rangle^\ddagger = \\
&= \quad \langle (f \circ (\mathbf{id}_A \times g))^\ddagger \circ g^\ddagger, g^\ddagger\rangle, \quad \text{by Prop. 10.2} \\
&= \quad \langle f^\ddagger \circ g \circ g^\ddagger, g^\ddagger\rangle, \quad \text{by the parameter identity} \\
&= \quad \langle f^\ddagger \circ g^\ddagger, g^\ddagger\rangle, \quad \text{by the fixed point identity} \\
&= \quad \langle f, g \circ \boldsymbol{\pi}_2^{A\times B}\rangle^\ddagger, \quad \text{by Prop. 10.2.}
\end{aligned}$$

More generally, it holds that

$$\langle f, g \circ \boldsymbol{\pi}_2^{A\times B}\rangle^\ddagger = \langle f \circ (\mathbf{id}_A \times \langle g, \boldsymbol{\pi}_2^{B\times C}\rangle), g \circ \boldsymbol{\pi}_2^{A\times(B\times C)}\rangle^\ddagger$$

where $f : A \times B \times C \stackrel{\bullet}{\to} A$ and $g : B \times C \stackrel{\bullet}{\to} B$.

This identity can be interpreted as a version of the fold/unfold transformation [27, 26]. For example, it yields that the logic programs

$$p :- \ q, r \quad r :- \ s, t$$

and

$$p :- \ q, s, t \quad r :- \ s, t$$

are equivalent for the well-founded semantics.

On the other hand, the following identity, which is a generalization of the above folding/unfolding identity, fails:

$$\langle f \circ \langle \boldsymbol{\pi}_1^{A\times B}, g\rangle, g\rangle^\ddagger \quad = \quad \langle f, g\rangle^\ddagger$$

where $f : A \times B \stackrel{\bullet}{\to} A$ and $g : A \times B \stackrel{\bullet}{\to} B$. And this again follows by standard equational reasoning using our positive and negative results. For suppose that the identity holds. Then the following special case obtained by letting $A = B$ and instantiating f with $h \circ \boldsymbol{\pi}_2^{A\times A}$ and g with $h \circ \boldsymbol{\pi}_1^{A\times A}$, where $h : A \stackrel{\bullet}{\to} A$, holds as well:

$$\langle h \circ h \circ \boldsymbol{\pi}_1^{A\times A}, h \circ \boldsymbol{\pi}_1^{A\times A}\rangle^\ddagger \quad = \quad \langle h \circ \boldsymbol{\pi}_2^{A\times A}, h \circ \boldsymbol{\pi}_1^{A\times A}\rangle^\ddagger.$$

Moreover, using Proposition 9, also

$$\langle h \circ \boldsymbol{\pi}_2^{A\times A}, h \circ h \circ \boldsymbol{\pi}_2^{A\times A}\rangle^\ddagger \quad = \quad \langle h \circ \boldsymbol{\pi}_2^{A\times A}, h \circ \boldsymbol{\pi}_1^{A\times A}\rangle^\ddagger.$$

But by Proposition 10,

$$\langle h \circ h \circ \boldsymbol{\pi}_1^{A\times A}, h \circ \boldsymbol{\pi}_1^{A\times A}\rangle^\ddagger \quad = \quad \langle (h \circ h)^\ddagger, h \circ (h \circ h)^\ddagger\rangle,$$

and by Proposition 10 and 9,

$$\langle h \circ \boldsymbol{\pi}_2^{A\times A}, h \circ h \circ \boldsymbol{\pi}_2^{A\times A}\rangle^\ddagger \quad = \quad \langle h \circ (h \circ h)^\ddagger, (h \circ h)^\ddagger\rangle.$$

We conclude that

$$h \circ (h \circ h)^\ddagger \quad = \quad (h \circ h)^\ddagger,$$

contradicting Proposition 14.

8 Conclusion

We extended the well-founded fixed point operation of [9, 28] to a parametric operation and studied its equational properties. We found that several of the identities of iteration theories hold for the parametric well-founded fixed point operation, but some others fail. By showing that some identities of iteration theories do not hold, we tried to have a better understanding why logic programs with the well-founded semantics cannot be manipulated using standard fixed point methods. And by showing that some other identities hold, we tried to understand to what extent the standard techniques can be used for manipulating logic programs.

Two interesting questions arise for further investigation. The first concerns the *algorithmic description* of the valid identities of the well-founded fixed point operation. Does there exist an algorithm to decide whether an identity (in the language of cartesian categories equipped with a dagger operation) holds for the well-founded fixed point operation? The second concerns the *axiomatic description* of the valid identities of the well-founded fixed point operation. These questions are also relevant in connection with modular logic programing, cf. [19, 23, 24].

An alternative semantics of logic programs with negation based on an infinite domain of truth values was proposed in [25]. The infinite valued approach has been further developed in the abstract setting of 'stratified complete lattices' in [4, 17, 18, 15, 16]. In particular, it has been proved in [15] that the stratified least fixed point operation arising in this approach *does* satisfy all identities of iteration theories. So in this regards, the infinite valued semantics behaves just as the Kripke-Kleene semantics [21], as it corresponds to the least fixed points. In fact, the iteration theory identities are sound and complete for both the Kripke-Kleene semantics and the infinite valued semantics.

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