An analysis of the equational properties of the well-founded fixed point

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Abstract

We study the logical properties of the (parametric) well-founded fixed point operation. We show that the operation satisfies several, but not all of the equational properties of fixed point operations described by the axioms of iteration theories.

1. Introduction

Fixed points and fixed point operations have been used in just about all areas of computer science. There has been a tremendous amount of work on the existence, construction and logic of fixed point operations. It has been shown that most fixed point operations, including the least (or greatest) fixed point operation on monotonic functions over complete lattices, satisfy the same equational properties. These equational properties are captured by the notion of iteration theories, or iteration categories, cf. (Bloom and Ésik 1993) or the recent survey (Ésik 2015a).

In this paper, we study the equational properties of the well-founded fixed point operation as defined in (Denecker, Marek, and Truszczyński 2000; 2004; Vennekens, Gilis, and Denecker 2006) with the aim of relating well-founded fixed points to iteration categories. We extend the well-founded fixed point operation to a parametric fixed point (or dagger) operation (Bloom and Ésik 1993; 1996) over the cartesian category of approximation function pairs between complete bilattices and offer an initial analysis of its equational properties. Our main results show that several identities of iteration theories hold for the well-founded fixed point operation, but some others fail. For proofs, we refer to (Carayol and Ésik 2015).

2. Preliminaries

Recall that a *complete lattice* (Davey and Priestley 1990) is a partially ordered set L, ordered by $\leq$, such that each $X \subseteq L$ has a supremum $\bigvee X$ and hence also an infimum $\bigwedge X$. In particular, each complete lattice has a least and a greatest element, respectively denoted either $\perp$ and $\top$, or 0 and 1. We say that a function $f : L \to L$ over a complete lattice L is monotonic (anti-monotonic, resp.) if for all $x, y \in L$, if $x \leq y$ then $f(x) \leq f(y)$ ($f(x) \geq f(y)$, resp.).

A *complete bilattice*[1] (Fitting 2002) $(B, \leq_p, \leq_t)$ is equipped with two partial orders, $\leq_p$ and $\leq_t$, both giving rise to a complete lattice. We will denote the $\leq_p$-least and greatest elements of a complete bilattice by $\perp$ and $\top$, and the $\leq_t$-least and greatest elements by 0 and 1, respectively.

An example of a complete bilattice is $\mathcal{FOUR}$, which has 4 elements, $\perp, \top, 0, 1$. The nontrivial order relations are given by $\perp \leq_p 0, 1 \leq_p \top$ and $0 \leq_t \perp, \top \leq_t 1$.

There are two closely related constructions of a complete bilattice from a complete lattice. Here we recall one of them. Suppose that $L = (L, \leq)$ is a complete lattice with extremal (i.e., least and greatest) elements 0 and 1. Then define the partial orders $\leq_p$ and $\leq_t$ on $L \times L$ as follows:

$$\begin{aligned} (x, x') \leq_p (y, y') &\quad\Leftrightarrow\quad x \leq y \,\wedge\, x' \geq y' \\ (x, x') \leq_t (y, y') &\quad\Leftrightarrow\quad x \leq y \,\wedge\, x' \leq y'. \end{aligned}$$

Then $L \times L$ is a complete bilattice with $\leq_p$-extremal elements $\perp = (0, 1)$ and $\top = (1, 0)$, and $\leq_t$-extremal elements $0 = (0, 0)$ and $1 = (1, 1)$. Note that when L is the 2-element lattice $\mathbf{2} = \{0 \leq 1\}$, then $L \times L$ is isomorphic to $\mathcal{FOUR}$. In this paper, we will mainly be concerned with the ordering $\leq_p$.

In any category, we usually denote the composition of morphisms $f : A \to B$ and $g : B \to C$ by $g \circ f$ and the identity morphisms by id_A. We let SET and CL respectively denote the category of sets and functions an the category of complete lattices and monotonic functions. Both categories are *cartesian categories* with the usual direct product $A_1 \times \cdots \times A_n$ (equipped with the pointwise order in CL) serving as categorical product. The categorical projection morphisms $\pi_i^{A_1 \times \cdots \times A_n} : A_1 \times \cdots \times A_n \to A_i$, $i \in [n] = \{1, \ldots, n\}$, are the usual projection functions.

Products give rise to a *tupling* operation. Suppose that $f_i : C \to A_i$, $i \in [n]$ in a cartesian category. Then there is a unique $f : C \to A_1 \times \cdots \times A_n$ with $\pi_i^{A_1 \times \cdots \times A_n} \circ f = f_i$ for all $i \in [n]$. We denote this unique morphism f by $\langle f_1, \ldots, f_n \rangle$ and call it the (target) tupling of the f_i (or pairing, when $n = 2$). And when $f : C \to A$ and $g : D \to B$,

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[1] Sometimes bilattices are equipped with a negation operation and the bilattices as defined here are called pre-bilattices.

then we define $f \times g$ as the unique morphism $h : C \times D \to A \times B$ with $\pi_1^{A\times B} \circ h = f \circ \pi_1^{C\times D}$ and $\pi_2^{A\times B} \circ h = g \circ \pi_2^{C\times D}$.

When $m, n \geq 0$, ρ is a function $[m] \to [n]$ and $A_1, \ldots, A_n$ is a sequence of objects in a cartesian category, we associate with ρ (and $A_1, \ldots, A_n$) the morphism

$$\rho^{A_1,\ldots,A_n} = \langle \pi_{\rho(1)}^{A_1\times\cdots\times A_n}, \ldots, \pi_{\rho(m)}^{A_1\times\cdots\times A_n} \rangle$$

from $A_1 \times \cdots \times A_n$ to $A_{\rho(1)} \times \cdots \times A_{\rho(m)}$ (Note that in SET and CL, $\rho^{A_1,\ldots,A_n}$ maps $(x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n$ to $(x_{\rho(1)}, \ldots, x_{\rho(m)}) \in A_{\rho(1)} \times \cdots \times A_{\rho(m)}$.) With a slight abuse of notation, we usually let ρ denote this morphism as well. Morphisms of this form are sometimes called *base morphisms*. When $m = n$ and ρ is a bijection, then the associated morphism $A_1 \times \cdots \times A_n \to A_{\rho(1)} \times \cdots \times A_{\rho(n)}$ is an isomorphism. Its inverse is the morphism associated with the inverse ρ^{-1} of the function ρ. For each object A, the base morphism associated with the unique function $[m] \to [1]$ is the *diagonal morphism* $\Delta_m^A = \langle \mathrm{id}_A, \ldots, \mathrm{id}_A \rangle : A \to A^m$, usually denoted just Δ_m.

3. The category CL

The objects of $\mathbf{CL}$ are complete lattices. Suppose that A, B are complete lattices. A morphism from A to B in $\mathbf{CL}$, denoted $f : A \stackrel{\bullet}{\to} B$, is a $\leq_p$-monotonic function $f : A \times A \to B \times B$, where $A \times A$ and $B \times B$ are the complete bilattices determined by A and B. Thus, $f = \langle f_1, f_2 \rangle$ such that $f_1 : A \times A \to B$ is monotonic in its first argument and anti-monotonic in the second argument, and $f_2 : A \times A \to B$ is anti-monotonic in its first argument and monotonic in its second argument. (Such functions f are called approximations in (Vennekens, Gilis, and Denecker 2006).) Composition is ordinary function composition and for each complete lattice A, the identity morphism $\mathbf{id}_A : A \stackrel{\bullet}{\to} A$ is the identity function $\mathrm{id}_{A\times A} = \mathrm{id}_A \times \mathrm{id}_A = \langle \pi_1^{A\times A}, \pi_2^{A\times A} \rangle : A \times A \to A \times A$.

The category $\mathbf{CL}$ has finite products. (Actually it has all products). Indeed, a terminal object T of $\mathbf{CL}$ is any 1-element lattice. Suppose that $A_1, \ldots, A_n$ are complete lattices. Then consider the direct product $A_1 \times \cdots \times A_n$ as an object of $\mathbf{CL}$ together with the following morphisms $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n} : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_i$, $i \in [n]$. For each i, $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$ is the function $A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \to A_i \times A_i$ defined by $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}(x_1, \ldots, x_n, x_1', \ldots, x_n') = (x_i, x_i')$, so that in SET, $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$ can be written as

$$\langle \pi_i^{A_1\times\cdots\times A_n\times A_1\times\cdots\times A_n}, \pi_{n+i}^{A_1\times\cdots\times A_n\times A_1\times\cdots\times A_n} \rangle = \\ = \pi_i^{A_1\times\cdots\times A_n} \times \pi_i^{A_1\times\cdots\times A_n}.$$

It is easy to see that the morphisms $\boldsymbol{\pi}_i^{A_1\times\cdots\times A_n}$, $i \in [n]$, determine a product diagram in $\mathbf{CL}$. To this end, let $f^i = \langle f_1^i, f_2^i \rangle : C \stackrel{\bullet}{\to} A_i$ in $\mathbf{CL}$, for all $i \in [n]$, so that each f^i is a $\leq_p$-monotonic function $C \times C \to A_i \times A_i$. Then let $h = \langle h_1, h_2 \rangle$ be the function $C \times C \to A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n$, where $h_1 = \langle f_1^1, \ldots, f_1^n \rangle$ and $h_2 = \langle f_2^1, \ldots, f_2^n \rangle$. Thus, h_1 and h_2 are functions $C \times C \to A_1 \times \cdots \times A_n$. The tupling of any sequence of morphisms $f^i = \langle f_1^i, f_2^i \rangle : C \stackrel{\bullet}{\to} A_i$ in $\mathbf{CL}$ is $h = \langle h_1, h_2 \rangle$. We will denote it by $\boldsymbol{\langle} f^1, \ldots, f^n \boldsymbol{\rangle} : C \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$.

Proposition 1 $\mathbf{CL}$ *is a cartesian category in which the product of any objects* $A_1, \ldots, A_n$ *agrees with their product in* CL.

For further use, we note the following. Suppose that $\rho : [m] \to [n]$ and $A_1, \ldots, A_n$ are complete lattices. Then the associated morphism $\boldsymbol{\rho}^{A_1,\ldots,A_n} : A_1 \times \cdots \times A_n \stackrel{\bullet}{\to} A_{\rho(1)} \times \cdots \times A_{\rho(m)}$ in $\mathbf{CL}$ is the function

$$A_1 \times \cdots \times A_n \times A_1 \times \cdots \times A_n \to \\ A_{\rho(1)} \times \cdots \times A_{\rho(m)} \times A_{\rho(1)} \times \cdots \times A_{\rho(m)}$$

given by

$$(x_1, \ldots, x_n, x_1', \ldots, x_n') \mapsto \\ (x_{\rho(1)}, \ldots, x_{\rho(m)}, x_{\rho(1)}', \ldots, x_{\rho(m)}').$$

Thus, $\boldsymbol{\rho}^{A_1,\ldots,A_n} = \rho^{A_1,\ldots,A_n} \times \rho^{A_1,\ldots,A_n}$, where $\rho^{A_1,\ldots,A_n}$ is the morphism associated with ρ and $A_1, \ldots, A_n$ in SET (or CL). This is in accordance with $\mathbf{id}_A = \mathrm{id}_A \times \mathrm{id}_A$.

Suppose that $f : C \stackrel{\bullet}{\to} A$ and $g : D \stackrel{\bullet}{\to} B$ in $\mathbf{CL}$, so that f is a function $C \times C \to A \times A$ and g is a function $D \times D \to B \times B$. Then $f \times g : C \times D \stackrel{\bullet}{\to} A \times B$ in the category $\mathbf{CL}$ is the function $(\mathrm{id}_A \times \langle \pi_2^{B\times A}, \pi_1^{B\times A} \rangle \times \mathrm{id}_B) \circ h \circ (\mathrm{id}_C \times \langle \pi_2^{D\times C}, \pi_1^{D\times C} \rangle \times \mathrm{id}_D)$ from $C \times D \times C \times D$ to $A \times B \times A \times B$, where h is $f \times g : C \times C \times D \times D \to A \times A \times B \times B$ in SET.

Some subcategories

Motivated by (Denecker, Marek, and Truszczyński 2000; 2004; Vennekens, Gilis, and Denecker 2006), we define several subcategories of $\mathbf{CL}$. Suppose that A, B are complete lattices. Following (Denecker, Marek, and Truszczyński 2000), we call an ordered pair $(x, x') \in A \times A$ *consistent* if $x \leq x'$. Moreover, we call $f : A \stackrel{\bullet}{\to} B$ in $\mathbf{CL}$ consistent if it maps consistent pairs to consistent pairs. The consistent morphisms in $\mathbf{CL}$ determine a cartesian subcategory of $\mathbf{CL}$ with the same product diagrams. Let $\mathbf{CCL}$ denote this subcategory. We define two subcategories of $\mathbf{CCL}$. The first one, $\mathbf{ACL}$, is the subcategory determined by those morphisms $f = \langle f_1, f_2 \rangle : A \stackrel{\bullet}{\to} B$ in $\mathbf{CL}$ such that $f_1(x, x) \leq f_2(x, x)$ for all $x \in A$. The second, $\mathbf{EACL}$, is the subcategory determined by those $f : A \stackrel{\bullet}{\to} B$ with $f_1(x, x) = f_2(x, x)$. These are again cartesian subcategories with the same product diagrams.

As noted in (Denecker, Marek, and Truszczyński 2000), most applications of approximation fixed point theory use *symmetric* functions. We introduce the subcategory of $\mathbf{CL}$ having complete lattices as object but only symmetric $\leq_p$-preserving functions as morphisms. Suppose that $f : A \stackrel{\bullet}{\to} B$ in $\mathbf{CL}$, say $f = \langle f_1, f_2 \rangle$, We call f symmetric if $f_2(x, x') = f_1(x', x)$, i.e., when

$$f_2 \quad = \quad f_1 \circ \langle \pi_2^{A\times A}, \pi_1^{A\times A} \rangle : A \times A \to B.$$

The symmetric morphisms determine a subcategory of $\mathbf{CL}$, denoted $\mathbf{SCL}$. In fact, $\mathbf{SCL}$ is a subcategory of $\mathbf{EACL}$. Moreover, it is again a cartesian subcategory with the same products.

4. Fixed points

Suppose that A and B are complete lattices, ordered by $\leq$, and let $f : A \times B \to A$ be a monotonic function. The least fixed point operation on CL maps f to the monotonic function $f^\dagger : B \to A$ such that for all $y \in B$, $f^\dagger(y)$, sometimes also denoted $\mu x.f(x,y)$, is the least solution of the fixed point equation $x = f(x,y)$. The existence of $f^\dagger(y)$ is guaranteed by the Knaster-Tarski fixed point theorem. It is also known that $f^\dagger(y)$ is the least $z \in A$ such that $f(z,y) \leq z$.

In this section, we recall from (Denecker, Marek, and Truszczyński 2000) the construction of stable and well-founded fixed points. Suppose that $f = \langle f_1, f_2 \rangle : A \stackrel{\bullet}{\to} A$ in $\mathbf{CL}$, so that f is a $\leq_p$-monotonic function $A \times A \to A \times A$. Then $f_1 : A \times A \to A$ is monotonic in its first argument and anti-monotonic in its second argument, and $f_2 : A \times A \to A$ is monotonic in its second argument and anti-monotonic in its first argument. Define the functions $s_1, s_2 : A \to A$ by

$$s_1(x') = \mu x.f_1(x,x') \quad \text{and} \quad s_2(x) = \mu x'.f_2(x,x'),$$

and let $S(f) : A \times A \to A \times A$ be the function $S(f)(x,x') = (s_1(x'), s_2(x))$. Since s_1 and s_2 are anti-monotonic, $S(f)$ is a morphism $A \stackrel{\bullet}{\to} A$ in $\mathbf{CL}$. We call $S(f)$ the *stable function* for f. It is known that every fixed point of $S(f)$ is a fixed point of f, called a *stable fixed point* of f. Since $S(f)$ is $\leq_p$-monotonic, there is a $\leq_p$-least stable fixed point $f^\ddagger$, called the *well-founded fixed* point of f.

The above construction can slightly be extended. When $f : A \times B \stackrel{\bullet}{\to} A$ is in $\mathbf{CL}$ and $(y,y') \in B \times B$, then let $g : A \stackrel{\bullet}{\to} A$ be given by $g(x,x') = f(x,y,x',y')$. Then we define $f^\ddagger(y,y') = g^\ddagger$.

Proposition 2 *Suppose that $f : A \times B \stackrel{\bullet}{\to} A$ is in $\mathbf{CL}$. Then $f^\ddagger : B \stackrel{\bullet}{\to} A$ is also in $\mathbf{CL}$. However, neither of the subcategories $\mathbf{CCL}$, $\mathbf{ACL}$, $\mathbf{EACL}$ and $\mathbf{SCL}$ is closed under ‡.*

5. Some valid identities

Iteration categories capture the equational properties of several fixed point operations including the least fixed point operation over CL. Axiomatizations of iteration categories can be conveniently divided into two parts, axioms for Conway categories and the commutative (Bloom and Ésik 1993) or group identities (Ésik 1999b), or the generalized power identities of (Ésik 1999a). Known axiomatizations of Conway categories include the group consisting of the parameter (1), composition (5) and double dagger (7) identities, and the group consisting of the fixed point (2), parameter (1), pairing (6) and permutation (3) identities. In this section we establish several of the above mentioned identities for the parametrized well-founded fixed point operation over $\mathbf{CL}$. In the next section we will show that several others fail.

Proposition 3 *The following parameter (1), fixed point (2) and permutation (3) identities hold:*

$$(f \circ (\mathbf{id}_A \times g))^\ddagger \quad = \quad f^\ddagger \circ g, \tag{1}$$

for all $f : A \times B \stackrel{\bullet}{\to} A$ and $g : C \stackrel{\bullet}{\to} B$.

$$f \circ \langle f^\ddagger, \mathrm{id}_B \rangle \quad = \quad f^\ddagger, \tag{2}$$

$f : A \times B \stackrel{\bullet}{\to} A$.

$$(\boldsymbol{\rho} \circ f \circ (\boldsymbol{\rho}^{-1} \times \mathbf{id}_B))^\ddagger \quad = \quad \boldsymbol{\rho} \circ f^\ddagger, \tag{3}$$

for all $f : A_1 \times \cdots \times A_n \times B \stackrel{\bullet}{\to} A_1 \times \cdots \times A_n$ and permutation $\rho : [n] \to [n]$.

Also, a special case of the pairing identity (6) holds:

Proposition 4 *The following identity holds:*

$$\langle f, g \circ (\boldsymbol{\pi}_2^{A \times B} \times \mathbf{id}_C) \rangle^\ddagger \quad = \quad \langle f^\ddagger \circ \langle g^\ddagger, \mathbf{id}_C \rangle, g^\ddagger \rangle, \tag{4}$$

where $f : A \times B \times C \stackrel{\bullet}{\to} A$ and $g : B \times C \stackrel{\bullet}{\to} B$.

The identity (4) has already been established in Theorem 3.11 of (Vennekens, Gilis, and Denecker 2006), see also the Splitting Set Theorem of (Lifschitz and Turner 1994).

Proposition 5 *The weak functorial dagger implication holds: for all $f : A^n \times B \stackrel{\bullet}{\to} A^n$ and $g : A \times B \stackrel{\bullet}{\to} A$ in $\mathbf{CL}$: if $f \circ (\boldsymbol{\Delta}_n \times \mathbf{id}_B) = \boldsymbol{\Delta}_n \circ g$, then $f^\ddagger = \boldsymbol{\Delta}_n \circ g^\ddagger$.*

Since the weak functorial implication holds, so do the commutative and group identities (Bloom and Ésik 1993; Ésik 1999b).

6. Some identities that fail

Proposition 6 *The composition identity*

$$(f \circ \langle g, \boldsymbol{\pi}_2^{A \times C} \rangle)^\ddagger \quad = \quad f \circ \langle (g \circ \langle f, \pi_2^{B \times C} \rangle)^\ddagger, \mathbf{id}_C \rangle, \tag{5}$$

$f : B \times C \stackrel{\bullet}{\to} A$, $g : A \times C \stackrel{\bullet}{\to} B$, fails in $\mathbf{CL}$, even in the following simple case: $f \circ (f \circ f)^\ddagger = (f \circ f)^\ddagger$, where $f : A \stackrel{\bullet}{\to} A$.

Proposition 7 *The squaring identity $(f \circ f)^\ddagger = f^\ddagger$ fails, where $f : A \stackrel{\bullet}{\to} A$.*

Since the fixed point, parameter and permutation identities hold but the composition identity fails, the pairing identity (6) also must fail, see (Bloom and Ésik 1993).

Proposition 8 *The pairing identity*

$$\langle f, g \rangle^\ddagger \quad = \quad \langle f^\ddagger \circ \langle h^\ddagger, \mathbf{id}_C \rangle, h^\ddagger \rangle, \tag{6}$$

where $h = g \circ \langle f^\ddagger, \mathbf{id}_{B \times C} \rangle$ fails, where $f : A \times B \times C \stackrel{\bullet}{\to} A$ and $g : A \times B \times C \stackrel{\bullet}{\to} B$.

Proposition 9 *The double dagger identity*

$$f^{\ddagger\ddagger} \quad = \quad (f \circ (\langle \mathbf{id}_A, \mathbf{id}_A \rangle \times \mathbf{id}_B))^\ddagger, \tag{7}$$

$f : A \times A \times B \stackrel{\bullet}{\to} A$, fails in $\mathbf{CL}$, even in the particular case when $B = T$ (terminal object).

7. Some applications

The established identities can be seen as abstract versions of transformations over logic programs that preserve the well-founded semantics (in the bilattice setting). For one example, consider the simple propositional logic program $p :- \ q, \sim r;\ q :- \ r, \sim p;\ r :- \ p, \sim q$. Identifying p, q, r, we obtain $p :- \ p, \sim p$. By the weak functorial implication, the two programs are equivalent in the sense that each component of the well-founded semantics of the first program agrees with the well-founded semantics of the second.

By formulating transformations as identities, one can use standard (many-sorted) equational logic to derive other identities that in turn give rise to new transformations. For example, the following identity is an equational consequence of those established in the paper:

$$\langle f, g \circ \pi_B^{A\times B} \rangle^\ddagger = \langle f \circ (\mathbf{id}_A \times \langle g, \pi_C^{B\times C} \rangle), g \circ \pi_{B\times C}^{A\times B\times C} \rangle^\ddagger$$

where $f : A \times B \times C \xrightarrow{\bullet} A$ and $g : B \times C \xrightarrow{\bullet} B$.

This identity is an abstract version of the fold/unfold transformation (Tamaki and Sato 1984; Seki 1993). For example, it yields that the following propositional logic programs $p :- q, r;\ r :- s, t$ and $p :- q, s, t;\ r :- s, t$ are equivalent for the well-founded semantics. On the other hand, the following identity, which is a generalization of the above folding/unfolding identity, fails: $\langle f \circ \langle \pi_A^{A\times B}, g \rangle, g \rangle^\ddagger = \langle f, g \rangle^\ddagger$, where $f : A \times B \xrightarrow{\bullet} A$ and $g : A \times B \xrightarrow{\bullet} B$. And this again follows by standard equational reasoning using our positive and negative results.

8. Conclusion

We extended the well-founded fixed point operation of (Denecker, Marek, and Truszczyński 2000; Vennekens, Gilis, and Denecker 2006) to a parametric operation and studied its equational properties. We found that several of the identities of iteration theories hold for the parametric well-founded fixed point operation, but some others fail. By showing that some identities of iteration theories do not hold, we tried to have a better understanding why logic programs with the well-founded semantics cannot be manipulated using standard fixed point methods. And by showing that some other identities hold, we tried to understand to what extent the standard techniques can be used for manipulating logic programs. Two interesting questions arise for further investigation. The first concerns the *algorithmic description* of the valid identities of the well-founded fixed point operation. The second concerns the *axiomatic description* of the valid identities. These questions are also relevant in connection with modular logic programing, cf. (Ferraris et al. 2009; Janhunen et al. 2009; Lifschitz and Turner 1994).

An alternative semantics of logic programs with negation based on an infinite domain of truth values was proposed in (Rondogiannis and Wadge 2005). The infinite valued approach has been further developed in the abstract setting of 'stratified complete lattices'. It has been proved in (Ésik 2015b) that the stratified least fixed point operation arising in this approach *does* satisfy all identities of iteration theories. So in this regards, the infinite valued semantics behaves just as the Kripke-Kleene semantics (Fitting 2002).

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References

Bloom, S. L., and Ésik, Z. 1993. *Iteration Theories - The Equational Logic of Iterative Processes*. EATCS Monographs on Theoretical Computer Science. Springer.

Bloom, S. L., and Ésik, Z. 1996. Fixed-point operations on ccc's. part I. *Theor. Comput. Sci.* 155(1):1–38.

Carayol, A., and Ésik, Z. 2015. An analysis of the equational properties of the well-founded fixed point. Report, arXiv:1511.09423.

Davey, B. A., and Priestley, H. A. 1990. *Introduction to lattices and order*. Cambridge: Cambridge University Press.

Denecker, M.; Marek, V. M.; and Truszczyński, M. 2000. Approximations, stable operators, well-founded fixpoints and applications in nonmonotonic reasoning. In *Logic-Based Artificial Intelligence*. Springer. 127–144.

Denecker, M.; Marek, V. W.; and Truszczyński, M. 2004. Ultimate approximation and its application in nonmonotonic knowledge representation systems. *Inf. Comput.* 192(1):84–121.

Ésik, Z. 1999a. Axiomatizing iteration categories. *Acta Cybern.* 14(1):65–82.

Ésik, Z. 1999b. Group axioms for iteration. *Inf. Comput.* 148(2):131–180.

Ésik, Z. 2015a. Equational properties of fixed point operations in cartesian categories: An overview. In *MFCS 2015, Proceedings, Part I*, volume 9234 of *LNCS*, 18–37. Springer.

Ésik, Z. 2015b. Equational properties of stratified least fixed points (extended abstract). In *WoLLIC 2015, Bloomington, IN, USA, July 20-23, 2015, Proceedings*, volume 9160 of *LNCS*, 174–188. Springer.

Ferraris, P.; Lee, J.; Lifschitz, V.; and Palla, R. 2009. Symmetric splitting in the general theory of stable models. In *IJCAI 2009, Proceedings*, 797–803.

Fitting, M. 2002. Fixpoint semantics for logic programming a survey. *Theor. Comput. Sci.* 278(1-2):25–51.

Janhunen, T.; Oikarinen, E.; Tompits, H.; and Woltran, S. 2009. Modularity aspects of disjunctive stable models. *J. Artif. Intell. Res. (JAIR)* 35:813–857.

Lifschitz, V., and Turner, H. 1994. Splitting a logic program. In *Logic Programming 1994, Proceedings*, 23–37. MIT Press.

Rondogiannis, P., and Wadge, W. W. 2005. Minimum model semantics for logic programs with negation-as-failure. *ACM Trans. Comput. Log.* 6(2):441–467.

Seki, H. 1993. Unfold/fold transformation of general logic programs for the well-founded semantics. *J. Log. Program.* 16(1):5–23.

Tamaki, H., and Sato, T. 1984. Unfold/fold transformation of logic programs. In *Second International Logic Programming Conference, Proceedings*, 127–138.

Vennekens, J.; Gilis, D.; and Denecker, M. 2006. Splitting an operator: Algebraic modularity results for logics with fixpoint semantics. *ACM Trans. Comput. Log.* 7(4):765–797.