On rational trees

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Abstract. Rational graphs are a family of graphs defined using labelled rational transducers. Unlike automatic graphs (defined using synchronized transducers) the first order theory of these graphs is undecidable, there is even a rational graph with an undecidable first order theory. In this paper we consider the family of rational trees, that is rational graphs which are trees. We prove that first order theory is decidable for this family. We also present counter examples showing that this result cannot be significantly extended both in terms of logic and of structure.

1 Introduction

The algorithmic study of infinite object has achieved many success through the use of finite automata. This concise and efficient model was first introduced to characterize word languages in the late fifties, since then it has been extended and generalized in order to define infinite words, relations, relational structures, group structures, or graphs.
In 1960 Büchi, [Büc60], used finite automata to characterize infinite words, and so proving the decidability of monadic second order logic of the integers with the successor relation. Almost ten years later, this result was extended to the complete binary tree by Rabin [Rab69]. For many years adhoc extensions were proposed. Later on, around the year 1990 Muller and Schupp, then Courcelle and finally Caucal proposed generalizations of Rabin's result based on transformation of the complete binary tree [MS85,Cou90,Cau96].
Another way of using finite automata in the theory of finitely presented infinite objects was introduced by Hodgson [Hod83], simply using finite automata to define relational structures, obtaining the decidability of first order logic. Later on, nurturing from group theory [ECH$^+$92], Khoussainov and Nerode formalized and generalized the notion of automatic structure (and graph) [KN94]. Independently Sénizergue, and later on Pelecq considered a slightly different notion of automatic structure, involving an automatic quotient [Sén92,Pél97]. Several investigations, as well as an extension of first-order logic were conducted by Blumensath and Grädel on automatic structures [BG00]. In 2000 the notion of rational graphs was investigated [Mor00], this general family had already been defined as asynchronous automatic by Khoussainov and Nerode, but it was not very satisfactory from the logical point of view.
In most of these cases the decidability of the logic comes from the underlying automaton, or more generally from closure properties. An interesting question

is to know whether some structural restriction of these families would yield better decidability results. For automatic structures recently Khoussainov *et alii* considered automatic trees [KRS05], and have been able to disclose properties of these trees, like their Cantor-Bendixson rank, or the existence of a rational infinite path.
In this paper we consider rational trees, that is rational graphs that are trees. We first define carefully this family, state a few basic results, and give simple examples. We then use Gaifman's theorem and compositional methods [She75,Zei94] to prove that their first order logic is decidable. As it is not the case for general rational graphs, it heavily relies on the tree structure, and need a deep investigation. Finally we explore the boundaries of this result by exhibiting a rational directed acyclic graph with an undecidable first-order theory, and also a rational tree with an undecidable first-order theory enriched with rational accessibility.

2 Preliminaries

In this section we will recall the definition of the family of rational graphs. More details can be found in [Mor00,MS01]. We also state some properties of automatic graphs [KN94].
For any set E, its powerset is denoted by 2^E; if it is finite, its size is denoted by $|E|$. Let the set of nonnegative integers be denoted by $\mathbb{N}$, and $\{1, 2, 3, \ldots, n\}$ be denoted by $[n]$. A monoid M is a set equipped with an associative operation (denoted $\cdot$) and a (unique) neutral element (denoted ε). A monoid M is *free* if there exist a finite subset A of M such that $M = A^* := \bigcup_{n\in\mathbb{N}} A^n$ and for each $u \in M$ there exists a unique finite sequence of elements of A, $(u(i))_{i\in[n]}$, such that $u = u(1)u(2)\cdots u(n)$. Elements of a free monoid will be called words. Let u be a word in M, $|u|$ denotes the length of u and $u(i)$ denotes its ith letter.

2.1 Rational graphs

The family of rational subsets of a monoid $(M, \cdot)$ is the least family containing the finite subsets of M and closed under union, concatenation and iteration.
A transducer is a finite automaton labelled by pairs of words over a finite alphabet X, see for example [AB88] [Ber79]. A transducer accepts a relation in $X^* \times X^*$; these relations are called rational relations as they are rational subsets of the product monoid $(X^* \times X^*, \cdot)$.
Now, let Γ and X be two finite alphabets. A *graph* G is a subset of $X^* \times \Gamma \times X^*$. An *arc* is a triple: $(u, a, v) \in X^* \times \Gamma \times X^*$ (denoted by $u \xrightarrow[G]{a} v$ or simply $u \xrightarrow{a} v$ if G is understood).
Rational graphs, denoted by $Rat(X^* \times \Gamma \times X^*)$, are extensions of rational relations, characterized by *labelled transducers*.

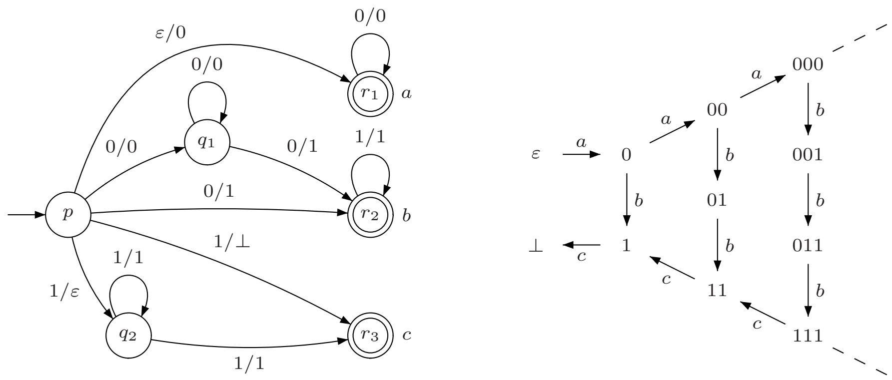
Definition 2.1. A *labelled transducer* $T = (Q, I, F, E, L)$ over X, is composed of a finite set of states Q, a set of initial states $I \subseteq Q$, a set of final states $F \subseteq Q$, a finite set of transitions (or edges) $E \subseteq Q \times X^* \times X^* \times Q$ and a mapping L from F into 2^Γ.

An arc $u \xrightarrow{a} v$ is *accepted* by a labelled transducer T if there is a path from a state in I to a state f in F labelled by (u, v) and such that $a \in L(f)$.

Definition 2.2. A graph in $2^{X^* \times \Gamma \times X^*}$ is *rational* if it is accepted by a labelled rational transducer.

Let G be a rational graph, for each a in Γ we denote by G_a the restriction of G to arcs labelled by a (it defines a rational relation between vertices); let u be a vertex in X^*, we denote by $G_a(u)$ the set of all vertices v such that $u \xrightarrow{a} v$ is an arc of G.

Example 2.3. The graph on the right is generated by the labelled transducer on the left.



The path $p \xrightarrow{0/0} q_1 \xrightarrow{0/1} r_2 \xrightarrow{1/1} r_2$ accepts the couple $(001, 011)$, the final state r_2 is labelled by b thus there is a arc $001 \xrightarrow{b} 011$ in the graph.

The *trace* of a graph G from an initial vertex i to a final vertex f is the set of path labels labelling a path from i to f. For example the trace of the graph from Example 2.3 between ε and $\perp$ is the set $\{a^n b^n c^n \mid n > 0\}$

Theorem 2.4 (Morvan, Stirling 01) *The traces of rational graphs from an initial to a final vertex is precisely the context-sensitive languages.*

2.2 Automatic graphs

A classical subfamily of rational graphs is formed by the set of automatic graphs [KN94,Pél97,BG00].

These graphs are accepted by letter-to-letter transducers with rational terminal functions completing one side of the accepted pairs and assigning a label to the arc.

As the terminal function is rational, it can be introduced in the transducer adding states and transitions. A *left-synchronized transducer* is a transducer such that each path leading from an initial state to a final one can be divided into two parts: the first one contains arcs of the form $p \xrightarrow{A/B} q$ with $A, B \in X$ while the second part contains either arcs of the form $p \xrightarrow{A/\varepsilon} q$ with $A \in X$ or of the form $p \xrightarrow{\varepsilon/B} q$ with $B \in X$ (not both). Right-synchronized transducers are defined conversely.

Definition 2.5. A graph over $X^* \times \Gamma \times X^*$ is *automatic* if it is accepted by a left-synchronized or right-synchronized labelled transducer T.

Example 2.6. The graph defined by Example 2.3 is automatic. The relation G_b is synchronized. And the relations G_a and G_c are right-automatic.

The next result follows from the fact that automatic relations form a boolean algebra.

Proposition 2.7 *The first-order theory of automatic graphs is decidable.*

The Theorem 2.4 was extended to automatic graphs by Rispal in [Ris02].

Theorem 2.8 (Rispal 02) *The traces of rational graphs from an initial to a final vertex is precisely the context-sensitive languages.*

3 Rational trees, examples and boundaries

Trees are natural structures in computer science. A lot of families of trees occurred outside of the study of infinite graphs. For example, regular trees that have only a finite number of sub-trees up to isomorphism, algebraic trees which are the unfolding of regular graphs [Cau02], or also trees that are solutions of higher order recursive program schemes [Dam77].

Definition 3.1. A *rational tree* is a rational graph satisfying these properties:

(i) it is connected;
(ii) every vertex is the target of at most one arc;
(iii) there is a single vertex with in-degree 1, called the *root*.

Each vertex of a rational tree is called a *node*. The *leaves* are vertices that are not source of any arc.

3.1 Elementary results

The properties *(ii)* and *(iii)* from Definition 3.1 are easy to verify: *(ii)* consists in checking that the relation $\bigcup_{a \in \Gamma} (\xrightarrow{a})^{-1}$ is functional. This is solved using Shutzenberger's theorem, see among others [Ber79]. The condition *(iii)* consist in checking that the rational set $Dom(T) \setminus Im(T)$ has only one element.
In order to prove that it is undecidable to check whether a rational graph is a tree, we use a variation of the classical uniform halting problem for Turing machines.

Proposition 3.2 *Given any deterministic Turing machine M, a deterministic Turing machine M' may be constructed such that: M halts on ε if and only if M' halts from any configuration.*

Proposition 3.3 *Given any deterministic Turing machine a rational (unlabelled) graph $G(M)$ may be constructed in such a way that: M halts from any configuration if and only if $G(M)$ is a tree.*

Proof. Let us consider the deterministic Turing machine $M = (Q, T, \delta, q_0)$, Q is the set of states (with $q_0 \in Q$ the initial state), T the set of tape symbols (including two special symbols \$ and # denoting the extremities of the tape) and $\delta : Q \times T \to Q \times T \times \{l, r, p\}$ the transition function.
We define the configuration of such a machine in the usual way: uqv, with $q \in Q, u \in \$(T + \Box)^*, v \in (T + \Box)^*\#$, and $\Box$ denoting the empty space.
We define $G(M)$ in this way: the *vertices* are precisely the configuration of the machine plus a special vertex \$#.
The *arcs* consist of the transitions of the machine going backwards, and of the set $\{\$\#\} \times \{\$uqAv\# \mid (q, A) \notin Dom(\delta) \wedge u, v \in (T + \Box)^*\}$.
The vertex \$# is the only vertex which is not the target of any arc (condition *(iii)*), and as the machine is deterministic and the arcs go backward, this graph satisfies also the condition *(ii)*. Furthermore this graph is connected if and only if the machine M reaches, from any configuration, a configuration in which there is no possible transition. $\Box$

From these two results considering a deterministic Turing machine we construct a second one that halts on every input if the first one stops from the empty word. Now using Proposition 3.3, we construct a rational graphs which is a tree if and only if the second machine halts one every input. This proves the following proposition.

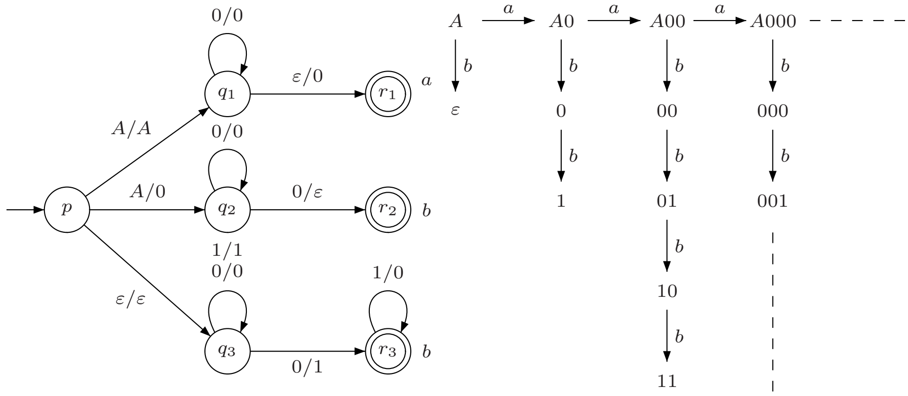
Proposition 3.4 *It is undecidable to know whether a rational graph is a tree.*

We conclude this subsection by a simple result, which is a direct consequence of the rationality of the inverse image of a rational relation, and the fact that all vertices are accessible from the root.

Proposition 3.5 *Given any rational tree, accessibility and rational accessibility are decidable for any given pair of vertices.*

3.2 The 2^n-tree

We give here a first example of rational tree. Indeed this tree is automatic. It is defined by a line of a's, and the nth vertex of this line is connected to a segment of 2^n b's.



The encoding of the vertices of this tree relies on the fact that there are 2^n n-tuples over $\{0, 1\}$. The transducer performs the binary addition.

3.3 A non-automatic rational tree

We now construct a rational tree of finite, yet unbounded, degree which is not automatic.

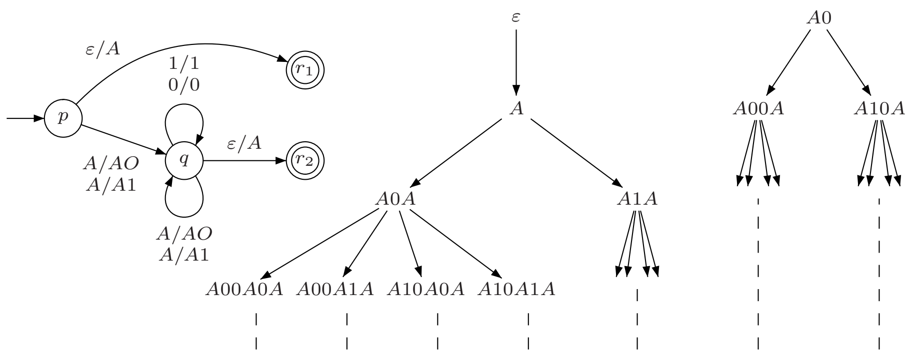
This tree is obtained from a rational forest by the adjunction of a line connecting the roots of each connected component. As these roots form a rational set of words, the following lemma allows construct such a line while still obtaining a rational tree.

Lemma 3.6 *Given a rational language L, the graph whose vertices are the words of L connected into a half line in length-lexicographic order is an automatic graph.*

This result is obtained by remarking that the length-lexicographic order (as a relation on words) is an automatic relation, and using closure properties of these relations.

Our example relies on the limit of the growth rate of automatic graphs of finite degree. For such an automatic tree, an obvious counting argument ensures that there exists p, q and s such that there are at most p^{qn+s} vertices at distance n of the root.

Therefore the tree (we call it *simplexp*) such that each vertex of depth n has 2^n sons, has precisely $2^{n(n-1)/2}$ vertices of depth n, and is therefore not automatic. Still it is the connected component of a rational forest F and the tree constructed from F using the Lemma 3.6 has the same growth and therefore is not automatic, up to isomorphism. For simplicity we only present the forest F and a transducers generating it:



In this forest the connected component of ε is simplexp. Each vertex of depth n has precisely n occurrence of A and thus 2^n sons. Furthermore this transducer is co-functional, and strictly increasing, therefore each connected component is a tree with root. We have, thus, constructed a rational tree of finite degree, which is, up to isomorphism, not automatic.

4 First-order theory of rational trees is decidable

In this section we use Gaifman's theorem (see, e.g., [EF95]) to prove that the first-order theories of rational trees are decidable. This result, which is not true for rational graphs in general, was conjectured in [Mor01]. We will see, in Section 5, that there are no obvious extensions of this result.

4.1 Logical preliminaries

We introduce basic notations on first-order logic over relational structures.
A *relational signature* Σ is a ranked alphabet. For every symbol $R \in \Sigma$, we write $|R| \geq 1$ the arity of R. A relational structure $\mathcal{M}$ over Σ is given by a tuple $(M, (R^{\mathcal{M}})_{R \in \Sigma})$ where M is the *universe* of $\mathcal{M}$ and where for all $R \in \Sigma$, $R^{\mathcal{M}} \subseteq M^{|R|}$.
Let $\mathcal{V}$ be a countable set of first-order variables. We use $x, y, z \ldots$ to range over first-order variables in $\mathcal{V}$ and $\bar{x}, \bar{y}, \bar{z}, \ldots$ to designate tuples of first-order variables. An atomic formula over Σ is either $R(x_1, \ldots, x_{|R|})$ for $R \in \Sigma$ and $x_1, \ldots, x_{|R|} \in \mathcal{V}$ or $x = y$ for $x, y \in \mathcal{V}$. Formulas over Σ (Σ-formulas) are obtained by closure under conjunction $\wedge$, negation $\neg$ and existential quantification $\exists$ starting from the atomic Σ-formulas. The bounded and free variables of a formula are defined as usual. A formula without free variables is also called a *sentence*. We write $\varphi(\bar{x})$ to indicate that the free variables of φ belong to $\bar{x}$.
For every relational structure $\mathcal{M}$, any formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n$ in M, we write $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$ if $\mathcal{M}$ satisfies the formula φ when x_i is interpretated as a_i. If φ is a sentence, we simply write $\mathcal{M} \models \varphi$. Two sentences φ and ψ are logically equivalent if for all structure $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \psi$.

The *quantifier rank* $\mathrm{qr}(\varphi)$ of a formula φ is defined by induction on the structure of φ by taking $\mathrm{qr}(\varphi) = 0$ for φ atomic, $\mathrm{qr}(\varphi \wedge \psi) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$, $\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$ and $\mathrm{qr}(\exists x\, \varphi) = \mathrm{qr}(\varphi) + 1$. For a fixed signature Σ, there are countably many Σ-sentences of a given quantifier rank. Up to logical equivalence there are only finitely many such sentences, but this equivalence is undecidable. A classical way to overcome this problem is to define a (decidable) syntactical equivalence on formulas such that, up to this equivalence, there are only finitely many formulas of a given quantifier rank (see e.g. [EF95]).
We define for all rank $k \geq 0$ a finite set Norm_k^Σ of normalized Σ-sentences such that for every Σ-sentence φ we can effectively compute a logically equivalent sentence $\mathrm{Norm}(\varphi)$ in Norm_k^Σ. Note that this set is finite and computable.

$$\begin{aligned}
\mathrm{ANA}^\Sigma(\bar{x}) &= \{\, \varphi, \neg\varphi \mid \varphi \text{ atomic over } \Sigma \text{ with free variables in } \bar{x} \,\} \\
\mathrm{Norm}_0^\Sigma(\bar{x}) &= \left\{ \bigvee_{R \in \mathcal{R}} \bigwedge_{\varphi \in R} \varphi \mid \mathcal{R} \subseteq 2^{\mathrm{ANA}^\Sigma(\bar{x})} \right\} \\
\mathrm{Norm}_{k+1}^\Sigma(\bar{x}) &= \left\{ \bigvee_{R \in \mathcal{R}} \bigwedge_{\varphi \in R} \varphi \mid \mathcal{R} \subseteq 2^{\left\{\, \exists y \varphi, \forall y \varphi \,\mid\, \varphi \in \mathrm{Norm}_k^\Sigma(\bar{x}, y) \,\right\}} \right\}
\end{aligned}$$

where $y \notin \bar{x}$.
The k-theory of a structure $\mathcal{M}$ over Σ is the finite set

$$\mathrm{Thm}_k(\mathcal{M}) := \left\{ \varphi \mid \varphi \in \mathrm{Norm}_k^\Sigma \text{ and } \mathcal{M} \models \varphi \right\}.$$

We write $\mathrm{Thm}_k^\Sigma = 2^{\mathrm{Norm}_k^\Sigma}$ the set of all possible k-theories[1].

4.2 Gaifman's Theorem for graph structures

We now focus our attention on graph structures and particularly on trees. A *graph structure* is a relational structure over a signature with symbols of arity 2. To every graph Σ-structure is associated a graph labelled by the symbols of Σ. We say that a graph structure is a *tree structure* if the associated graph is a tree. For all tree structure $\mathcal{T}$, we write $r(\mathcal{T}) \in T$ the root of $\mathcal{T}$. For all $u \in T$, we write $\mathcal{T}_{/u}$ the subtree of $\mathcal{T}$ rooted at u and for all $n \geq 0$, $\mathcal{T}_{/u}^n$ the tree $\mathcal{T}_{/u}$ restricted to the elements of depth at most n.
We recall Gaifman's Theorem, which states that every first-order formula is logically equivalent to a *local formula*.
In order to define local formulas, it is first necessary to define a notion of *distance*. In the following, we write $d(x, y) \leqslant n$ (resp. $d(x, y) < n$) the first-order formula expressing that the distance, without taking the orientation of the arcs into account, between x and y is less or equal to n (resp. less than n).
We denote by $S(r, x)$ the ball of radius r centered at x: $\{y \mid d(x, y) \leqslant r\}$
We now need to restrict a formula $\varphi(x)$ to a ball of radius r centered at x: we denote by $\varphi^{S(r,x)}$ the restriction of formula $\varphi(x)$ to the ball of center x and

[1] Remark that Thm_k contains elements that are not the k-theory of any structure. For instance, an element of Thm_k may contain both φ and $\neg\varphi$.

radius r. This notation is defined by renaming each bound occurrence of x in φ by a new variable, and localizing each quantification:

$$[\exists z \varphi]^{S(r,x)} := \exists z (d(x,z) \leqslant r \wedge \varphi^{S(r,x)})$$

A basic local formula is of the the form:

$$\exists x_1 \ldots \exists x_n \bigwedge_{1 \leqslant i < j \leqslant n} (d(x_i, x_j) > 2r \wedge \psi^{S(r,x_i)}(x_i))$$

A local sentence is a boolean combination of basic local sentences.

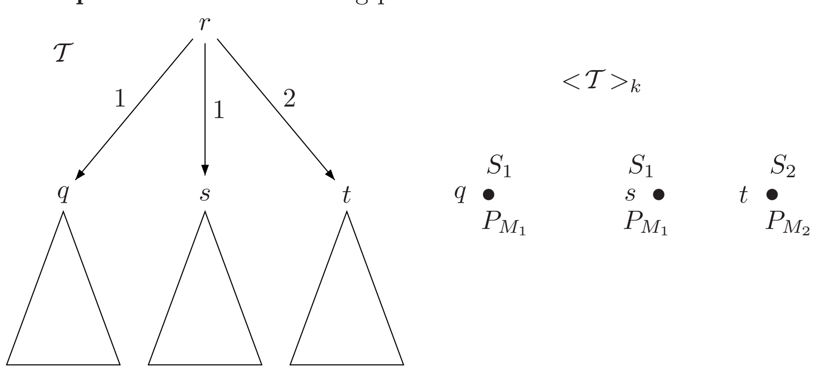
Theorem 4.1 (Gaifman) *Every first-order sentence is logically equivalent to a local sentence.*

Note that the equivalence stated in this theorem is effective.

4.3 Compositional results for trees

We present basic compositional results for trees that will allow us to characterize the center of the balls involved in the definition of basic local formulas. The compositional method is a powerfull way to obtained decidability results mainly developed in [She75] (see [Zei94,Rab06] for a survey). The results presented here are not new and could, for example, be derived from the general templates presented in [Zei94,Rab06].
For every tree structure $\mathcal{T}$ over the signature $\Sigma = \{E_1, \ldots, E_\ell\}$ and for every $k \geq 1$, we define *the reduced tree of* $\mathcal{T}$, the structure $<\mathcal{T}>_k$ over the monadic signature $<\Sigma>_k := \{S_1, \ldots, S_\ell\} \cup \left\{ P_M \mid M \in \mathrm{Thm}_k^{\Sigma} \right\}$. The universe of $<\mathcal{T}>_k$ is the set of successors of the root of $\mathcal{T}$. The predicats in $<\Sigma>_k$ are interpreted as follows: for all $i \in [\ell]$, $u \in S_i^{<\Sigma>_k}$ iff $(r(\mathcal{T}), u) \in E_i^{\mathcal{T}}$ and for all $M \in \mathrm{Thm}_k^{\Sigma}$, $u \in P_M$ iff $\mathrm{Thm}(\mathcal{T}_{/u}) = M$.

Example 4.2. In the following picture we illustrate a reduced tree.



The tree depicted on the left is defined over $\Sigma = \{E_1, E_2\}$, the reduced tree $<\mathcal{T}>_k$ is defined over $<\Sigma>_k := \{S_1, S_2\} \cup \left\{ P_M \mid M \in \mathrm{Thm}_k^{\Sigma} \right\}$; and $\mathrm{Thm}_k(\mathcal{T}_{/q}) = \mathrm{Thm}_k(\mathcal{T}_{/s}) = M_1, \mathrm{Thm}_k(\mathcal{T}_{/t}) = M_2$

Lemma 4.3 *For all tree structure $\mathcal{T}$ over $\Sigma = \{E_1, \ldots, E_\ell\}$ and all $k \geq 1$,* $\mathrm{Thm}_k(\mathcal{T})$ *can be effectively computed from* $\mathrm{Thm}_k(<\mathcal{T}>_{k+1})$.

Remark 4.4. As the signature of $<\mathcal{T}>_k$ is monadic, every formula is equivalent to a boolean combinaison of formulas of the form

$$\exists x_1, \ldots, x_\ell \bigwedge_{i \neq j} x_i \neq x_j \ \wedge \bigwedge_{P \in R, i \in [\ell]} P(x_i) \ \wedge \bigwedge_{P \notin R, i \in [\ell]} \neg P(x_i).$$

where $P \subseteq <\Sigma>_k$. See, for example, the Exercise 2.3.12 of [EF95].

The following lemma allows to compute the theory of a ball in a tree from the theories of some subtrees contained in that ball.

Lemma 4.5 *For all tree $\mathcal{T}$ over a signature $\Sigma = \{E_1, \ldots, E_\ell\}$ and any vertex $u \in T$ with a path $u_0 a_1 u_1 \ldots u_m$ (with $u_m = u$), from the root of T and any rank $k \geq 1$ and any depth $n \geq 0$, there exists a constant p effectively computable from m, n and k such that for any formula $\varphi(x)$ with $\mathrm{qr}(\varphi) = k$, we can decide whether $T \models \varphi^{S(n,x)}[u]$ from the sequence of labels $a_1 \ldots a_m$ and from* $(\mathrm{Thm}_p(<\mathcal{T}^n_{/u_i}>_p))_{i \in [0,m]}$.

4.4 First-order theory of rational trees

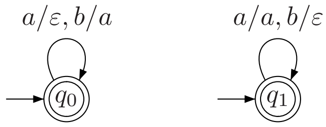
We now tackle the proof of the decidabilty of the first-order theories of rational trees using Gaifman's Theorem.
The first step is to use the results from Subsection 4.3 to prove that for all $r \geq 1$ and for all formula $\varphi(x)$, the set of centers of a ball of radius r satisfying $\varphi(x)$ (where x is interpreted as the center of the ball) form a rational set of words.
We start by showing that the set of roots of a subtree of a certain depth having a given k-theory form a rational set of words. In order to apply Lemma 4.3, we need the following key lemma concerning rational trees.

Lemma 4.6 *For all rational tree T labelled by Γ and over X^*, all $i \in \Gamma$ and $L \in \mathrm{Rat}(X^*)$, the set of $u \in \mathrm{Dom}(T)$ having a least ℓ successors by i in L is rational and can be effectively constructed.*

Proof (Sketch). The proof relies on the fact that the in-degree of a tree is of at most one. We use the uniformazition of rational relations [Eil74,Ber79] which states that for every transducer[2] H there exists a functional transducer $\overrightarrow{H}$ such that $\overrightarrow{H} \subseteq H$ and $\mathrm{Dom}(H) = \mathrm{Dom}(\overrightarrow{H})$. As the in-degree of T is at most one, if we restrict H_i (the transducer accepting the i-labelled arcs of T restricted in image to L) to the rational set $X^* \setminus \mathrm{Im}(\overrightarrow{H_i})$ to obtain a transducer H'_i, we have decreased the out-degree of H_i by exactly 1. Hence the set of vertices having at least 2 successors by i is $\mathrm{Dom}(H'_i)$. The proof then follows by a straightforward induction. □

[2] We do not distinguish between the transducer and the relation it accepts.

Remark 4.7. Note that this result does not hold when the in-degree is greater than 1. Consider for example, the transducer H depicted bellow. The set of words having exactly 1 image by H is the context-free language containing the words having the same number of a's and b's.



Lemma 4.8 *For all rational tree T labelled by $\Gamma = [\ell]$, all $k \geqslant 1$, $n \geqslant 1$, and all sentence φ over $\Sigma = \{E_1, \ldots, E_\ell\}$ or over $<\Sigma>_k$, the sets:*

- $L_\varphi^{n,k} := \{\, u \in \mathrm{Dom}(T) \mid\ <\mathcal{T}^n_{/u}>_k \models \varphi \,\}$
- $L_\varphi^{n} := \{\, u \in \mathrm{Dom}(T) \mid \mathcal{T}^n_{/u} \models \varphi \,\}$

are rational and effectively computable.

Proof (Sketch). We prove both properties simultaneously by induction on the depth n.
For the basis case $n = 0$, remark that for all rational tree T, T^0 is reduced to a single vertex and for all $k \geqslant 1$, $<T^0>_k$ is empty. As these structures are finite, we can decide for all formula φ if it is satisfied by the structure. Accordingly, L^0_φ and $L^{0,k}_\varphi$ are either the $\emptyset$ or $\mathrm{Dom}(T)$.
For the induction step $n+1$. Let $k \geqslant 1$ be a rank and φ be $<\Sigma>_k$-sentence. By Remark 4.4, we can restrict our attention to formulas stating there exists at least m elements belonging to $S_i^{<T>_k}$ and $P_M^{<T>_k}$ for some $i \in [\ell]$ and $M \in \mathrm{Thm}^\Sigma_k$.
Let $m \geqslant 0$, $i \in [\ell]$, $M \in \mathrm{Thm}^\Sigma_k$ and ψ the corresponding formula. By induction hypothesis, the set of vertices $X := \left\{\, u \in \mathrm{Dom}(T) \mid \mathrm{Thm}_k(\mathcal{T}^n_{/u}) = M \,\right\}$ is rational and computable. It is easy to check that for all $u \in \mathrm{Dom}(T)$, $<T^{n+1}_{/u}>_k$ satifies ψ if and only if u has m successors by i belonging to X. By Lemma 4.6, the set $L_\psi^{n+1,k}$ is rational.
The second property follows then by Lemma 4.3. □

It then follows by Lemma 4.5 and 4.8 that:

Lemma 4.9 *For all rational tree T labelled by $\Gamma = [\ell]$,all formula $\varphi(x)$ over $\Sigma = \{E_1, \ldots, E_\ell\}$ and $n \geq 1$, the set $\left\{\, u \in \mathrm{Dom}(T) \mid \mathcal{T} \models \varphi^{S(n,x)}[u] \,\right\}$ is rational and can be effectively computed.*

Before applying Gaifman's theorem we need a last property of rational trees.

Lemma 4.10 *For all rational tree T with vertices in X^*,$L \subseteq \mathrm{Dom}(T) \in \mathrm{Rat}(X^*)$ and for all $r \geq 1$, we can decide if there exists $u_1, \ldots, u_m \in L$ such that for all $i \neq j \in [m]$ $d(u_i, u_j) > r$.*

We can now use Gaifman's theorem to obtain the decidability of the first-order theory of rational trees.

Proposition 4.11 *Every rational tree has a decidable first-order theory.*

Proof. By Gaifman's theorem 4.1, it is enough to decide basic local sentences. Let T be a rational tree and $\varphi = \exists x_1 \ldots \exists x_n \bigwedge_{1 \le i < j \le n} (d(x_i, x_j) > 2r \wedge \psi^{S(r,x_i)}(x_i))$ be a basic local sentence.
By Lemma 4.9, the set $L = \left\{ u \in \mathrm{Dom}(T) \mid \mathcal{T} \models \psi^{S(r,x)}[u] \right\}$ is rational.
To conclude, by Lemma 4.10, we can decide if there exists $u_1, \ldots, u_n \in L$ such that for all $i \neq j \in [n]$ $d(u_i, u_j) > 2r$.
Combinning these two results , we can decide wether T satisfy φ. □

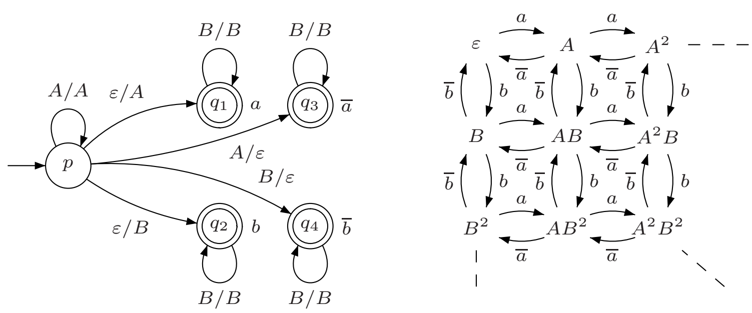
Due to the use of Gaifman's Theorem, the complexity of this decision procedure is non-elementary. However if we only consider rational trees of bounded out-degree, we can obtain an elementary decision procedure using the same technic as for the automatic graphs of bounded degree [Loh03].

5 Discussion on extension of this result

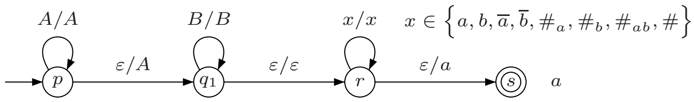
In this section, we illustrate that the result we have proved in previous section is in some sense maximal. We will first show that first-order theory together with rational accessibility is undecidable for rational trees. Then we will construct a rational directed acyclic graph with an undecidable first-order theory.

5.1 Finding a wider decidable logic

An obvious extension of first-order logic is first-order logic with accessibility, which is simply first-order theory in the transitive closure of the original structure. A broader extension is first-order logic with rational accessibility. For every rational language $L \in Rat(\Gamma^*)$ we add, to the first-order logic, a binary predicate $\mathbf{reach}_L$ meaning that the first vertex is connected to the second by a path in L.
We now prove that, even though Proposition 3.5 states that accessibility and rational accessibility are decidable for rational trees, first-order logic with rational accessibility is undecidable.
We use the grid (a quarter plane), with backward arcs. It is a rational graph:



We simulate two counters machines on the unfolding of this graph. As these machines may test for zero, we add a loop on each vertex expressing that either counter, both or none is empty (denoted respectively by $\#_a, \#_b, \#_{ab}, \#$).
In order to unfold the resulting graph we transform the transducer to add the path leading to the vertex. Because the graph is both deterministic and co-deterministic, this yields a deterministic rational forest. This forest is composed of rooted connected components. The connected component with root ε is isomorphic to the unfolding of the grid with backward arcs (like each connected component with root in $\left\{a, b, \overline{a}, \overline{b}\right\}^*$). The transducer for arcs labelled a is the following:



The transducers for $b, \overline{a}$ and $\overline{b}$ are similar. The transducers for $\#_a, \#_b, \#_{ab}, \#$ are the identity for the first part, and correspond to empty A, B, both or none.
Now we have a rational forest. We simply have to transform it into a rational tree. Again we use Lemma 3.6. Finally for each Minsky machine M we define a rational language L_M of its behaviour, and use this first-order formula to check whether it reaches empty counters (which is undecidable):

$$\exists u \exists v (\mathbf{reach}_{L_M}(u, v) \wedge \mathbf{root}(u) \wedge \neg(\exists w (v \xrightarrow{\overline{a}} w \vee v \xrightarrow{\overline{b}} w))).$$

We have, thus, found a rational tree with undecidable first-order theory with rational accessibility.

Remark 5.1. Indeed it is possible to improve this result in creating an ad hoc graph for encoding each machine. In this case it is first-order with accessibility which is undecidable for the whole family (and not just a single graph). Also it is possible to transform the tree in order to have an automatic tree.

5.2 Broaden the graph family

The first-order theory of rational graphs is undecidable. Indeed there are rational graphs with an undecidable first-order theory. Now we construct such a graph that is a directed acyclic graph (dag for short). This emphases the fact that the decidability of first-order theory of rational trees is deeply connected to the tree structure of these graphs.

Proposition 5.2 *There exists a rational directed acyclic graph with an undecidable first-order theory.*

Proof (Sketch). The construction of this dag (denoted G_{pcp}) relies on an encoding of *every* instance of the Post correspondence problem (pcp for short).
The precise construction of G_{pcp} is intricate. Thomas gives a similar construction in [Tho02], he construct a rational graph with undecidable first-order theory. It

relies on the encoding of a universal Turing machine, and a simple formula detecting a loop depending on the instance of pcp, this example does not translate obviously for dag.
An instance of pcp is a sequence $((u_i, v_i))_{i\in[n]}$, and the problem is to determine whether there is a word w such that $w = u_{i_1}u_{i_2}\dots u_{i_k} = v_{i_1}v_{i_2}\dots v_{i_k}$, for some integer k, and a sequence $(i_\ell)_{\ell\in[k]}$ of elements of $[n]$.
The graph G_{pcp} is oriented so that no cycle can occur. There are three components in this graph. The first one is the initialisation that produce all possible sequence of indices. The second part, on one side substitutes k by u_k simultaneously everywhere it occurs, on the other side substitutes k by v_k. These two paths are done separately. The third and final part of the graph joins the u branches to the v branches.
Now for any instance of pcp we construct a first-order sentence whose satisfaction in G_{pcp} implies the existence of a solution of pcp for the corresponding instance. Indeed the formula ensures that the initialisation process is done, that the correct u_i's and v_i's are followed, and that both path meet. □

5.3 Conclusion

In this paper we have investigated some properties of rational trees. The main result is that these graphs have a decidable first-order theory. This result is interesting because it mostly relies on structural properties of this family.
It is well known that the first-order theory of automatic graphs is also decidable. It should be interesting to determine if there are larger families of rational graphs with decidable first-order theory. It would also be interesting to be able to isolate a family having first-order theory with accessibility decidable. It is neither the case for automatic graphs, rational trees, and even automatic trees (see Remark 5.1).
An unexplored aspect of this study is to consider the traces of these graphs. The traces of automatic and rational graphs are context sensitive languages [MS01,Ris02]. Our conjecture is that there are even context-free languages that can not be obtained by rational trees, for instance the languages of words having the same number of a and b.

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