Solution to Exercise I.3.4

Let \((M, +)\) be a commutative monoid, with the subinvariant ultrametric \(d\).

Given \(a, b \in M\), and a fixed \(\epsilon > 0\), \(a \sim b\) denotes that \(a, b\) are \(\epsilon\)-near of each other, this means \(d(a, b) < \epsilon\). Observe that \(\sim\) is an equivalence relation using the fact that \(d\) is an ultrametric:

1. Obviously \(\sim\) is reflexive and symmetric.
2. If \(a \sim b\) and \(b \sim c\), then \(d(a, c) \leq \max(d(a, b), d(b, c)) < \epsilon\), hence \(a \sim c\).
   Hence \(\sim\) is transitive.
3. Also note that if \(a \sim b\) then by the fact that \(d\) is subinvariant we get \(a + c \sim b + c\) for all \(c\).
4. Moreover if \(c \sim a\) and \(c \sim a + b\), then \(c \sim a + b\). This is because \(d(c, a + b) \leq \max(d(c, a + b), d(a + b, c + b)) \leq \max(d(c, a + b), d(a, c)) < \epsilon\).

Now suppose \(\sum a_n = L\), then given any permutation \(\pi\) of the elements of this series, we have \(\sum a_{\pi(n)} = L\).

Proof. Fix \(\epsilon > 0\). There exists \(K\) such that for all \(n, \ell > K\) we have \(S_n \sim L\) and \(S_n \sim S_\ell\), where \(S_n\) is the partial sum of \(n\) terms. Fix any \(n_1 > K\). Take any \(n_3 > n_2 > n_1\). We have the following:

1. We have \(S_{n_1} \sim S_{n_2}\) and \(S_{n_1} \sim S_{n_3}\). By the above item number 4, we get \(S_{n_1} \sim S_{n_1} + a_{n_2+1} + \ldots + a_{n_3}\).
2. Similarly \(S_{n_1} \sim S_{n_2-1}\) and \(S_{n_1} \sim S_{n_3}\) entail that \(S_{n_1} \sim S_{n_1} + a_{n_2} + \ldots + a_{n_3}\).
3. and thus from \(S_{n_1} \sim S_{n_1} + a_{n_2+1} + \ldots + a_{n_3}\) and \(S_{n_1} \sim S_{n_1} + a_{n_2} + \ldots + a_{n_3}\) conclude that \(S_{n_1} \sim S_{n_1} + a_{n_2}\).

So far we have shown that for any \(n_1, n_2\) such that \(n_2 > n_1 > K\) we have \(S_{n_1} \sim S_{n_1} + a_{n_2}\). We can generalize this iteratively and inductively as follows. Take any finite set \(I \subseteq \mathbb{N}\), such that for all \(i \in I\) we have \(i > n_1\). Then we have

\[ S_{n_1} \sim S_{n_1} + \sum_{i \in I} a_i. \]

Let’s see this in the simple case \(I = \{n_2, n_3\}\), where \(n_3 > n_2 > n_1 > K\). By the basis of the induction: \(S_{n_2} \sim S_{n_1} + a_{n_2}\). Since \(S_{n_2} \sim S_{n_3}\) and \(S_{n_1} \sim S_{n_3-1}\), by the transitivity of \(\sim\) we get \(S_{n_3} \sim S_{n_1} + a_{n_2}\) and \(S_{n_3-1} \sim S_{n_1} + a_{n_2}\). Now again by the item number 4 above, we have \(S_{n_1} + a_{n_2} + a_{n_3} \sim S_{n_1} + a_{n_2}\). Since \(S_{n_1} \sim S_{n_1} + a_{n_2}\), by the transitivity of \(\sim\) we get \(S_{n_1} \sim S_{n_1} + a_{n_2} + a_{n_3}\).

Now let \(M\) be large enough such that \(\{\pi(1), \pi(2), \ldots, \pi(M)\} \supseteq \{1, \ldots, n_1\}\).

Take any \(\ell > M\). We have:

\[
\sum_{1 \leq i \leq \ell} a_{\pi(i)} = S_{n_1} + \sum_{i \in I} a_i
\]
for some finite set $I$. Thus
\[ \sum_{1 \leq i \leq \ell} a_{\pi(i)} \sim S_{n_1}, \]
and since $L \sim S_{n_1}$, we get by transitivity:
\[ \sum_{1 \leq i \leq \ell} a_{\pi(i)} \sim L. \]
This completes the proof. \qed