Erratum to various proofs of Christol’s theorem

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Abstract

This note presents an erratum to the proofs of Christol's theorem found in [Ber02], [AS03], and [BR11].

1 The problem

In the proofs of Christol’s theorem in [Ber02], [AS03], [BR11], a problem occurs revolving around a division of two formal power series, which is not guaranteed to result in a formal power series, but only a Laurent series.

• In the proof of [BR11, Theorem 5.4.1], the division is made as ‘Set $v = u/c_0$.’ Although we are guaranteed that $c_0 \neq 0$, we do not have the guarantee that $c_0(0) \neq 0$, so it is possible that $v$ only exists as a Laurent series. Later in the proof, it is implicitly assumed that $v = u/c_0$ is an ordinary power series, and the proof relies on two earlier results (Lemma 5.4.2 and Corollary 5.4.3) that are only given for ordinary power series, as well as the operator $\circ$ which is only defined for ordinary power series.

• In the proof of [AS03, Theorem 12.2.5], the problem is essentially the same. Here the division resulting in a Laurent series is given as ‘Put $G = A(X)/B_0(X)$’, and the underlying definitions and lemmas that need to be adjusted are Definition 12.2.1 and Lemma 12.2.2.

• In the proof of [Ber02, Theorem 3.2.1], the problem and the solution are, again, essentially the same, and centers around the usage of formal power series in a context where Laurent series are needed.

This problem, however, does not occur in the original papers [Chr79] and [CKFR80]. However, unlike the above books, neither of these papers gives an explicit formulation of the underlying lemma, which this note tries to make explicit.
2 The solution

We will use a representation of Laurent series as (equivalence classes) of pairs consisting of a power series and an integer, the offset. The Laurent series

\[ \sum_{i \geq k} a_i x^i \]

is represented by the equivalence class of pairs of the form:

\[ \left[ \sum_{i \geq 0} a_{k+i} x^i, k \right] \]

Note that:

1. The equivalence relation can be given by

   \[ [x^j S, n - j] \sim [x^k S, n - k] \]

   for all \( j, k \in \mathbb{N}, n \in \mathbb{Z}, \) and all power series \( S \).

2. Addition and multiplication of Laurent series can be given in terms of (in the case of addition, suitably chosen) representatives of the equivalence classes as

   \[ [S, n] + [T, n] = [S + T, n] \]

   and

   \[ [S, n][T, m] = [ST, n + m] \]

   (1)

   and these operations respect the equivalence relation and define the usual operations on Laurent series. The definition of the product gives rise to the exponentiation rule:

   \[ [S, n]^k = [S^k, nk] \]  

   (2)

We identify a Laurent series with any of its representations.

The operator \( A_{i,q} \), corresponding to \( \Lambda_i \) in [AS03] and (with some conceptual differences) the operator/monoid action \( \diamond \) in [BR11] is defined (first on power series) by

\[ A_{i,q} \left( \sum_{j \geq 0} a_j x^j \right) = \sum_{j \geq 0} a_{qj+i} x^j \]

We first state the power series version of the lemma, which is part b) of [AS03 Lemma 12.2.2] and has an easy correspondence to [BR11 Lemma 5.4.2, Corollary 5.4.3]:

Lemma 1. For all power series \( S \) and \( T \), all \( q, i \in \mathbb{N} \) with \( q \geq 1 \) and \( 0 \leq i < q \), we have:

\[ A_{i,q}(ST^q) = A_{i,q}(S)T \]
We now define the operation $B_{i,q}$ extending $A_{i,q}$ to Laurent series as follows:

$$B_{i,q}([S, aq]) = [A_{i,q}(S), a]$$

(Note that $B$ is not defined for all representatives of a Laurent series, but always for some representatives, and where it is, it again respects the equivalence relation.)

**Lemma 2.** For all Laurent series $V$ and $W$, all $q, i \in \mathbb{N}$ with $q \geq 1$ and $0 \leq i < q$, we have:

$$B_{i,q}(VW^q) = B_{i,q}(V)W$$

**Proof.** Note that $V$ must have some representation as a pair $V = [S, aq]$ for some power series $S$ and natural number $a$. Furthermore, let $W = [T, b]$. We now have:

\begin{align*}
B_{i,q}([S, aq][T, b]^q) & = B_{i,q}([S, aq][T^q, bq]) \quad \text{by (2)} \\
& = B_{i,q}([ST^q, (a + b)q]) \quad \text{by (1)} \\
& = [A_{i,q}(ST^q), a + b] \quad \text{definition of } B_{i,q} \\
& = [A_{i,q}(S)T, a + b] \quad \text{by Lemma (2)} \\
& = [A_{i,q}(S), a][T, b] \quad \text{by (1)} \\
& = B_{i,q}([S, aq])[T, b] \quad \text{definition of } B_{i,q} \\
\end{align*}

With this extended lemma, we can fix the problem in the proofs as follows:

- In [AS03], note that (given $q$), we have

$$\Lambda_r(S) = A_{r,q}(S)$$

and thus, the definition of $B_{i,q}$ and Lemma 2 give the extension of the definition of $\Lambda$ and of Lemma 12.2.2 (b) to Laurent series.

The proof of (the right to left direction of) Christol’s theorem (Theorem 12.2.5) now is fixed by

- Observing that in ‘Put $G = A(X)/B_0(X)$’, $G$ is defined as a Laurent series.
- Replacing ‘$H \in GF(q[[X]])$’ by ‘$H \in GF(q)((X))$’ in the definition of $\mathcal{H}$. (Note that $\mathcal{H}$ still is finite.)
- Appealing to the extended Lemma 12.2.2 (b) and the extended version of $\Lambda_r$ in the chain of equations starting with $\Lambda_r(H)$ near the end of the proof.

Because $\Lambda_r$ is consistent with its extension on power series, it follows that the restriction of $\mathcal{H}$ to power series is closed under the (ordinary) $\Lambda_r$ operators.
In [BR11], note that the operation
\[(r \circ u)\]
on power series, in the case where \(|r| = 1\), or equivalently \(r \in q\), is the same as
\[A_{r,q}(u)\]
as defined in this note. More generally, for words \(w \in q^*\), with \(|w| = n\), there is always some \(i\) with \(0 \leq i < q^n\) such that:
\[(w \circ u) = A_{i,q^n}(u)\]
If we extend this operation to Laurent series \(u\) by defining
\[(r \circ u) = B_{r,q}(u)\]
for elements \(r \in q\), the extension to words of arbitrary length again holds. Thus it follows from Lemma 2 that the extended versions of Lemma 5.4.2 and Corollary 5.4.3 again hold.

In the proof of Theorem 4.4.1, the problem can now be fixed by making the following observations/modifications:

– Note that \(v = u/c_0\) defines a Laurent series instead of a power series.
– \(F\) should now be defined as an (again finite) set of Laurent series.
– The appeal to Corollary 5.4.3 should be replaced by an appeal to its extension to Laurent series.

Finally observe that, because \(u\) is in fact a power series, so is \((r \circ u)\) for every \(r \in q^*\).

References


