The D₁ example. (1)

Let $D₁$ be the Dyck set on one letter. Then $D₁ \subseteq \Sigma^*$ where $\Sigma^* = \{\varepsilon, \sigma, \tau\}^*$. We propose to prove that the set

$$A = \{ \varepsilon, \tau_1, \tau_2, \tau_1^p \tau_2^q \mid p \geq 0, q \geq 0 \}$$

is not in the principal cone $C(D₁)$. This follows from the following proposition 1. Let $f : \Sigma^* \rightarrow \Sigma^*$ be a non-linear relational relation where $\Sigma^* = \tau_1 \cup \tau_2 \cup \varepsilon$, $\Sigma$ is a finite alphabet and $\tau_1$ and $\tau_2$ are two distinct letters not in $\Sigma$. Then one of the following two possibilities prevails.

1. [Eilenberg's notation]

Now, notons habituellement $D₁ \star$.
(ii) There exist integers \( m_1, m_2 \leq k \) such that
\[
\begin{align*}
\tau_1^{m_1} \times \tau_2^{m_2} \in fD_1, & \quad x \in \mathbb{E}^x \\
\text{then either } m_1 \leq k & \text{ or } m_2 \leq k.
\end{align*}
\]

(iii) There exist integers \( m_1, m_2, e_1, e_2 \in \mathbb{N} \)
\[
l_1 + e_2 > 0, \quad \text{and an element } x \in \mathbb{E}^x\]
such that
\[
\tau_1^{m_1 + e_1 n} \times \tau_2^{m_2 + e_2 n} \in fD_1
\]
for all \( n > 0 \).

Proof. Let \( \mathcal{O} = (Q, \mathcal{I}, T) \) be a \( \Sigma \)-\( \Sigma \) automaton computing the relation \( f \). Let \( p \) and \( q \) be a pair of states of \( \mathcal{O} \). We define the automata \( \mathcal{O}^p, \mathcal{O}^q, \mathcal{O}^q \) as follows
\[
\mathcal{O}^p = (Q, \mathcal{I}, p), \quad \mathcal{O}^q = (Q, q, \bar{T})
\]
\[
\mathcal{O}^{q_0} = (Q, \bar{q}, q)
\]
Further in $\text{Or}_p$ all edges starting at $p$ as well as all edges carrying a label $\Sigma \neq \Sigma_1$ are removed. Thus $\text{Or}_p$ is a $\Sigma, \Sigma, \Sigma_1$-automaton with $p$ as exit. Similarly in $\text{Or}_q$ all edges terminating at $q$ and all edges carrying a label $\Sigma \neq \Sigma_2$ are removed. Thus $\text{Or}_q$ is a $\Sigma, \Sigma, \Sigma_2$-automaton with $q$ as entry. In $\text{Or}_g$ all edges with labels $\Sigma_1$ or $\Sigma_2$ are removed. Thus $\text{Or}_g$ is a $\Sigma, \Sigma, \Sigma_2$-automaton.

Consider the composite automaton $(\cdot_1) (\text{Or}_p) (\text{Or}_g) (\cdot_2)$

and let

$$f_{\text{Or}_g} : \Sigma^* \rightarrow \Sigma_2^*$$

be the relation that it computes.
The following assertion is clear
\[ x_1^* \subseteq x_2^* \text{ if } D_1 = \bigcup \text{fp}_g D_i \]
the union extended over all pairs \((p,g)\) of states of \(D_2\). In view of this it suffices to prove the conclusion for each of the relations \(\text{fp}_g\). Thus we may assume that \(f = \text{fp}_g\) and that \(D_2\) is the composite automaton \(\ldots\).

Let \(D_i'\) be the set of all initial segments of \(D_i\) and consider the set
\[ A_i = (D_i' \cup x_1^*) \cap |\Omega_p| \]
Since \(D_i'\) is algebraic the set \(A_i\) is algebraic. Define the "norm"
\[ s_1 : \{\sigma, \sigma, x_1^*\}^* \to N \]
by \(s_1 \sigma = s_1 \sigma = 0, s_1 x_1 = 1\). By the theorem \(\ldots\) there exists an
integer \( k_1 \geq 0 \) such that if \( a_1 \in A_1 \) and \( s_1 a_1 > k_1 \), then \( a_1 \) admits a factorization

\[(\ast) \quad a_1 = b_1 c_1 d_1 e_1 f_1, \quad s_1 c_1 + s_1 e_1 > 0\]
such that

\[(\ast\ast) \quad b_1 c_1^n d_1 e_1^n f_1 \in A_1 \]
for all \( n > 0 \).

Similarly let \( D_1'' \) be the set of all terminal segments of \( D_1 \), and let

\[A_2 = (D_1'' \cup \tau_2) \cap A_1\]

Then \( A_2 \) is algebraic. Using the norm

\[s_2 : \{ \bar{\sigma}, \sigma, \tau_2 \}^* \rightarrow \mathbb{N} \]
given by \( s_2 \bar{\sigma} = s_2 \sigma = s_2 \tau_2 = 1 \), we obtain an integer \( k_2 \geq 0 \) such that if \( a_2 \in A_2 \) and \( s_2 a_2 > k_2 \), then

\[(\ast\ast\ast) \quad a_2 = s_2 e_2 d_2 e_2 b_2, \quad s_2 e_2 + s_2 c_2 > 0\]
such that

\[(\ast\ast\ast\ast) \quad s_2 e_2^n d_2 c_2^n b_2 \in A_2 \]
for all \( n > 0 \).
Let \( k = \text{max} (k_1, k_2) \), and suppose that (contrary to (2)) there is an element
\[
\tau_1^{m_1} \times \tau_2^{m_2} \in D, \quad x \in E^*
\]
with \( k < m_1 \) and \( k < m_2 \). Then there exist elements
\[
a_1 \in A_1, \quad a \in \{0, 1\}, \quad a_2 \in A_2
\]
such that
\[
a_1 a a_2 \in D \cup \tau_1^{m_1} \times \tau_2^{m_2}
\]
Consequently,
\[
s_1 a_1 = m_1 > k_1 \quad s_2 a_2 = m_2 > k_2
\]
Since
\[
a_1 \in A_1, \quad a_2 \in A_2
\]
we have factorizations (0.2) and (0.4) satisfying (0.3) and (0.5).

Now extend the function \( \gamma : \Sigma^0 \rightarrow Z \)
to \( \gamma : \Sigma^0 \cup S_2^* \rightarrow Z \) by setting
\[
\gamma z_1 = \gamma z_2 = \gamma z = 0 \quad \text{for all} \quad z \in E^*.
\]
Condition (2.3) implies:

\[ \gamma c_1 \geq 0, \quad \gamma e_1 = \gamma c_1 = 0 \]

Assume \( \gamma c_1 + \gamma e_1 = 0 \). Then

\[ \gamma a_1 = \gamma (b_1 c_1 + d_1 e_1 f_1) \]

for all \( n \geq 0 \). It follows that

\[ b_1 c_1 + d_1 e_1 f_1 = 0 \]

Thus \( a_1 \) is in \( D \) and also in \( D_1 \). Consequently

\[ \tau_1^{m_1} + \tau_2^{m_2} \in f \ D_1 \]

for all \( n \geq 0 \) where \( \tau_1 = \delta c_1 + \delta e_1 > 0 \);

thus (2.3) holds. We may therefore assume

\[ \gamma c_1 + \gamma e_1 = \gamma_1 > 0 \]

Similarly, condition (2.3) implies

\[ \gamma c_2 \leq 0, \quad \gamma c_2 + \gamma e_2 \leq 0 \]

and by an argument dual to the above we may assume

\[ \gamma c_2 + \gamma e_2 = -\gamma c_2 < 0 \]
The elements 
\[ b_1 c_1 e_1^2 f_1 a f_2 e_2^2 d_2 c_2^2 b_2 \]
are then in \( 1021 \) and also in \( D_1 \) if \( \tau_1 \equiv \tau_2 \). Consequently,
\[ \tau_1^{m_1 + l_1 n} \times \tau_2^{m_2 + l_2 n} \in f \cdot D_1 \]
for all \( n > 0 \) where
\[ \xi_1 = \tau_2 (s_1 c_1 + s_1 e_1), \quad \xi_2 = \tau_1 (s_2 c_2 + s_2 e_2) \]