The Franklin Leader Election Algorithm

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Outline

Introduction

We start with a set of n players. We assign a classical permutation of $\{1, \ldots, n\}$ to the set, all players corresponding to a peak stay alive, the other ones are killed. If there are no peaks, we choose a player at random (this is assumed to have 0 cost), indeed in the original game, one deals with circular permutations, so there always exists at least one peak, here we first approach the problem with a classical permutation.
What is the distribution of the number X_n of phases before getting only one player? This is the essence of a Leader Election Algorithm proposed by W.R. Franklin [3]. We present a first probabilistic analysis of this algorithm.

First analysis

Let

$$Y_n := \text{ number of peaks, starting with } n \text{ players,}$$
$$P(n,k) := \mathbb{P}[Y_n = k] = \mathbb{P}[k \text{ peaks, starting with } n \text{ players}],$$
$$\Pi(n,j) := \mathbb{P}(X_n = j) = \mathbb{P}[j \text{ phases are necessary to end the game, starting with } n \text{ players}],$$
$$\Lambda(n,j) := \sum_0^j \Pi(n,k).$$

First of all, we know see (see [1]), that the pentavariate generating function (GF) of valleys (u_0), double rise (u_1), double fall (u_1') and peaks (u_2) is given by

$$I(z,\mathbf{u}) = \frac{\delta}{u_2}\frac{v_1+\delta\tan(z\delta)}{\delta - v_1\tan(z\delta)} - \frac{v_1}{u_2},$$

with

$$v_1 = (u_1+u_1')/2, \delta = \sqrt{u_0u_2 - v_1^2}.$$

This gives the GF of the number of peaks:

$$\frac{\tan[z(u-1)^{1/2}]}{(u-1)^{1/2} - \tan[z(u-1)^{1/2}]}, \qquad (1)$$

hence the mean m and variance v of the number of peaks:

$$m(n) = (n-2)/3, v(m) = 2(n+1)/45.$$

Moreover, from [2], we know that the distribution P is asymptotically Gaussian.

Let $x(n)$ be the mean number of phases, $\mathbb{E}(X_n)$, starting with n players. As we shall see, the initial values are

$$x(0) = x(1) = 0, x(2) = x(3) = x(4) = 1.$$

The mean number of players $c(j)$ still alive after j phases is given by

$$c(j) = 3^{-j}[n - (3^j - 1)].$$

If we want $c(j) = 1$, this leads to

$$x(n) = j \sim \log_3 n - \log_3 2,$$

but the initial conditions are less simple, so the constant will differ from $-\log_3 2$.

Let us now construct Π. We have

$$\Pi(n,j) = \sum_{0}^{\lfloor (n-1)/2 \rfloor} P(n,k)\Pi(k,j-1). \qquad (2)$$

We have the initial values

$$\Pi(0,0) = 1, \Pi(0,j) = 0, j > 0, \Pi(1,0) = 1, \Pi(1,j) = 0, j > 0.$$

Also

$$\Pi(2,1) = \Pi(3,1) = \Pi(4,1) = 1.$$

A picture of P and Π is given in tables 1 and 2.

$P(n,k)$

$n \backslash k$	0	1	2	3	4
1	1	0	0	0	0
2	1	0	0	0	0
3	2/3	1/3	0	0	0
4	•	•	0	0	0
5	•	•	•	0	0
6	•	•	•	0	0
7	•	•	•	•	0

Table 1: $P(n,k)$

$\Pi(n, k)$

$n \backslash j$	0	1	2	3
0	1	0	0	0
1	1	0	0	0
2	0	1	0	0
3	0	1	0	0
4	0	1	0	0
5	0	$\bullet$	$\bullet$	0
.				
20	0	$< 10^{-7}$	$\bullet$	$\bullet$

Table 2: $\Pi(n, j)$

Denoting the jth column of Π by $\pi^{(j)}$, we have

$$\pi^{(j)} = P^{j-1}\pi^{(1)},$$

where $\pi^{(j)}$ and P are used on $n \geq 2, k \geq 2$. In these new coordinates, P is triangular: $P(n,j) = 0, j > \lfloor (n-1)/2 \rfloor$. So is P^j. But $\pi^{(1)}(n) < 10^{-7}, n > 20$, so numerically, the significant columns of P^j are the first 20 columns. Also, we see the importance of the initial first column of Π. Moreover, for $n > 75$, $P(n,k)$ is indistinguishable from the Gaussian limit. So we have used the expansion of the GF (1) for $n \leq 75$ and the Gaussian limit afterwards. Of course we have

$$x(n) = \sum_{0}^{\infty} \Pi(n,j)j,$$

and

$$x(n) = 1 + \sum_{3}^{\infty} P(n,k)x(k)$$

A plot of $x(n) - \log_3 n$ versus $\log_3 n$ is given in Figure 1. The oscillations will be clear later on.

$x(n) - \log_3 n$ versus $\log_3 n$

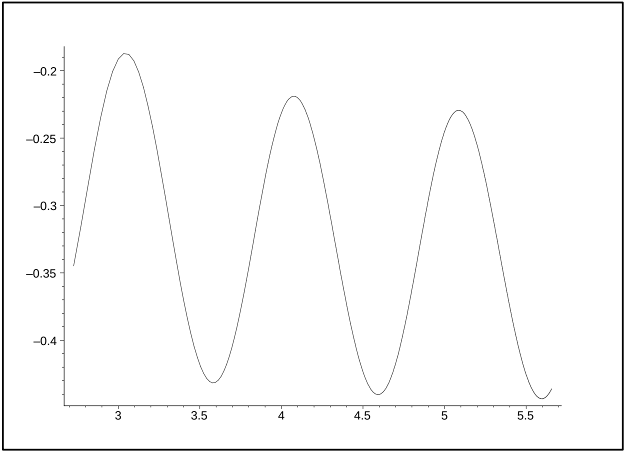
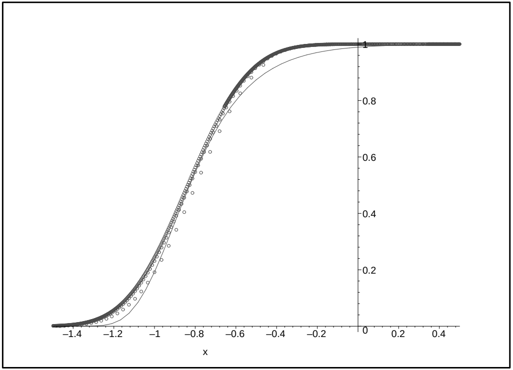


Figure 1: $x(n) - \log_3 n$ versus $\log_3 n$

We have observed that at most two values carry the main part of the probability mass $\Pi(n, j)$. Again this will be explained later on. The next Figure 2 gives the observed distribution function (DF) of X_n: $\Lambda(n, j), n = 20..500$ plotted against $j - \log_3 n$. There seems to exist a limiting distribution. We have also plotted an adjusted Gumbel distribution, the fit is bad.

$\Lambda(n, j)$ versus $j - \log_3 n$



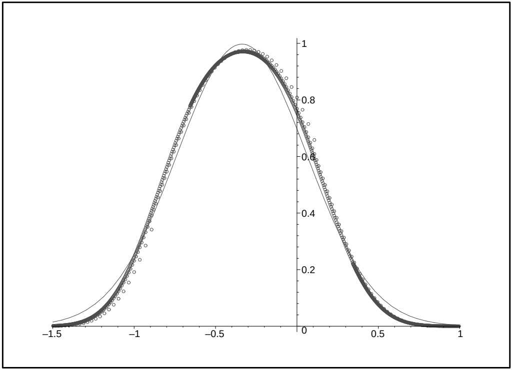
○ : observed

— : Gumbel distribution

Figure 2: $\Lambda(n, j)$ versus $j - \log_3 n$

Similarly, in Figure 3, we show the probability $\Pi(n, j), n = 20..500$ plotted against $j - \log_3 n$. The fit with an adjusted Gaussian distribution is equally bad.

$\Pi(n, j)$ versus $j - \log_3 n$



$\circ$: observed

— : Gaussian distribution

Figure 3: $\Pi(n, j)$ versus $j - \log_3 n$

The few spare points of both figures are actually due to the propagation of two discrete distributions shown in Figure 4, for $n = 0..20$.

$\Pi(n,j)$ versus $j - \log_3 n$, $n = 0..20$

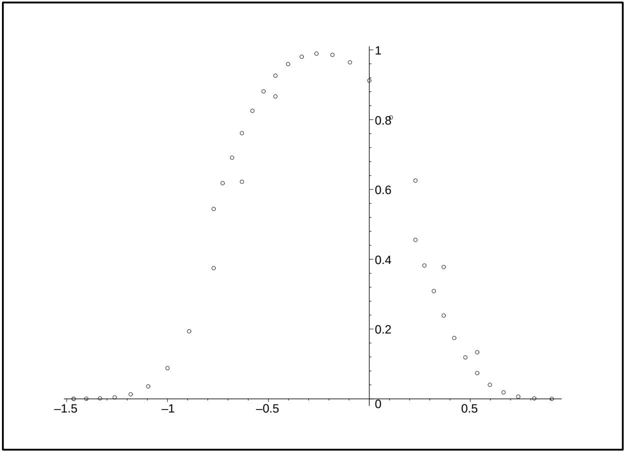


Figure 4: $\Pi(n,j)$ versus $j - \log_3 n$, $n = 0..20$

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6. if $F(\eta)$ is smooth, we can derive, as in [5], all periodic moments of X_n, in particular $x(n)$.

$n \backslash j$	$\cdot$	$j-1$	j	$j+1$	$\cdot$
$\cdot$					
$\cdot$					
$n/3 - \alpha\sqrt{n}$		$\bullet$	$\bullet$		
$\cdot$		$\log_3 n - 1$			
$\cdot$					
$n/3$		$\bullet$	$\bullet$		
$\cdot$					
$\cdot$					
$n/3 + \alpha\sqrt{n}$		$\bullet$	$\bullet$		
$\cdot$					
$\cdot$					
n			$\bullet$	$\bullet$	
			$\log_3 n$		

Table 3: $\Pi(n,j)$

The effect of initial values is now clear. To illustrate this we have changed to $\Pi((0,0) = \Pi(0,1) = \Pi(1,0) = \Pi(1,1) = 1/2$: Table 4. The equivalent of Figures 2,3,4, is given in Figures 5,6,7.

$\Pi(n, k)$, other initialization

n \ j	0	1	2	3
0	1/2	1/2	0	0
1	1/2	1/2	0	0
2	0	1/2	1/2	0
3	0	1/2	1/2	0
4	0	1/2	1/2	0
5	0	•	1/2	•
.				
20	0	$< 10^{-7}$	•	•

Table 4: $\Pi(n, j)$, other initialization

$\Lambda(n, j)$ versus $j - \log_3 n$, other initialization

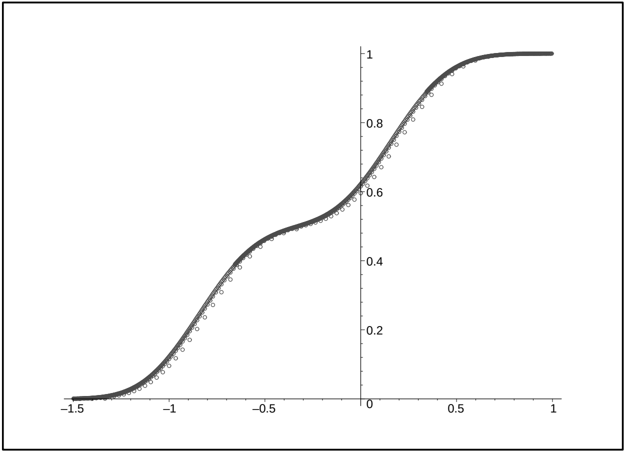

Figure 5: $\Lambda(n, j)$ versus $j - \log_3 n$, other initialization

$\Pi(n,j)$ versus $j - \log_3 n$, other initialization

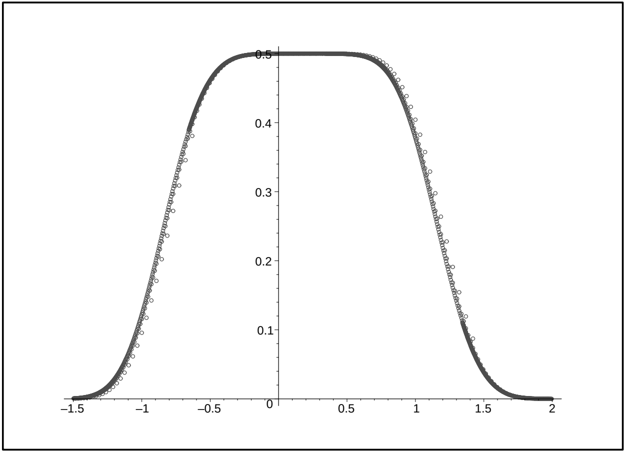

Figure 6: $\Pi(n,j)$ versus $j - \log_3 n$, other initialization

$\Pi(n, j)$ versus $j - \log_3 n$, $n = 0..20$, other initialization

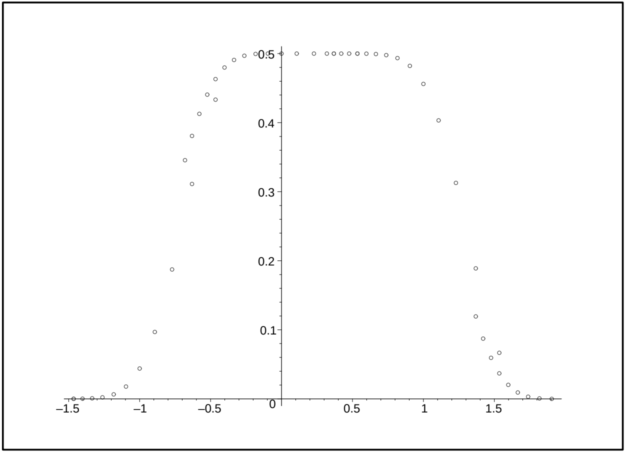

Figure 7: $\Pi(n, j)$ versus $j - \log_3 n$, $n = 0..20$, other initialization

Proof of convergence by S.Janson

For each n, the number of peaks Y_n is a random variable (r.v.) with values in $\{1, \ldots, n\}$. Assume $\mathbb{P}(Y_n < n) > 0$. Start with n individuals. In the first round, we get Y_n survivors. Continue (with independent $Y's$) until one survives. Let X_n be the number of rounds. (Thus $X_1 = 0$).
Equivalently: consider a Markov chain on $\{1, 2, \ldots\}$ where $P_{i,j} = 0$ if $j > i$ (and $P_{i,i} < 1, i > 1$). Let X_n be the number of steps to absorption in 1. Let Y_n be the distribution of the next position if we start at n.

Assumptions:
(i) Y_n stochastically monotone, i.e.
$\mathbb{P}(Y_n \leq k) \geq \mathbb{P}(Y_{n+1} \leq k), n \geq 1, k \geq 1$. Equivalently, we may couple such that $Y_n \leq Y_{n+1}$.
For some constant $\alpha \in (0, 1$ and $\varepsilon > 0$ and a sequence $\delta_n = 1/\log^{\varepsilon} n$ (or simpler, $n^{-\varepsilon}$),
(ii)

$$\mathbb{E}Y_{n+1} - \mathbb{E}Y_n = \alpha + \mathcal{O}(\delta_n),$$

and
(iii)

$$\mathbb{P}(|Y_n - \alpha n| > \delta_n n) = \mathcal{O}(n^{-2-\varepsilon}).$$

Note $\mathbb{E}|Y_n - \alpha n|^p = \mathcal{O}(n^{p/2})$ for some $p > 4$ suffices for (iii). For example $\mathbb{E}|Y_n - \alpha n|^6 = \mathcal{O}(n^3)$, which is true for Franklin algorithm.

By (iii)

$$\mathbb{E}|Y_n - \alpha n| \leq \delta_n n + \mathcal{O}(n^{-1-\varepsilon}) = \mathcal{O}(n\delta_n). \tag{3}$$

Theorem

There exists a distribution function F such that

$$\sup_k |\mathbb{P}(X_n \leq k) - F(k - \log n)| \to 0$$

or, equivalently, if $Z \sim F$,

$$d_{TV}(X_n, \lceil Z + \log n \rceil) \to 0.$$

This is illustrated in Figure 8

Illustration of the Theorem

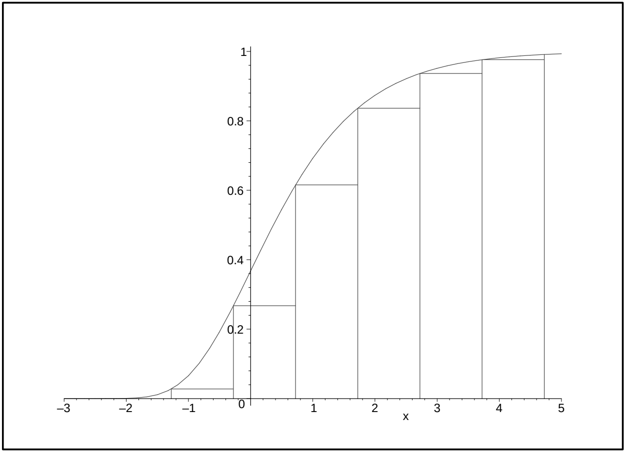

Figure 8: Illustration of the Theorem

Proof

First, $\mathbb{P}(Y_n = n) \to 0$ by (iii), so since $\mathbb{P}(Y_n < n) > 0, q_b := \sup_n \mathbb{P}(Y_n = n) < 1$. Hence X_n is stochastically dominated by a geometric r.v. and thus $\mathbb{E}X_n = \mathcal{O}(n)$. In particular, $\mathbb{E}X_n < \infty$ for every n.
We may couple s.t. $X_{n+1} \geq X_n$ a.s.
Let

$$
\begin{aligned}
x_n &:= \mathbb{E}X_n, \\
d_n &:= \mathbb{E}X_{n+1} - \mathbb{E}X_n = x_{n+1} - x_n, \\
b_n &:= \max_{1 \leq k \leq n} k d_k.
\end{aligned}
$$

We have

$$x_n = \mathbb{E}X_n = 1 + \mathbb{E}x_{Y_n}.$$

Thus

$$d_n = \mathbb{E}(x_{Y_{n+1}} - x_{Y_n}) = \mathbb{E} \sum_{Y_n}^{Y_{n+1}-1} d_j = \sum_{j=1}^{n} d_j \mathbb{P}(Y_n \le j < Y_{n+1}). \quad (4)$$

Equ. (4) yields first, with $d_n^* = \max_{k \le n} d_k$, noting that

$$\sum_{j=1}^{n} \mathbb{P}(Y_n \le j < Y_{n+1}) = \mathbb{E} Y_{n+1} - \mathbb{E} Y_n \le 1, \text{ for } n \ge n_0, \text{ by (ii)} ,$$

$$\begin{aligned} d_n(1 - \mathbb{P}(Y_n \le n < Y_{n+1})) &\le \sum_{j=1}^{n-1} \mathbb{P}(Y_n \le j < Y_{n+1}) d_{n-1}^* \\ &\le d_{n-1}^*(1 - \mathbb{P}(Y_n \le n < Y_{n+1})). \end{aligned}$$

Hence $d_n \le d_{n-1}^*$ so $d_n^* = d_n \vee d_{n-1}^* = d_{n-1}^*, n \ge n_0$, and $d_n^* = d_{n_0}^* < \infty$, all $n \ge n_0$.
In other words, $d^* := \sup_n d_n < \infty$.

Let $\beta = (1+\alpha)/2$. If n is large enough, so that $(\alpha - \delta_n)n < \beta n$, then (4) yields, using (ii) and (iii),

$$
\begin{aligned}
d_n \le\ & d^* \sum_{j<(\alpha-\delta_n)n} \mathbb{P}(Y_n \le j) + \frac{1}{n(\alpha-\delta_n)} b_{\beta n} \sum_j \mathbb{P}(Y_n \le j < Y_{n+1}) \\
& + d^* \sum_{j>\beta n} \mathbb{P}(Y_{n+1} > j) \\
& \le d^* \mathcal{O}(n n^{-2-\varepsilon}) + \frac{1}{n(\alpha-\delta_n)} b_{\beta n} (\mathbb{E} Y_{n+1} - \mathbb{E} Y_n) \\
& = \mathcal{O}(n^{-1-\varepsilon}) + \frac{1}{n} \frac{\alpha + \mathcal{O}(\delta_n)}{\alpha - \delta_n} b_{\beta n}.
\end{aligned}
$$

Thus

$$nd_n \le (1 + \mathcal{O}(\delta_n))b_{\beta n} + \mathcal{O}(n^{-\varepsilon}). \qquad (5)$$

Replace n by k and take sup over $\beta n < k \le n$. Since b is increasing, choosing $\varepsilon = 2$,

$$b_n \le (1 + \mathcal{O}(\delta_{\beta n}))b_{\beta n} + \mathcal{O}(n^{-\varepsilon}) = \left(1 + \mathcal{O}\left(\frac{1}{\log^2 n}\right)\right) b_{\beta n}.$$

It follows by induction over m that if

$$(1/\beta)^m \le n < (1/\beta)^{m+1},$$

and $\beta < 1$. Then

$$b_n \le C_1 \prod_{j=1}^{m} (1 + \frac{C_2}{j^2})$$

and thus

$$b_n = \mathcal{O}(1).$$

In other words

Lemma

$$d_n = \mathcal{O}(1/n).$$

Let d_w := Wasserstein distance:

$$d_w(\mu_1, \mu_2) = \sup_{f \in \mathcal{F}_w} \left| \int f d\mu_1 - \int f d\mu_2 \right|,$$

$$\mathcal{F}_w := \{f : |f(x) - f(y)| \leq d(x, y)\},$$

the set of Lipschitz functions .

Since $X_{n+1} \geq X_n$, $d_w(X_n, X_{n+1}) = \mathbb{E}(X_{n+1} - X_n) = d_n$. Thus

$$d_w(X_n, X_m) \leq \frac{|n-m|}{n \wedge m} \sup b_k = \mathcal{O}\left(\frac{|n-m|}{n \wedge m}\right) \qquad (6)$$

Define, with $\log := \log_{1/\alpha}$

$$\tilde{X}_t = X_{\lfloor t \rfloor} - \log t, t \geq 1.$$

Then, for $t \geq 1/\alpha$

$$\begin{aligned}
d_w(\tilde{X}_t, \tilde{X}_{\alpha t}) &= d_w(X_{\lfloor t \rfloor} - 1, X_{\lfloor \alpha t \rfloor}) \leq \mathbb{E} d_w(X_{Y_{\lfloor t \rfloor}}, X_{\lfloor \alpha t \rfloor}) \\
&= \mathcal{O}\left(\mathbb{E}\left(\frac{Y_{\lfloor t \rfloor} - \lfloor \alpha t \rfloor}{Y_{\lfloor t \rfloor} \wedge \lfloor \alpha t \rfloor}\right)\right) \\
&= \mathcal{O}\left(\frac{\mathbb{E}(Y_{\lfloor t \rfloor} - \lfloor \alpha t \rfloor}{t}\right) + \mathcal{O}\left(t\mathbb{P}\left(Y_{\lfloor t \rfloor} < \frac{\alpha}{2}\lfloor t \rfloor\right)\right) \\
&\text{as } Y_{\lfloor t \rfloor} \geq 1 \\
&= \mathcal{O}(\delta_{\lfloor t \rfloor}) + \mathcal{O}(t.t^{-2-2}) = \mathcal{O}\left(\frac{1}{\log^2 t}\right).
\end{aligned}$$

For any $t \geq 1$, thus

$$\sum_{j=0}^{\infty} d_w(\tilde{X}_{\alpha^{-j}t}, \tilde{X}_{\alpha^{-j-1}t}) < \infty,$$

and thus there exists a limit in distribution $Z(t)$, say, such that

$$\tilde{X}_{\alpha^{-j}t} \xrightarrow{d} Z(t) \text{ as } j \to \infty.$$

Clearly, $Z(\alpha t) \stackrel{d}{=} Z(t)$, so the distribution $\mathcal{L}(Z(t))$ is a periodic function of $\log t$. We also find

$$d_w(\tilde{X}_t, Z(t)) = \mathcal{O}\left(\frac{1}{\log t}\right),$$

and, in particular

$$\mathbb{E}\tilde{X}_t = \mathbb{E}X_{\lfloor t \rfloor} - \log t = \mathbb{E}Z(t) + \mathcal{O}\left(\frac{1}{\log t}\right).$$

Define, for every real x, $F(x) = \mathbb{P}(Z(t) \leq x)$ for any t such that $x + \log(t)$ is an integer. Since $Z(t)$ is periodic in $\log(t)$, this does not depend on the choice of t. Then, for any sequence n of integers,

$$\begin{aligned}\mathbb{P}(X_n \leq k) &= \mathbb{P}(X_n - \log n \leq k - \log n) \\ &= \mathbb{P}(Z(n) \leq k - \log n) + o(1) = F(k - \log n) + o(1).\end{aligned}$$

It now follows that F is monotone, and thus a distribution function, by Lemma 4.6 in Janson [4].
Note further that, for $1 \leq \gamma \leq \alpha^{-1}$, $b^* = \sup_k b_k$,

$$\begin{aligned}d_w(\tilde{X}_t, \tilde{X}_{\gamma t}) &\leq d_w(X_{\lfloor t \rfloor}, X_{\lfloor \gamma t \rfloor}) + |\log t - \log(\gamma t)| \\ &\leq (\lfloor \gamma t \rfloor - \lfloor t \rfloor)\frac{b^*}{\lfloor t \rfloor} + \log \gamma = \mathcal{O}(\gamma - 1) + \frac{1}{\lfloor t \rfloor}.\end{aligned}$$

Letting $t \to \infty$,

$$d_w(Z(t), Z(\gamma t)) = \mathcal{O}(\gamma - 1) = \mathcal{O}(\log \gamma),$$

so $t \to \mathcal{L}(Z(t))$ is continuous and Lipschitz in the Wasserstein metric.

Assume Z has distribution F. We have

$$\mathbb{P}[Z(t) \leq \lfloor u + \log t \rfloor - \log t] = F(\lfloor u + \log t \rfloor - \log t)$$

But

$$\mathbb{P}[\lceil Z + \log t \rceil - \log t \leq \lfloor u + \log t \rfloor - \log t] \\ = \mathbb{P}[Z + \log t \leq \lfloor u + \log t \rfloor] = F(\lfloor u + \log t \rfloor - \log t),$$

so $\lceil Z + \log t \rceil - \log t$ has the same distribution as $Z(t)$. hence by the Lipschitz continuity of $Z(t)$, Z has a bounded density F'. ■

Periodicities

With the observed values of $\Pi(n, j)$ (see Figure 3), we have computed a (numerical) Laplace transform (with a variable step Euler-MacLaurin). With the usual machiney (see [5]), we have obtained the periodicities of the mean. This fits quite well with the observed periodicities (see Figure 1), the comparison is given in Figure 9.

observed (○) versus computed (line) periodicities

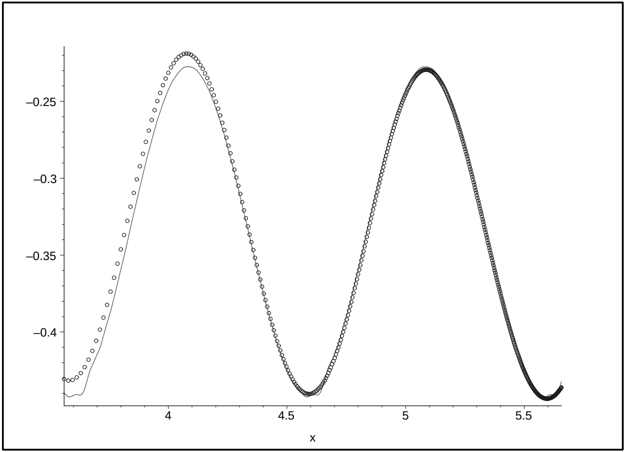

Figure 9: observed (○) versus computed (line) periodicities

Conclusion

WHAT IS $F(\eta)$????

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