A link between the Matsumoto-Yor property and an independence property on trees

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- The generalized inverse Gaussian distributions
- The Matsumoto-Yor independence property (M-Y)
- An independence property on trees (B-K)
- M-Y as a consequence of B-K in a particular case

The generalized inverse Gaussian distributions

Let $\mu \in \mathbb{R}$, $a > 0$ and $b > 0$. The GIG distribution with parameters μ, a, b is the probability measure :

$$GIG(\mu; a, b)(dx) = \left(\frac{b}{a}\right)^{\mu} \frac{x^{\mu-1}}{2K_{\mu}(ab)} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)} \mathbf{1}_{(0,\infty)}(x)dx$$

where K_{μ} is the classical McDonald special function. One can have $a = 0$ if $\mu > 0$ or $b = 0$ if $\mu < 0$.

☞ If $\mu = -\frac{1}{2}$, the law $GIG(\mu; a, b)$ is the classical inverse Gaussian distribution with density

$$IG(a, b)(dx) = \frac{a}{\sqrt{2\pi}} e^{ab} x^{-\frac{3}{2}} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)} \mathbf{1}_{(0,\infty)}(x)dx,$$

- If $\mu = \frac{1}{2}$ we have the reciprocal inverse Gaussian distribution with density

$$RIG(a,b)(dx) = \frac{b}{\sqrt{2\pi}} e^{ab} x^{-\frac{1}{2}} e^{-\frac{1}{2}(a^2x^{-1}+b^2x)} \mathbf{1}_{(0,\infty)}(x)dx.$$

- $RIG(0,b)$ is the gamma distribution with shape parameter 1/2 and scale parameter $2/b^2$, with density $\frac{b}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{b^2}{2}x} \mathbf{1}_{(0,\infty)}(x)dx$
- $X \sim IG(a,b) \Longleftrightarrow X^{-1} \sim RIG(b,a)$
- $IG(a_1,b) * IG(a_2,b) = IG(a_1+a_2,b)$
- $IG(a_1,b) * RIG(a_2,b) = RIG(a_1+a_2,b)$
- IG and RIG laws are respectively the distribution of the first and the last hitting time for a Brownian motion.
- The GIG density exists also as a probability measure on the set of positive definite matrices, the case $a = 0$ defining Wishart matrices (see Letac and Wesolowski, 2000).

The Matsumoto-Yor independence property

Proposition 1 *Let $\mu < 0$, $a > 0$ and $b > 0$. Consider two independent random variables X and Y such that X follows the law GIG(μ, a, b) while Y follows a gamma distribution: GIG($-\mu, 0, b$).*
Then the random variables $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are independent.

Proof by Matsumoto and Yor (*Nagoya Math. J.*, 2001) in the case $a = b$.
Letac and Wesolowski (*Ann. of Prob.* , 2000) noticed that it is true also if $a \neq b$ and proved that property to be in fact a characterization of GIG laws.

An independence property on trees (B-K)

Finite tree T, sets of vertices V, set of edges E, root s inducing a natural order on the tree. The set of all terminal vertices, called the *boundary* of T, is denoted by ∂T.

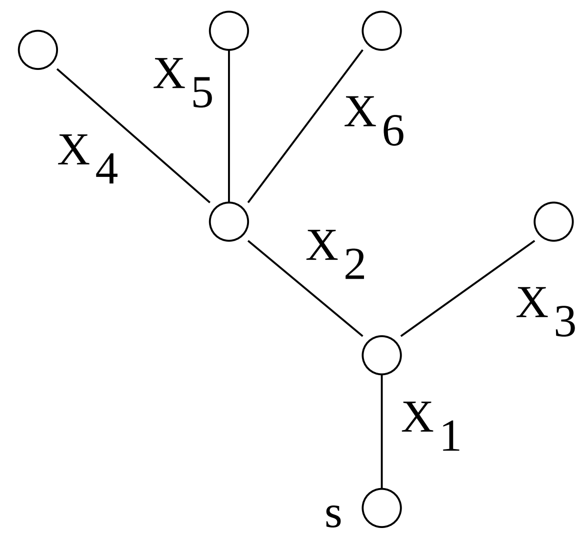

Figure 1: A tree-network with six edges equipped with resistances $X_1, \ldots, X_6$.

Suppose that each edge ending at a vertex v of T has a resistance X_v. Then, the total resistance R of the network is defined according to the Kirchoff-Ohm laws.

For instance, for the tree illustrated by Figure 1, with edge resistances X_1 to X_6, the total resistance is

$$R = X_1 + \left[\left\{ X_2 + (X_4^{-1} + X_5^{-1} + X_6^{-1})^{-1} \right\}^{-1} + X_3^{-1} \right]^{-1}.$$

To each vertex $v \in V \setminus \{s\}$ are associated two positive real numbers a_v and b_v fulfilling the following consistency conditions:

- For each $v \in V$, $b_v = \sum_{(v,v') \in E} b_{v'}$
- the sum $a = \sum_{v \in \pi \setminus \{s\}} a_v$ is the same along all paths π starting from the root s and ending at a terminal vertex.

Proposition 2 *Barndorff-Nielsen and K. (Adv. in Applied Prob , 1998) Suppose that the distributions of the random variables* $(X_v,\ v \in V \setminus \{s\})$ *are as follows:*

$$\begin{cases} X_v & \sim \quad RIG(a_v, b_v) = GIG(\frac{1}{2}, a_v, b_v) \quad \text{if } v \in \partial T \\ X_v & \sim \quad IG(a_v, b_v) = GIG(-\frac{1}{2}, a_v, b_v) \quad \text{otherwise.} \end{cases}$$

Then

(i). *the total resistance R of the tree follows the $RIG(a,b)$ distribution, where* $a = \sum_{v\in\pi\setminus\{s\}} a_v$ *and* $b = \sum_{v\in\partial T} b_v$.

(ii). *the variables*

$$W = \left(\sum_{v\in V} b_v^2 X_v\right) - b^2 R$$

and

$$Z = \left(\sum_{v\in V} a_v^2 X_v^{-1}\right) - a^2 R^{-1},$$

are independent, **the vector (W,Z) and the variable R are independent**. *Furthermore, W is gamma-distributed with parameters A and 2, Z is gamma-distributed with parameters B and 2, where $A = \frac{|V|-|\partial T|}{2}$ is half the number of internal vertices and $B = \frac{|\partial T|-1}{2}$.*

Our result : Link between M-Y and B-K

Proposition 3 *The independence property established in B-K (1998) implies the case* $\mu = -\frac{1}{2}$ *of the Matsumoto-Yor independence property.*

PROOF: Let $a > 0$ and $b > 0$. We recall that the inverse Gaussian law $IG(a, b)$ considered in B-K is the case $\mu = -\frac{1}{2}$ of the law GIG(μ, a, b). Consider two independent random variables X and Y such that X follows the law GIG($-\frac{1}{2}, a, b$) while Y follows a gamma distribution: GIG($\frac{1}{2}, 0, b$).

Let us apply B-K to the trivial tree of Figure 2.

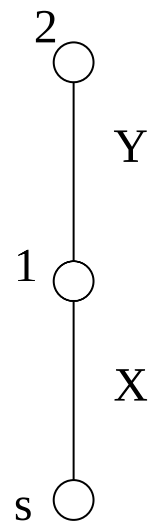

Figure 2: A two-edge tree with resistances X and Y.

Suppose the edge $(s, 1)$ has resistance X while the terminal edge $(1, 2)$ has resistance Y. Then the consistency conditions and the assumption of B-K are fulfilled, and it comes from (ii) of Proposition 3 that Z and R are independent.

But we have for this trivial tree,

$$R = X + Y$$

and

$$Z = a^2 X^{-1} - a^2 R^{-1} = a^2 (X^{-1} - (X+Y)^{-1}).$$

Thus $(X+Y)^{-1}$ and $X^{-1} - (X+Y)^{-1}$ are independent, which is the Matsumoto-Yor property.

□