**PARISTIC**

**NOV 2006**

*Quantitative Algorithmics*
*of Massive Data Sets*

# Algorithmique quantitative de grands flux de données

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`http://algo.inria.fr/flajolet`

Routers in the range of Terabits/sec ($10^{14}$b/s).

Google indexes 6 billion pages & prepares to index 100 Petabytes of data ($10^{17}$B).

*Can get a few key characteristics, QUICK and EASY*

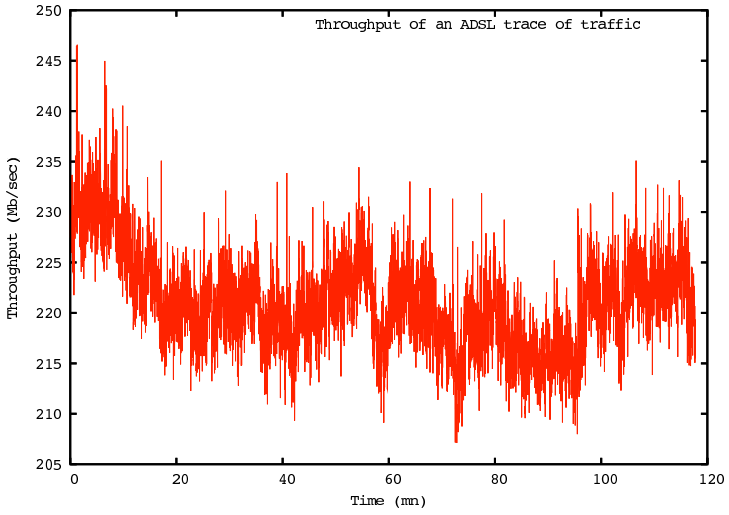
This talk:

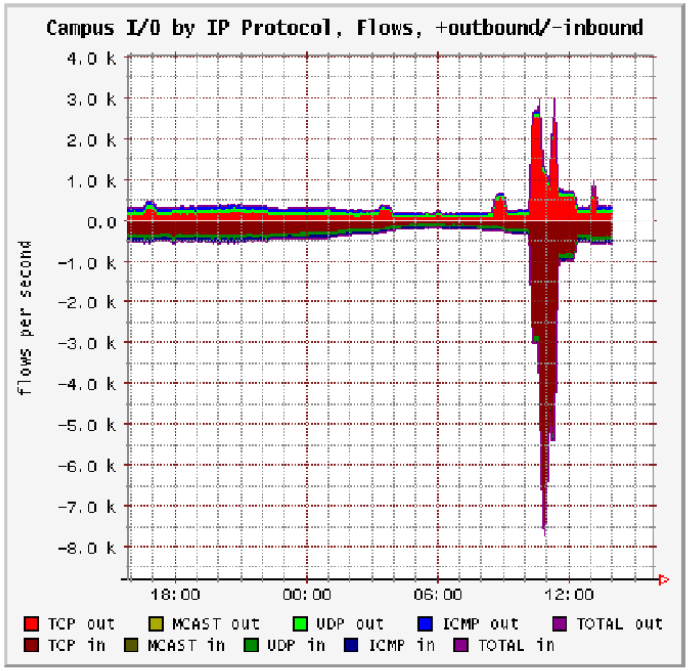
**Zoom** on some recent research directions $\neq$ geopolitics.

+ Interplay between theory and practice.

Combinatorics +algorithms +probabilities +analysis are useful!

Traces of attacks: Number of active connections in time slices.




*(Raw ADSL traffic)* *(Attack)*

Incoming/Outgoing flows at 40Gbits/second. Code Red Worm: 0.5GBytes of compressed data per hour (2001). CISCO: in 11 minutes, a worm infected 500,000,000 machines.

Left: ADSL FT@Lyon $1.5 \times 10^8$ packets (21h–23h). Right: (Estan-Varghese-Fisk) different incoming/outgoing connections

The situation is like listening to a play of Shakespeare and at the end *estimate the number of different words*.

Rules: Very little computation per element scanned, very little auxiliary memory = **Stream algorithms**.

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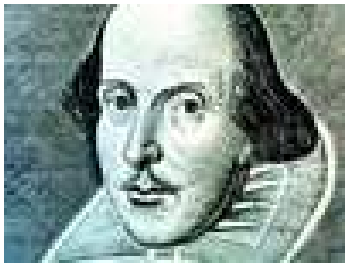


From Durand-Flajolet, LogLog Counting (ESA-2003):

Whole of Shakespeare, $m = 256$ small "bytes" of 4 bits each = 128bytes

```
ghfffghfghgghgggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhgggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhiggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $n^\circ \approx 30,897$ vs $n = 28,239$ distinct words. Error: $+9.4\%$ w/ 128 bytes!

— Routers & Networks: attacks, flow monitoring & control

— Databases: Query optimization = estimating the size of queries; also "sketches".

— "Raw" statistics on massive data: on the fly, fast and with little memory even on textual data $\in$ "*data mining*".

(Gains by a factor of 400 in data mining of the internet graph.)

♡♡♡♡♡♡

Produce **short algorithms & programs** with $\mathcal{O}(10)$ instructions based on mathematically oriented engineering.

• *Estimating characteristics of large data streams*

— counters: "beats" information theory;

— sampling: avoid negative effects

— cardinality estimators;

—- gather statistics for mice and elephants;

— profile indicators ($F_2$, etc)

$\rightsquigarrow \diamond$ Gains by factors in range 100-1000 (!)

• No analysis $\Longrightarrow$ no algorithm! Algorithms are inherently probabilistic.

— analytic combinatorics, complex asymptotics, Mellin transforms

— Interplay of analysis and design $\rightsquigarrow$ highly optimized algorithms.

# 1 ICEBERGS (& COMBINATORICS)

abcdaaaabcafcccaacccaaacacbbbbaabbbbbbb

Definition: *A $k$–iceberg is present in proportion $> 1/k$.*

One pass detection of icebergs for $k = 2$ using $1$ registers is possible .

— Trigger a gang war: equip each individual with a gun.

— Each guy shoots a guy from a different gang, then commits suicide: Majority gang survives.

— Implement sequentially & adapt to $k \geq 2$ with $\mathbf{k-1}$ registers. (Karp, Shenker, Papadimitriou. 2003)

# 2 APPROXIMATE COUNTING

*How to find an integer while posing few questions?*

— Ask if in (1—2), (2—4), (4—8), (8–16), etc?
— Conclude by binary search: cost is $2 \log_2 n$.

*The* **dyadic paradigm** *for unbounded search:*

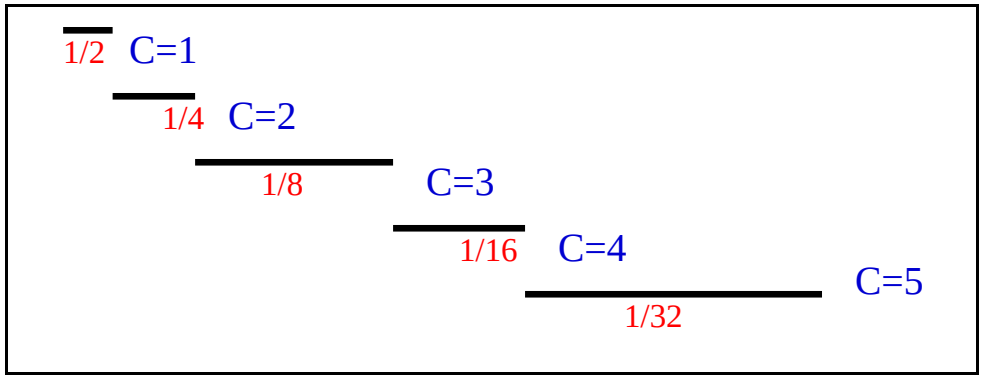
- Ethernet proceeds by period doubling + randomization.
- Wake up procedures for mobile communication (Lavault$^+$)
- Adaptive data structures: e.g., extendible hashing tables.

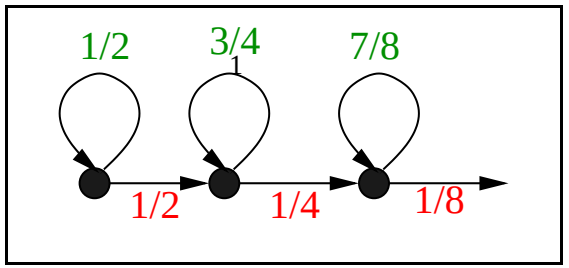
♡ Approximate Counting

# Approximate counting & randomization

The oldest algorithm (Morris CACM:1977), analysis (F, 1985).

Maintain counter subject to $X := X + 1$.




ALGORITHM: Approximate Couting

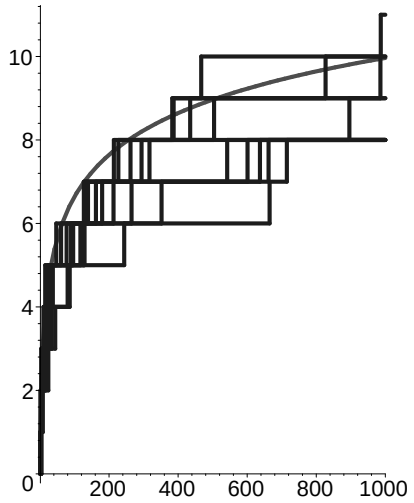
Initialize: $C := 1$;

Increment: do $C := C + 1$ with probability $2^{-C}$;

Output: $2^C - 2$.

Expect $C$ near $\log_2 n$ after $n$ steps, then use only $\log_2 \log n$ bits.

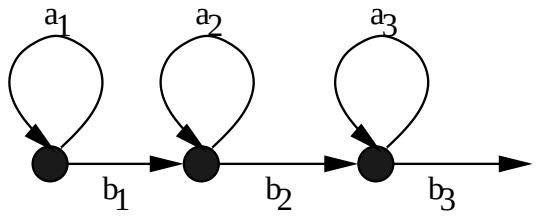
10 runs of of APCO: value of $C$ ($n = 10^3$)




**Theorem:** Using alternate bases, count till $n$ *probabilistically* using $\log_2 \log n + \delta$ bits, with accuracy about $0.59 \cdot 2^{-\delta/2}$.

Beats information theory(!?): 8 bits for counts $\leq 2^{16}$ w/ accuracy $\approx 15\%$.

**Symbolic methodology**:




♡ Generating Functions: $\boxed{(f_n) \mapsto f(z) := \sum_n f_n z^n.}$

$$\frac{(a_1)^\star b_1 (a_2)^\star b_2 (a_3)^\star}{\frac{1}{1-a_1} b_1 \frac{1}{1-a_2} b_2 \frac{1}{1-a_3}} \quad \text{since} \quad \frac{1}{1-f} = 1 + f + f^2 + \cdots \simeq (f)^\star.$$

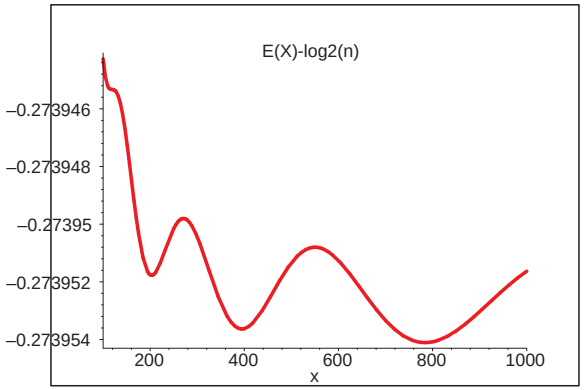
Perform probabilistic valuation $a_j \mapsto q^j$; $b_j \mapsto 1 - q^j$:

$$H_3(z) = \frac{q^{1+2} z^2}{(1-(1-q)z)(1-(1-q^2)z)(1-(1-q^3 z))}.$$

*cf* F.+Sedgewick, *Analytic Combinatorics* C.U.P., 2007.

*Approximate Counting*

Mean$(X) - \log_2 n$:




**Asymptotic methodology:** Mellin transform (FISe*)

$$f^\star(s) := \int_0^\infty f(x) x^{s-1}\, dx.$$

Need **singularities** in **complex plane**.

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♣ Dyadic superpositions of models $\Longrightarrow$ Asymptotics with fluctuations.

Mellin: Probabilistic counting, loglog counting + Lempel-Ziv compression (Jacquet-Szpa) + dynamic hashing + tree protocols (Jacquet+) + Quadtries &c.

## Cultural flashes

— Morris (1977): Couting a large number of events in small memory.
— The power of probabilistic machines & approximation (Freivalds)
— The TCP protocol: Additive Increase Multiplicative Decrease (AIMD) leads to similar functions (Robert et al, 2001+)
— Probability theory: Exponentials of Poisson processes (Yor et al, 2001)

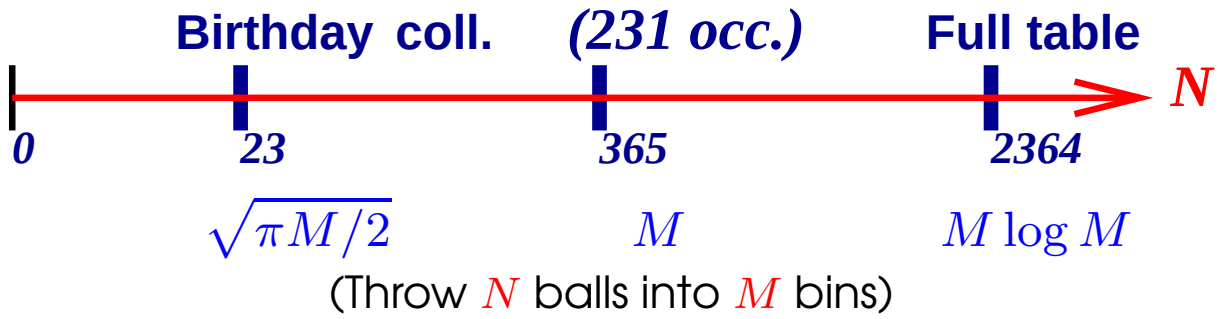
— Ethernet is unstable (Aldous 1986) but tree protocols *are* stable! Cf (Jacquet+).

# Probabilities: Coupons & Birthdays

Let a flow of people enter a room.

— *Birthday Paradox:* It takes on average **23** for a **birthday collision**

— *Coupon Collector*: After 365 persons, expect a **partial collection** of $\sim$**231 different days** in the year; it would take more than 2364 to reach a full collection.



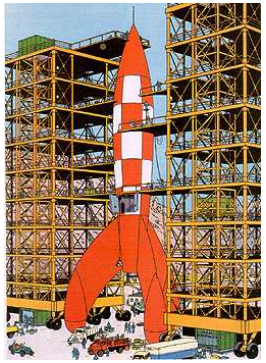

(Throw $N$ balls into $M$ bins)

Semi-clasical occupancy problems; generating functions; saddle points.

# Birthday Paradox Counting (Gedanken!)

Take your space ship to remote planet.

*How many days ($M$) in the year?*



**Gedanken Algorithm:**

— Interview people:

`39 Propergol, 42 Sargol, 11 Mordor, ..., 42 Sargol`

— Record $B$ = time till first birthday collision;

— $\mathbb{E}(B) \sim \sqrt{\frac{\pi M}{2}} \Longrightarrow$ return estimate $M \approx \frac{2B^2}{\pi}$;

— do some maths and improve!

*Sublinear algorithms . . .*

# Randomization and hashing

**Randomization** *is a major algorithmic paradigm.*

- Cryptography (implementation, attacks, cf RSA)
- Combinatorial optimization (smoothing, random rounding).
- Hashing and direct access methods

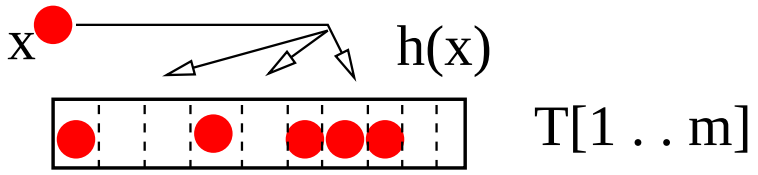
— Produce (seemingly) uniform data from actual ones;

— Provide reproducible chance

*To be or not to be. . .* $\rightsquigarrow$ 

# 3 HIT COUNTING

First Counting Algorithm: Estimate *cardinalities* $\equiv$ # of **distinct** elements.
Based on Coupon Collector phenomena & motivated by *query optimization in data bases.* (Whang$^+$, ACM TODS 1990)




ALGORITHM Hit Counting

Set up a table $T[1\,..\,m]$ of $m$ bit-cells.
— for $x$ in $S$ do mark cell $T[h(x)]$;
Return $-m \log V$, where $V$:=fraction of empty cells.

— Algorithm is *independent of replications*.

— Let $n$ be sought cardinality and $\alpha := n/m$, *filling ratio*. Expect

$$V \approx e^{-\alpha} \quad \text{fraction of empty cells,}$$

by classical occupancy theory.

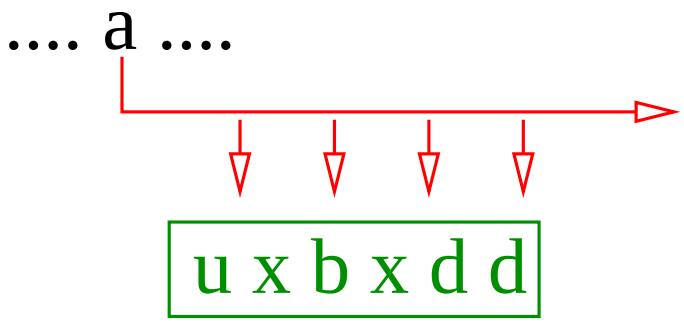
— Distribution is concentrated. Invert: $n \approx m \log(1/V)$.

**Linear storage complexity**: Count cardinalities till $N_{\max}$ using $\frac{1}{10} N_{\max}$ bits, for accuracy (standard error) = 2%.

Generating functions for occupancy; Stirling numbers; basic depoissonization.

# 4 SAMPLING

Classical sampling (Vitter, ACM TOMS 1985)



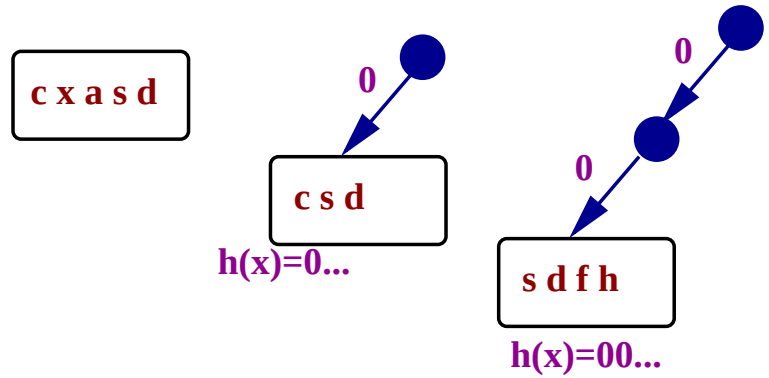

ALGORITHM Reservoir Sampling (*with multiplicities*)
Sample $m$ elements from $S = (s_1, \ldots, s_N)$; ($N$ unknown a priori)
Maintain a cache (reservoir) of size $m$;
— for each coming $s_{t+1}$:
place it in cache with probability $m/(t+1)$; drop random element;

Sample *values* (i.e., without multiplicity)? (Wegman+F).

ALGORITHM: Adaptive Sampling (*without multiplicities*)
Get a sample of size $\leq m$ according to **values.**
— Increase sampling depth and decrease sampling probabilities.

Sample of size $\leq m$:

depth $d = 0, 1, 2, \ldots$




Also: get $b$-sample by mild oversampling ($m = 4b$).

Analysis makes use of randomness digital trees, generating functions and Mellin transforms.

Adaptive Sampling = Second counting algorithm for *cardinalities*.

Let $d$ := sampling *depth*; $\xi$ :=sample size.

**Theorem**: $X := 2^d \xi$ estimates the cardinality of $S$ using $m$ words: unbiased, with standard error $\approx 1.20/\sqrt{m}$.

- $1.20 \doteq 1/\sqrt{\log 2}$: with $b = 1,000$W, get $4$% accuracy.
- Related to folk algorithm for leader election on channel: "*Talk, flip coin if noisy; sleep if Tails; repeat!*
- Related to "tree protocols with counting" $\gg$ Ethernet. Cf (Greenberg-F-Ladner JACM 1987).

# Cardinality Estimators

$F_0$ = *Number of different values*

— 1983–1985: (F-Martin, FOCS+JCSS) Probabilistic Counting
— 1987–1990: (Whang et al) Hit Counting
— 1984–1990: (Wegner) (F90 COMP.) Adaptive Sampling
— 1996: (Alon et al, STOC) $F_p$ statistics $\rightsquigarrow$ *later*
— 2000: (Indyk FOCS) Stable Law Counting $\rightsquigarrow$ *later*
— 2001: (Estan-Varghese SIGCOMM) Multiresolution Bitmap
— 2003: (Durand-F ESA) Loglog Counting
— 2005: (Giroire) MinCount
— 2006: (DFFM) HyperLoglog

Note: *suboptimal* algorithms can be **useful!**

# 5 PROBABILISTIC COUNTING

Observables of hashed values!

Third Counting Algorithm

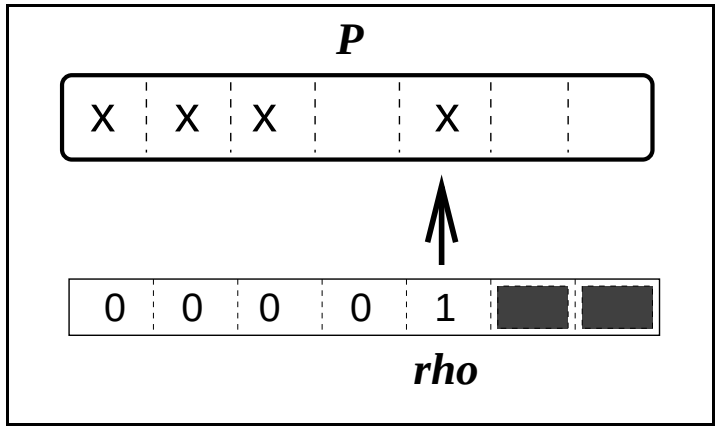
for *cardinalities*

$0.1xxxx \rightarrow \mathbb{P} = \frac{1}{2}$

$0.01xxx \rightarrow \mathbb{P} = \frac{1}{4}$

$0.001xx \rightarrow \mathbb{P} = \frac{1}{8}$

:




ALGORITHM: Probabilistic Counting

Input: a stream $S$; Output: cardinality $|S|$

For each $x \in S$ do /* $\rho \equiv$ position of leftmost 1-bit */

Set BITMAP[$\rho(\mathrm{hash}(w))$] := 1; od;

Return $P$ where $P$ is position of first 0.

— $P$ estimates $\log_2(\varphi n)$ for $\varphi \doteq 0.77351 = \dfrac{e^\gamma}{\sqrt{2}} \prod_{m\geq 2}^{\star} m^{\epsilon(m)}$, $\epsilon(m) := (-1)^{\sum \text{bits}(m)}$.

**A.** Average over $m$ trials $A = \frac{1}{m}[A_1 + \cdots + A_m]$; return $\frac{1}{\varphi} 2^A$. Error$\approx \frac{1}{\sqrt{m}}$.

**B.** Stochastic averaging needs only *one* hash function.

$$x \mapsto h(x) \longrightarrow \begin{cases} 00 & \text{count } n_{00} \\ 01 & \text{count } n_{01} \\ 10 & \text{count } n_{10} \\ 11 & \text{count } n_{11}. \end{cases}$$

**Implementation:** Hash + switch + update.

**Theorem [FM]**: Prob. Count. is *asymptotically unbiased*. Accuracy is $\frac{0.78}{\sqrt{m}}$ for $m$ Words of size $\log_2 N$.

## Data mining of the Internet graph

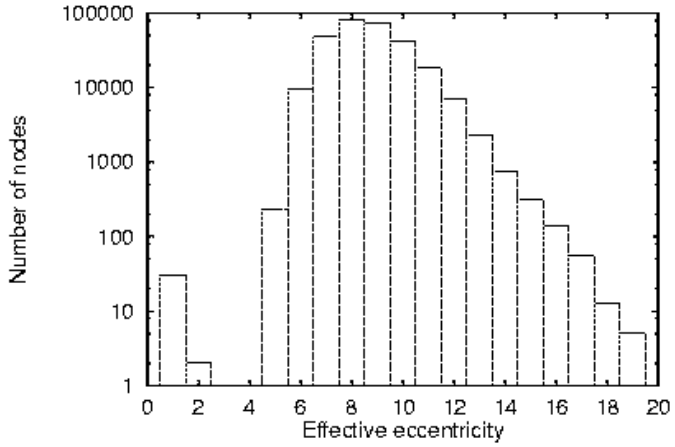
(Palmer, Gibbons, Faloutsos$^2$, Siganos 2001)

Internet graph: 285k nodes, 430kedges.

For each vertex $v$, define ball $B(v; R)$ of radius $R$.

Want: histograms of $|B(v, R)|$ $R = 1 \ldots 20$

Get CPU **400** speedup factor (minutes rather than day)



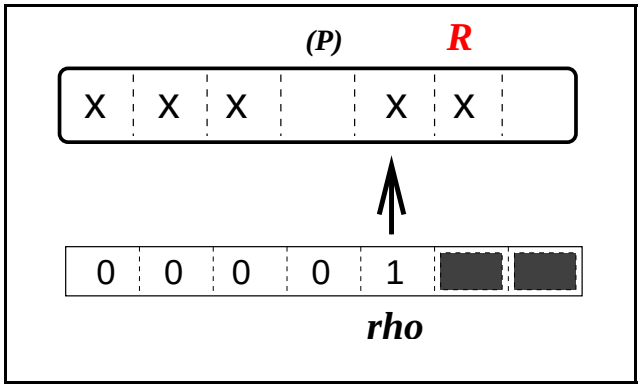

b) Histogram of diameters

+ Sliding window usage (Motwani et al) (Fusy & Giroire for MinCount).

# 6 LOGLOG COUNTING

Fourth Counting Algorithm for *cardinalities*: (Durand-F, 2003/DFFM, 2006)

- Hash values $\longrightarrow \rho(h(x))$ = position of leftmost 1-bit = a geometric RV $G(x)$.
- To set $S$ associate $R(S) := \max_{v \in S} G(v)$.




Max of geometric RVs are well-known (Prodinger*). $R(s)$ estimates $\sim \log(\widehat{\varphi}\,\mathrm{card}(S))$, with $\widehat{\varphi} := e^{-\gamma}\sqrt{2}$.

- Do stochastic averaging with $m = 2^{\ell}$:

Return $\boxed{\frac{m}{\widehat{\varphi}} 2^{\mathrm{Average}}}$.

# Engineering HyperLogLog

©Durand+F+Fusy+Meunier, 2006.

++ Optimize by limiting effect of discrepant values: use harmonic means of $2^R \rightsquigarrow$ HyperLogLog ($\ll$Chassaing-Gerin, 2006). Asymptotically near-optimal?.

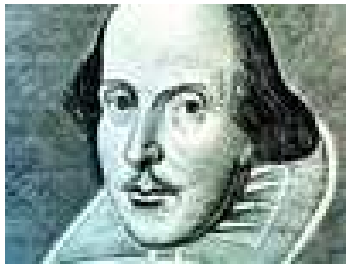
+ Correct nonasymptotic regime by switching to Hit Counting.

+ Extend beyond nominal capacity by random allocation correction.

**Theorem.** HyperLogLog needs $m$ "bytes", each of length $\log_2 \log N$. Accuracy is: $\frac{1.05}{\sqrt{m}}$.

**Distributed implementation:** exchange registers (a few kilobytes) and can even **compress**, to reduce storage by extra factor of 3.

Whole of Shakespeare:



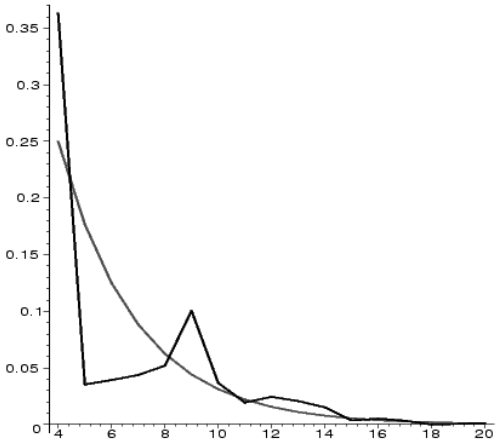
$m = 256$ small "bytes" of 4 bits each = 128bytes

```
ghfffghfghgghgggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhgggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhiggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Estimate $n^\circ \approx 30,897$ against $n = 28,239$ distict words

Error is $+9.4\%$ for 128 bytes(!!)

**Features:** Errors $\approx$ *Gaussian*, seldom more than $2\times$ standard error.




Mahābhārata: 8MB, 1M words, 177601 diff.

HTTP server: 400Mb log pages 1.8 M distinct req.

| $m$ | $2^6$ (50by) | $2^{10}$ (0.8kby) | $2^{14}$ (12kb) |
|---|---|---|---|
| Obs: | 8.9% | 2.6% | 1.2% |
| $\sigma$: | 11% | 2.8% | 0.7% |

# Example: Mining the human genome

(by Frédéric Giroire)

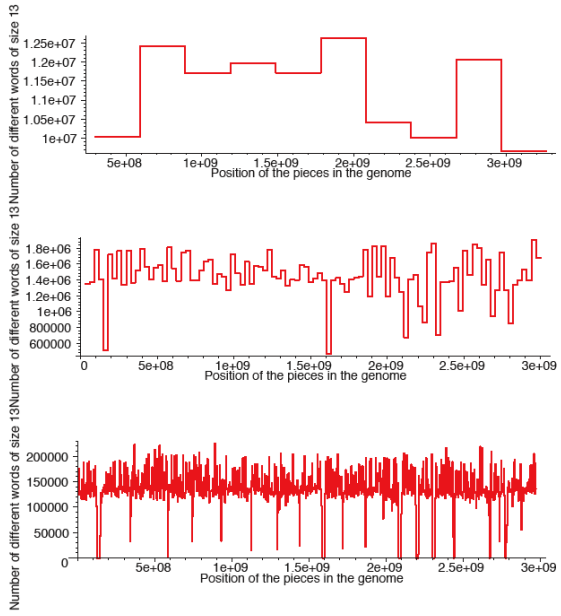
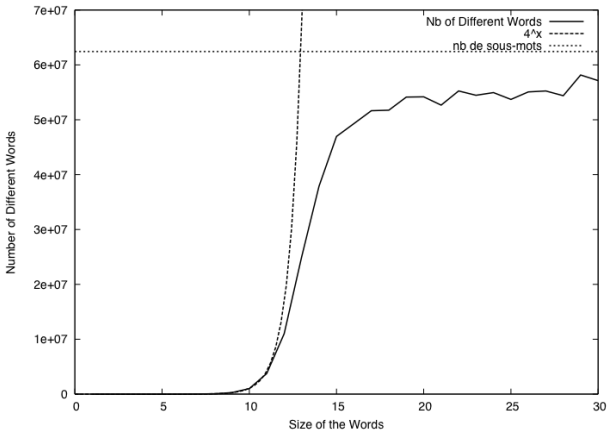



Figure 1: Mots différents de taille 13 dans des séquences consécutives du génome de $3M$, $30M$ et $300M$ de paires de bases.




Patterns of length 13 in sections of human genome (3Gbp).

Number of different patterns of length $\ell$ in Chr. 20 (60Mbp).

## Summary

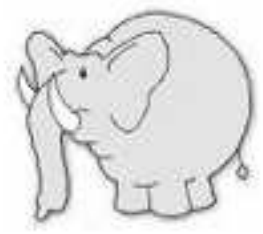
*Analytic results* ($\lg \equiv \log_2$): Alg/Mem/Accuracy

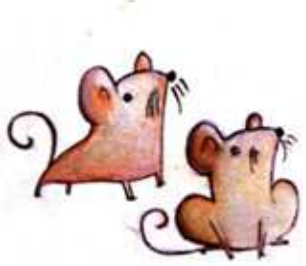
| CouponCC | AdSamp | ProbC | LogLog |
|---|---|---|---|
| $\approx \frac{N}{10}$ bits | $m \cdot \lg N$ Words | $m \cdot \lg \frac{N}{m}$ Words | $m \cdot \lg \lg \frac{N}{m}$ Bytes |
| $\approx 2\%$ | $\frac{1.20}{\sqrt{m}}$ W | $\frac{0.78}{\sqrt{m}}$ W | $\approx \frac{1.30--1.05}{\sqrt{m}}$ By |

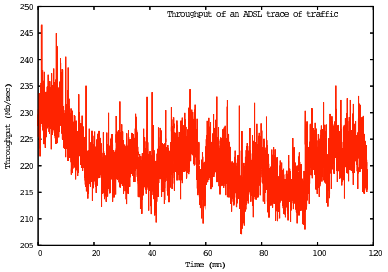
### $F_0$ statistics, $N = 10^8$ & 2% error

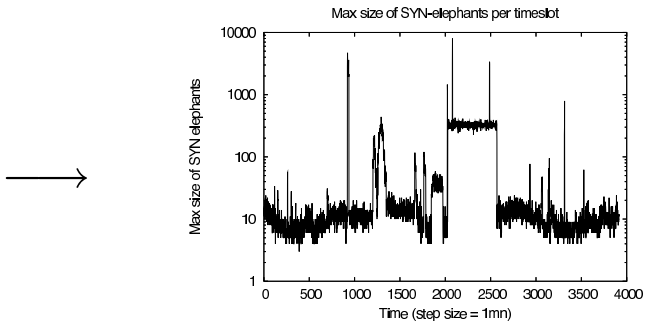
— Coupon Collector Counting = 1 Mbyte + used for corrections
— Adaptive Sampling = 16 kbytes + sampling, unbiased
— Probabilistic Counting: = 8 kbytes + sliding window
— Multiresolution bitmap (analysis?) = 5 kbytes?
— MinCount ©Giroire = 4 kbytes + sliding window
— Loglog Counting = 2 kbytes + mice/elephants

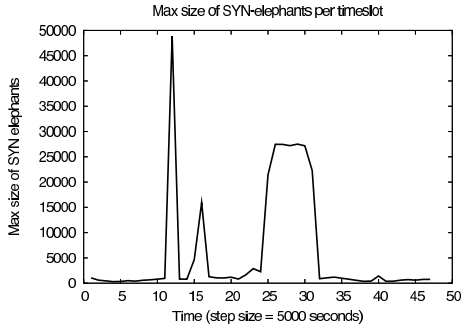
# 7 ELEPHANTS AND MICE








DoS attacks: Largest elephants; (here) elephant = number of packets to a given destination.

— By LogLog Counting and $p$-filtering: estimate $\mathrm{card}(\Phi)$ and $\mathrm{card}(\Phi_{pM} \cup \Phi_{pE})$ (Jean-Marie+Gandouët06).

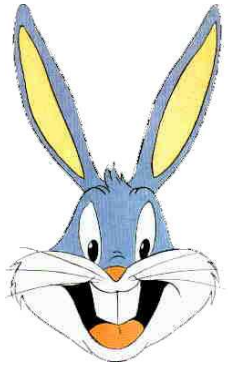
— By Bloom filters (Azzana+Robert+Chabchoub)

# 8 PROFILING: FREQUENCY MOMENTS

Alon-Matias-Szegedy: $F_p := \sum_v (f_v)^p$ where $f_v$:= frequency of value $v$.

dim = $n$ $\longrightarrow$ dim $\approx \log n$

Johnson–Lindenstrauss
embeddings
dimension reduction
**Indyk**'s beautiful ideas



Use of random Gaussian projections for $F_2$; Stable laws for $0 \leq p \leq 2$.

# Indyk's $F_p$ algorithm

• Stable law of parameter $p \in (0,2)$: $\mathbb{E}(e^{itX}) = e^{-|t|^p}$.

No second moment; no 1st moment if $p \in (0,1)$.

$c_1 X_1 + c_2 X_2 \stackrel{\mathcal{L}}{\cong} \mu X$, with $\mu := (c_1^p + c_2^p)^{1/p}$.

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ALGORITHM: Frequency Moments F$p$;

Initialize Z:=0;

For each $x$ in $S$ do $Z := Z + \mathrm{Stable}_\alpha(x)$.

Return $Z$.

Estimate $F_p$ parameter from $m$ copies of $Z$-values.

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Remark: Use of $\log(|Z|)$ to estimate seems better than median(?)

# Conclusions

For huge data streams, using reduced/minimal storage, one can:
— Sample positions and (distinct) values;
— Count events and cardinalities ($F_1, F_0$);
— Estimate profiles of data ($F_p$), $0 < p \leq 2$;
— Get informations on elephants and mice.

♡♡♡♡♡♡

The algorithms are based on randomization $\mapsto$ Analysis fully applies
— They tend to work **exactly** as predicted on real-life data;
— They often have a very elegant structure;
— Their analysis involves numerous methods of AofA: "Symbolic modelling by generating functions, Singularity analysis, Saddle-point and analytic depoissonization, Mellin transforms, Stable laws, etc."

*That's All, Folks!*