Threshold for Virus Spread on Networks

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What are the features of the topology that determine how virulent the epidemic is?

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- Model description

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- Ball et al. (1997) Structured networks with two levels of mixing: local (within a household) and global (between households)

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Eventually there are no more infectives in the population: How many nodes are removed?

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If we assume that the graph is connected, then the spectral radius $\lambda_1(A)$ has multiplicity one

Theorem Suppose $\beta\lambda_1(A) < 1$. Then

$$\mathbb{E}[|Y(\infty)|] \leq \frac{1}{1 - \beta\lambda_1(A)} \sqrt{n|X(0)|}$$

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Moreover, if the graph G is regular (i.e., each node has the same number of neighbours), then $\mathbb{E}[|Y(\infty)|] \leq \frac{1}{1-\beta\lambda_1(A)} |X(0)|$

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To infect node v at t, there must be a chain of distinct nodes $u_0, u_1, \ldots, u_t = v$ along which the infection passes from some initial infective u_0 to v

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Consequently, for $\beta\lambda_1(A) < 1$,

$$\begin{aligned}
\mathbb{E}[|Y(\infty)|] = \sum_{v \in V} \mathbb{P}(Y_v(\infty) = 1) &\leq \sum_{v \in V} \sum_{t=0}^{\infty} \sum_{u \in V} (\beta A)^t_{uv} X_u(0) \\
&= \sum_{t=0}^{\infty} \mathbf{1}^T (\beta A)^t X(0) \\
&= \mathbf{1}^T (I - \beta A)^{-1} X(0) \\
&\leq \|\mathbf{1}\| \, \|(I - \beta A)^{-1}\| \, \|X(0)\| \\
&\leq \frac{1}{1 - \beta\lambda_1(A)} \sqrt{n|X(0)|}
\end{aligned}$$

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Regular graphs: Using the spectral decomposition and the fact that the positive vector $\frac{1}{\sqrt{n}}(1,\ldots,1)^T$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1(A) = d$, we have

$$\begin{aligned}
\mathbb{E}[|Y(\infty)|] &\leq \frac{1}{1-\beta\lambda_1(A)}\mathbf{1}^T\sum_{i=1}^{n} x^i(x^i)^T X(0) \\
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General infectious periods: If $\left(1-\mathbb{E}[e^{-\lambda I}]\right)\lambda_1(A) < 1$ then the Theorem holds.

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$$\mathbb{E}[|Y(\infty)|] = 1 + \beta(n-1) = 1 + c\sqrt{n-1}$$

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- Thus the upper bound of the theorem is close to the best possible. Moreover this illustrates the effect of initial conditions

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$$\mathbb{P}\big(|Y(\infty)| > (1+o(1))\gamma n\big) = \gamma, \quad \mathbb{P}\big(|Y(\infty)| = O(\log(n))\big) = 1-\gamma$$

for any $|X(0)| \geq 1$, where $\gamma > 0$ solves $\gamma + e^{-\gamma c} = 1$. Hence $\mathbb{E}[|Y(\infty)|] \geq (1+o(1))\gamma^2 n$.

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  for any $|X(0)| \geq 1$, where $\gamma > 0$ solves $\gamma + e^{-\gamma c} = 1$. Hence $\mathbb{E}[|Y(\infty)|] \geq (1+o(1))\gamma^2 n$.

Threshold $c = 1$ for the final size of the epidemic starting with a constant number of initial infectives, the final size is a constant independent of n if $c < 1$, and a fraction of n if $c > 1$

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The size of the epidemic is stochastically dominated by a (subcritical) branching process with offspring distribution $Binom(n-1, \beta p_n)$

Lemma If $\limsup_{n\to\infty} c_n = c < 1$, then for all n sufficiently large, $\mathbb{E}[|Y(\infty)|]$ is bounded by a constant that does not depend on n.

On the other hand, if $\liminf_{n\to\infty} c_n = c > 1$, then $\mathbb{E}[|Y(\infty)|] \geq (1+o(1))\gamma^2 n$ where $\gamma > 0$ solves $\gamma + e^{-\gamma c} = 1$ (Giant component).

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- Models for power law graphs: preferential attachment, Yule branching processes

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Let $G(w)$ be a graph and $w = (w_0, w_1, \ldots, w_{n-1})$ the sequence of its average degrees. The edge between the pair of vertices (i, j) is present with probability

$$p_{ij} = \frac{w_i w_j}{\sum_{k=1}^{n} w_k}, \qquad w_i = c(i_0 + i)^{-\frac{1}{\gamma - 1}}$$

where

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By Chung, Lu and Vu 2004, under mild conditions the spectral radius of the graph is

$$\lambda_1(A) = \begin{cases} (1+o(1))\sqrt{m}\,, & \gamma > 2.5, \\ (1+o(1))Cm^{3-\gamma}\,, & 2 < \gamma < 2.5\,. \end{cases}$$

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By Theorem, if $\beta\lambda_1(A) < 1$, then the size of the epidemic is bounded by $\sqrt{n}$ times the size of the initial infective population

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Fix k (as a function of n) and consider the subgraph induced by the k nodes with the largest weight in the random graph $G(\beta w)$. The average expected degree of this subgraph is easily seen to be

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Suppose first that $\gamma > 3$. Then d_k is a non-decreasing function of k, and its maximum value, attained at $k = n$, is βd. This only yields the weak result that there is a large epidemic if $\beta d > 1$.

Large outbreak ($2 < \gamma < 3$)

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d_k is a decreasing function of k. Fix $\delta > 0$. Defining N_δ to be the largest value of k for which $d_k > 1 + \delta$, we see that

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- $2 < \gamma < 3$, studying the epidemic on the subgraph of high degree nodes (close to an Erdös-Rényi graph) and using above result one can show that there is a large epidemic with high probability.

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- Conjecture: the right threshold appears to be $\tilde{d} = \frac{\sum_i w_i^2}{\sum_i w_i}$.
- Duration of such the epidemic (in the supercritical case) until it reaches an absorbing state (diameter is a trivial lower bound)