Exit problem of a two-dimensional risk process from a cone: exact and large deviations results

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1 The Cramer-Lundberg, renewal and perturbed reserves models:

$$X_t = x + p\,t - S(t) + Y(t) \tag{1}$$
$$S(t) = \sum_{i=1}^{N(t)} Z_i$$

where x denotes the initial reserve of an insurance company, p is the premium rate per unit of time, the i.i.d. random variables Z_i with distribution function $B(x)$ are the "claims", and N_t is a counting process [1]). Alternatively, we have a **SDE with constant coefficients** $\boxed{dX_t = pdt - dS(t) + dY(t)}$.

A more realistic model, including an interest rate r, is the GOU

$$\boxed{dX_t = (p + rX_t)dt - dS(t) + dY(t)}$$

[1] Quite similar to the classical risk model is the M/G/1 workload model used in queueing theory, where only **upward** jumps are allowed (the stationary distribution of the M/G/1 workload coincides with that of the ruin problem).

2 The ruin problem

Let τ_y^+, τ_y denote the times of first passage:

$$\tau_y^+ := \inf\{t \geq 0 : X_t \geq y\}$$

and

$$\tau_y := \inf\{t \geq 0 : X_t < y\}$$

When $y = 0$, $\tau := \tau_0 = \inf\{t \geq 0 : X_t < 0\}$ is also called **ruin time**.

Classical risk theory is concerned with the harder downwards first-passage in the "non-smooth exit" direction, especially with the finite-time ruin or survival probabilities

$$\psi(t,x) = P_x[\tau \leq t]$$
$$\overline{\psi}(t,x) = P_x[\tau > t] = 1 - \psi(t,x).$$

Analytical results are sometimes available for the "ultimate/perpetual" ruin probability

$$\boxed{\psi(x) = P_x[\tau < \infty]}$$

and for the "killed" probabilities of passage before an independent exponential horizon $\mathcal{E}(q)$ of rate q:

$$\boxed{\psi_q(x) = P_x[\tau \leq \mathcal{E}(q)]} \tag{2}$$

$$= \int_0^\infty q\mathrm{e}^{-qt} P_x[\tau \leq t]dt = \int_0^\infty \mathrm{e}^{-qt}\psi(\mathrm{d}t, x) = E_x \mathrm{e}^{-q\tau}$$

Since killed ruin probabilities are the Laplace-Carson transform in time of $\psi(t, x)$, their knowledge allows recovering the finite time ruin probabilities by inversion of the Laplace transform numerically/by Erlangization/by Laguerre transforms.

Example 1 With exponential claim sizes of intensity μ, the ultimate ruin probability is:

$\psi(x) = C\mathrm{e}^{-\gamma x}$ where $\gamma = \mu - \lambda/p > 0,\ C = \dfrac{\lambda}{\mu p} < 1.$

Formulas involving matrix exponentials are available more generally when the claims are phase-type (with rational Laplace transform).

The distinction between crossings up and down appears already in **one-dimensional diffusion/BD theory**, where the **space of "$q-$ harmonic functions**, i.e. solutions of the "Sturm-Liouville equation" with given boundary values

$$\Gamma f(x) - qf(x) = 0, q > 0 \qquad (3)$$

is two-dimensional, and may be generated by an **increasing and a decreasing positive solutions** (unique up to a constant) $\varphi_q^{\pm}(x)$, which will be called monotone solutions.

In diffusion theory, these monotone solutions intervene for example in the

Laplace transform of the hitting time:

$$\psi_q(y|x) = \begin{cases} E_x e^{-q\tau_y^+} = \frac{\varphi_q^+(x)}{\varphi_q^+(y)}, x \le y \\ E_x e^{-q\tau_y} = \frac{\varphi_q^-(x)}{\varphi_q^-(y)}, x \ge y \end{cases}$$

(these may be derived either from (3) + boundary conditions, or by optional stopping of the martingales $e^{-qt}\varphi_q^{\pm}(X_t)$).

"Spectrally-negative" processes like (1) continue to have a special increasing $q-$ harmonic function (which in the Levy case is simply exponential), and a multiplicative structure for first-passage upwards, in the direction of the "smooth exit".

Below, we will study more stringent survival requirements, like the "regulated survival probability"

$\psi^{(d)}(x)$ that the risk process $X(t)$ never crosses under the piecewise-linear barrier $b(t) = (x - dt)_+$.

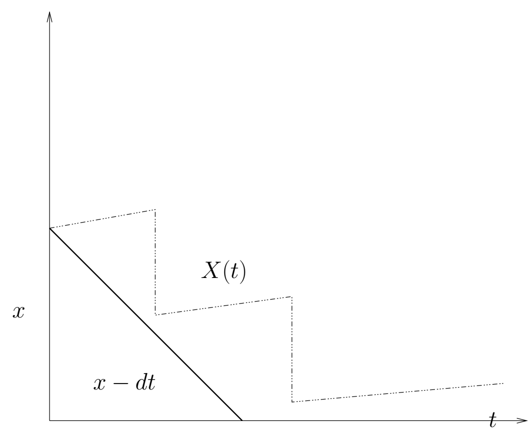

Figure 1: **Regulated survival**

3 Levy processes, Levy-Khyncin, Wiener-Hopf.

When N_t is Poisson, the reserves process (1) is an example of **Levy process without upward jumps/spectrally negative**. Typically, results about Levy process maybe expressed in terms of the **cumulant generating function/symbol/kernel** $\kappa(\theta)$:

$$\kappa(\theta) := t^{-1} \log\Big(E\big[\mathrm{e}^{\theta X(t)}\big]\Big)$$

A related important function is the inverse $q_+ = \Phi(q)$ of $\kappa(\theta)$, i.e. the largest nonnegative solution $\theta = \Phi(q)$ of "Lundberg's equation"

$$\kappa(\theta) - q = 0, \qquad \forall q \geq 0, \tag{4}$$

included the analyticity domain $\Theta = \{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}$.

Example 2 The cumulant exponent of the Cramer-Lundberg model $X(t) = X(0) + \mu\, t + \sigma W_t - \sum_{i=1}^{N_t} Z_i$ "perturbed" by an independent standard Brownian motion W is:

$$\kappa(\theta) = \frac{\sigma^2\theta^2}{2} + \mu\theta + \lambda(\hat{b}[\theta] - 1)$$

The absence of jumps or the presence of only phase-type jumps in one direction simplifies considerably the analysis of Levy models [2].

4 Asymptotic one dimensional results

The cgf $\kappa(\theta)$ of a spectrally negative Levy process X_t is well defined on some maximal open domain (l, ∞) including the positive half axis. The cases $l < 0, \kappa(l) \geq 0$, $l < 0, \kappa(l) < 0$, and $l = 0$, known respectively as Cramer's type, "almost exponential" and subexponential, give rise to different asymptotic behavior.

[2] While general Levy processes require a "Wiener-Hopf factorization" of (**??**), in the "spectrally one-sided/phase-type cases this reduces to finding a finite number of nonnegative/nonposative roots; furthermore, this may be achieved via matrix computations, by representing the symbol in matrix-exponential form. The generator of a Levy process, which is an integro-differential operator maybe also be viewed as $G = \kappa(D)$, where D is the first derivative, and where this expression is to be interpreted in the sense of pseudo-differential operators (in the case of Feller processes, this becomes $G = \kappa(x, D)$).

Under the **Cramer assumption** $\exists\gamma > 0, \kappa(-\gamma) = 0, \kappa'(-\gamma) > -\infty$ the asymptotic behavior is exponential

$$\lim_{x\to\infty} \frac{\psi(x)}{e^{-\gamma x}} = C, \qquad \boxed{C = -\kappa'(0)/\kappa'(-\gamma)} \tag{5}$$

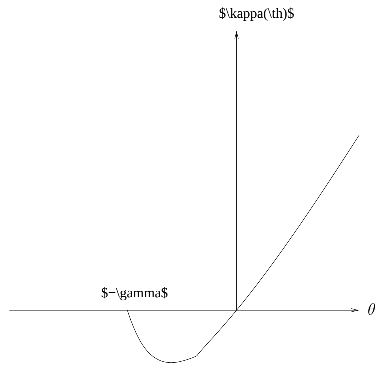

Figure 2: spectrally negative cgf/symbol/kernel

Assume that $\kappa(\theta)$ is steep, let $v > 0$, and let $\theta_v, \theta_v' \in \Theta^o$ be defined by $\theta_v < \theta_v'$, $\kappa'(\theta_v) = -v$ and $\kappa(\theta_v) = \kappa(\theta_v')$. θ_v is known as Cramer shift, and θ_v' as its conjugate

The **asymptotic behavior of the finite-time ruin probabilities** $\psi(t,x) = P_x(\tau < t)$ has first been obtained by Arfwedson (1955) via the saddle-point method. Later, Hoglund (1990) noted similar results for the "late ruin probabilities"

$$\boxed{w(t,x) = P(t \le \tau(x) < \infty) = \psi(x) - \psi(t,x)}$$

The modern proof uses the **exponential family** of measures $\{P^{(c)}\}$ defined by shifting the cgf

$$\boxed{\kappa^{(c)}(\theta) := \kappa(\theta + c) - \kappa(c)}$$

or by the Radon-Nikodym derivative $\Lambda^{(c)}$:

$$\left.\frac{\mathrm{d}P^{(c)}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \Lambda^{(c)}(t) := \exp\left(c(X(t) - X(0)) - \kappa(c)t\right) \tag{6}$$

for $\theta \in \Theta := \{\theta : \kappa(\theta) < \infty\}$. The basic idea is that asymptotic conditioning on rare events may often be expressed in terms of the exponential family.

4.1 Approximations for finite-time ruin probabilities and the limit laws for the process before/after ruin

We show now that the **limit laws of $X(t)$ conditioned on $t < \tau$ and on $t > \tau$**, respectively, as x and t tend to infinity such that $x/t = v$, are given by

$$\begin{aligned}\overline{\Psi}_v(\mathrm{d}x) &= c(v)^{-1}[\mathrm{e}^{\theta'_v x} - \mathrm{e}^{\theta_v x}]\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x, \qquad (7)\\ \Psi_v(\mathrm{d}x) &= |c(v)|^{-1}[\mathrm{e}^{\theta_v x}\mathbf{1}_{(0,\infty)}(x) + \mathrm{e}^{\theta'_v x}\mathbf{1}_{(-\infty,0)}(x)]\mathrm{d}x,\end{aligned}$$

where $0 < v < -\kappa'(-\gamma), \theta_v < \theta'_v < 0$ for $\overline{\Psi}_v$ and $v > -\kappa'(\gamma), \theta_v < 0 < \theta'_v$, for Ψ_v, respectively, and where $c(v)$ is the function appearing in Arfwedson's correction term (53).

Theorem 1 *Suppose that there exists a $\gamma > 0$ such that $\kappa(-\gamma) = 0, \kappa'(-\gamma) < 0$, and that $\kappa(\theta)$ is steep.*

Then the following hold true:

$$\overline{\Psi}_v \;=\; d - \lim_{t\to\infty} P_{vt}(X(t) \in \cdot\ |\tau > t), \qquad (8)$$
$$\Psi_v \;=\; d - \lim_{t\to\infty} P_{vt}(X(t) \in \cdot\ |\tau < t), \qquad (9)$$

where $d - \lim$ *denotes convergence in distribution.*

Proof of Theorem 1: We show that the mgfs of the probability measures on the right-hand side of (8) converge to that of $\overline{\Psi}_v$, by using the mgf of $\overline{\Psi}_v$: $f_v(c) := \frac{\theta_v}{\theta_v+c} \cdot \frac{\theta'_v}{\theta'_v+c}, \qquad c < -\theta_v$, and

$$P_x^{(c)}(t < \tau < \infty) = P_x^{(c)}(t < \tau)$$
$$\kappa^{*,(c)}(v) = \kappa^*(v) + \kappa(c) + cv$$
$$E_x[\mathrm{e}^{cX(t)}\mathbf{1}_{\{t<\tau\}}] = \mathrm{e}^{cx+\kappa(c)t} P_x^{(c)}(t < \tau < \infty).$$

Using now $\theta_v^{(c)} = \theta_v - c$ and $\theta_v^{(c)\prime} = \theta'_v - c$, we find from Theorem 5 that for $c < -\theta_v$:

$$\lim_{t\to\infty} E_{vt}(\mathrm{e}^{cX(t)}|t < \tau) = D^{(c)}(v)/D(v) =$$
$$\frac{\theta_v^{(c)\prime} - \theta_v^{(c)}}{|\theta_v^{(c)}\theta_v^{(c)\prime}|}\frac{|\theta'_v\theta_v|}{\theta'_v - \theta_v} = f_v(c),$$

$\square$

5 The " degenerate" proportional reinsurance model

Our motivation is a particular two-dimensional risk model in which two companies split the cumulative claim process $S(t)$ in proportions δ_1 and δ_2, and receive premiums at rates p_1 and p_2, respectively. Assuming w.l.o.g. $\delta_1 = \delta_2 = 1$, we arrive by scaling at the two-dimensional "degenerate" risk process $X = (X_1, X_2)$ where X_i, the risk processes of the i'th company, are:

$$\boxed{X_1(t) = x_1 + p_1 t - S(t), \quad X_2(t) = x_2 + p_2 t - S(t)}$$

where $S(t)$ is a spectrally negative Levy process.

We let $\rho := ES(1)$ (which in the classical case compound Poisson case is given by $\lambda \mathbb{E} \sigma_1$). As usual in perpetual ruin problems, we assume that $p_i > \rho$, which implies that $X_i(t) \to \infty$ a.s. as $t \to \infty$ $(i = 1, 2)$.

To avoid trivialities, we assume that $p_1 \neq p_2$; if, say the first company receives larger premiums per claimed amount, i.e.:

$$\boxed{p_1 > p_2}$$

the second company (to be called reinsurer) will be more prone to ruin. We show however that this may be counterbalanced by larger initial reserves $x_2 > x_1$.

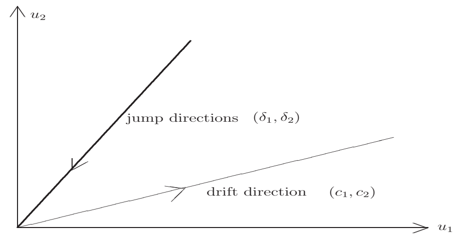

Figure 3: Geometrical considerations

6 Multidimensional ruin problems in the quadrant

Several ruin times may be of interest, like the first time τ_{or} when (at least) one insurance company is ruined, or the first time τ_{sim} when the insurance companies experience simultaneous ruin:

$$\begin{aligned} \tau_{\text{or}}(x_1, x_2) &:= \inf\{t \geq 0 : X_1(t) < 0 \quad \text{or} \quad X_2(t) < 0\} \\ \tau_{\text{sim}}(x_1, x_2) &:= \inf\{t \geq 0 : X_1(t) < 0 \quad \text{and} \quad X_2(t) < 0\}. \end{aligned}$$

We study the ultimate/killed ruin probabilities:

$$\psi_{\text{or}}(x_1, x_2) = P\left[\tau_{\text{or}}(x_1, x_2) < \mathcal{E}(q)\right] \tag{10}$$

$$\psi_{\text{sim}}(x_1, x_2) = P\left[\tau_{\text{sim}}(x_1, x_2) < \mathcal{E}(q)\right] \tag{11}$$

$$\begin{aligned} \psi_{\text{and}}(x_1, x_2) &= P_{(x_1,x_2)}\left[(\tau_1(x_1) < \mathcal{E}(q)) \cap (\tau_2(x_2) < \mathcal{E}(q))\right] \\ &= \psi_1(x_1) + \psi_2(x_2) - \psi_{\text{or}}(x_1, x_2) \geq \psi_{\text{sim}}(x_1, x_2). \end{aligned} \tag{12}$$

7 Exact ultimate/killed ruin probabilities

A key observation is that the "or" and "sim" ruin times τ are also equal to

$$\boxed{\tau(x_1, x_2) = \inf\{t \geq 0 : S(t) > b(t)\}}$$

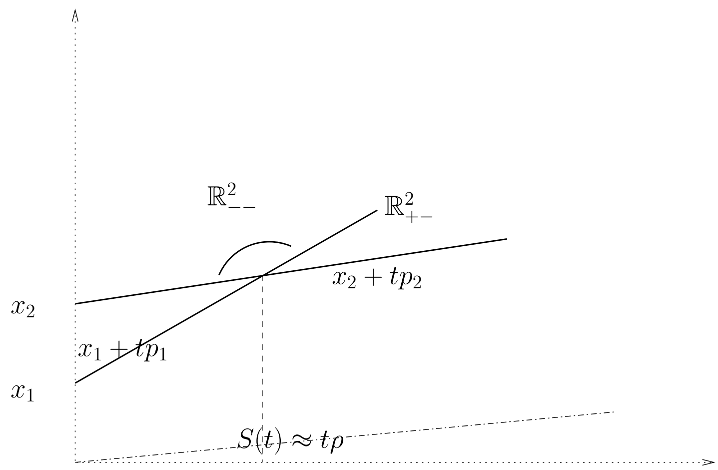

Figure 4: The piecewise-linear barrier corresponding to the degenerate two-dimensional first passage problem: $b_{\min}(t) = \min_{i=1,2}\{x_i + p_i t\}, b_{\max}(t) = \max_{i=1,2}\{x_i + p_i t\}$.

t

If $x_2 \leq x_1$, the barriers are linear

$$b_{\min}(t) = x_2 + p_2 t, \qquad b_{\max}(t) = x_1 + p_1 t$$

$$\psi_{\text{or}}(x_1, x_2) = \psi_2(x_2), \qquad \psi_{\text{sim}}(x_1, x_2) = \psi_1(x_1)$$

Solution in the upper cone $\mathcal{C}^c = \{x : x_2 > x_1\}$. Let

$$T = T(x_1, x_2) = \frac{(x_2 - x_1)_+}{p_1 - p_2} \tag{13}$$

denote the **deterministic time** of entering the lower cone $x_2 \le x_1$. In the "or" case, survival requires staying below the barrier $x_1 + p_1 t$ between the times 0 and T and subsequently staying below the barrier $x_2 + p_2 t$ after time T. Therefore, we find by conditioning at time T:

$$\boxed{\overline{\psi}_{\text{or}}(x_1, x_2) = \int_0^\infty \overline{\psi}_1(\mathrm{d}z, T|x_1)\overline{\psi}_2(x_2 + p_2 T - z)}$$

where

$$\overline{\psi}_i(\mathrm{d}z, T|x) := P_0(S(t) \le x + p_i t, \forall t \in [0, T], S(T) \in dz)$$

is the density at time T of the paths $S(T)$ which "survive" the upper barrier $x + p_i t$. It it convenient to reformulate this in terms of the one dimensional survival characteristics of the two coordinates of our reserves process $X_i(t)$.

Theorem 2 *Let $X(t) = (x_i + p_i t - S(t), \quad i = 1, 2)$ be a two-dimensional Lévy process with common cumulative claims $S(t)$ given by an arbitrary Lévy process, let*

$$I_i(t) = \inf_{0 \le s \le t} X_i(s) \wedge 0$$

denote the coordinates' infima, and suppose that $x_2 > x_1, p_2 < p_1$.

a) The two-dimensional ruin probabilities (10) are given by:

$$\begin{aligned}
\psi_{\rm sim}(x_1,x_2) &= P(\tau_2(x_2)\le T)+P(\tau_2(x_2)>T, \inf_{s>T} X_1(s)<0) \\
&= P(\tau_2(x_2)\le T)+\mathbb{E}[1_{\tau_2(x_2)>T}\ \psi_1(X_2(T))] \qquad (14)\\
\psi_{\rm or}(x_1,x_2) &= P(\tau_1(x_1)\le T)+P(\tau_1(x_1)>T, \inf_{s>T} X_2(s)<0) \\
&= P(\tau_1(x_1)\le T)+\mathbb{E}[1_{\tau_1(x_1)>T}\ \psi_2(X_1(T))] \qquad (15)\\
\psi_{\rm and}(x_1,x_2) &= P(T<\tau_1(x_1)<\infty)+P(\tau_1(x_1)\le T, \tau_2(x_2)<\infty) \qquad (16)
\end{aligned}$$

b) The two-dimensional survival probabilities associated to the or/sim ruin problems (10) are given by:

$$\begin{aligned}
\overline{\psi}_{\rm sim}(x_1,x_2) &= \int_0^\infty \overline{\psi}_2(\mathrm{d}z,T|x_2)\,\overline{\psi}_1(z),\\
\overline{\psi}_{\rm or}(x_1,x_2) &= \int_0^\infty \overline{\psi}_1(\mathrm{d}z,T|x_1)\,\overline{\psi}_2(z)
\end{aligned}$$

where $\overline{\psi}_i(\mathrm{d}z,T|x_i)$ *are the coordinate-wise densities of the "non-ruined" paths* $\overline{\psi}_i(\mathrm{d}z,T|x_i) = P_{x_i}(I_i(T)\ge 0, X_i(T)\in dz)$.

Not surprisingly, we have explicit killed versions $\psi_{\mathrm{or},q}(x_1,x_2), ...$ as well. We are also able to compute their double Laplace transform in space (and even invert it analytically by complex integration in the case of exponential claims).

8 Phase type and exponential claims

Corollary 1 *Suppose S is a compound Poisson process with phase-type jumps $(\boldsymbol{\beta}, \boldsymbol{B})$, i.e. $P[\sigma > x] = \boldsymbol{\beta} e^{\boldsymbol{B}x}\mathbf{1}$. If $x_2 > x_1$, it holds that*

$$\psi_{\mathrm{or}}(x_1, x_2) = P_{x_1}(I_1(T) < 0) + \boldsymbol{\eta}_2 \int_0^\infty e^{\boldsymbol{Q}_2 z}\overline{\psi}_1(\mathrm{d}z, T|x_1)\mathbf{1},$$

$$\psi_{\mathrm{sim}}(x_1, x_2) = P_{x_2}(I_2(T) < 0) + \boldsymbol{\eta}_1 \int_0^\infty e^{\boldsymbol{Q}_1 z}\overline{\psi}_2(\mathrm{d}z, T|x_2)\mathbf{1},$$

where $\boldsymbol{Q}_i = \boldsymbol{B} + \boldsymbol{b}\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}_i = \frac{\lambda}{p_i}\boldsymbol{\beta}(-\boldsymbol{B})^{-1}$.

Notes: 1) In this particular case (with 0 being the only nonnegative root of the symbol, the Wiener-Hopf factorization requires only a matrix inversion.

2) The expression for the "downwards phase-generator" Q has an obvious probabilistic interpretation [5].

In the case of exponential ultimate ruin probabilities, the previous result may be further simplified:

Corollary 2 *Suppose S is a compound Poisson process with exponential jumps, or spectrally positive.*

Then it holds that

$$\begin{aligned}\psi_{sim}(x_1,x_2) &= \psi_2(x_2,T)+\psi_1(x_1)\overline{\psi}_2^{(-\gamma_1)}(x_2,T)\\ \psi_{or}(x_1,x_2) &= \psi_1(x_1,T)+\psi_2(x_2)\overline{\psi}_1^{(-\gamma_2)}(x_1,T)\\ \psi_{and}(x_1,x_2) &= w_1(x_1,T)+\psi_2(x_2)\psi_1^{(-\gamma_2)}(x_1,T)\end{aligned}$$

Below, we will obtain a "three terms asymptotic expansion theorem", corresponding roughly to:

$$\psi(x_1,x_2)\approx P[\tau_1<\tau_2]+P[\tau_2<\tau_1]+P[\tau_1\approx\tau_2],$$

which will follow from the fact that the exact results established in the exponential case continue to be true for the Cramer case, asymptotically!

Proposition 1 *Fix $v\in(0,\infty)$, and let $x_1,x_2\to\infty$ such that $x_2/T(x_1,x_2)=v$, $v\neq-\kappa_2'(-\gamma_i)$ $i=1,2$. If there exist exist $\theta_v,\theta_v^{(1)},\theta_v^{(2)}\in\Theta^o$ such that $\theta_v<\theta_v^{(1)}<\theta_v^{(2)}$ and $\kappa_2'(\theta_v)=-v$, $\kappa_2(\theta_v)=\kappa_2(\theta_v^{(2)})$, $\kappa_1(\theta_v)=\kappa_1(\theta_v^{(1)})$, it holds that*

$$\psi_{\rm or}(x_1,x_2)\approx\psi_1(x_1,T)+\widetilde{C}_2(v)\mathrm{e}^{-\gamma_2x_2}\overline{\psi}_1^{(-\gamma_2)}(x_1,T) \tag{17}$$

$$\psi_{\rm sim}(x_1,x_2)\approx\psi_2(x_2,T)+\widetilde{C}_1(v)\mathrm{e}^{-\gamma_1x_1}\overline{\psi}_2^{(-\gamma_1)}(x_2,T) \tag{18}$$

where, for $i=1,2$,

$$\widetilde{C}_i(v)=\begin{cases}C_i & \text{if } -\kappa_2'(-\gamma_i)<v\\ c_{3-i}(v,\gamma_i)^{-1}[\psi_i^*(\theta_v)-\psi_i^*(\theta_v^{(3-i)})] & \text{if } 0<v<-\kappa_2'(-\gamma_i),\end{cases}$$

where

$$c_1(v,c) = \frac{\theta_v^{(1)} - \theta_v}{(\theta_v^{(1)} + c)(\theta_v + c)}, \quad c_2(v,c) = \frac{\theta_v^{(2)} - \theta_v}{(\theta_v^{(2)} + c)(\theta_v + c)}$$

and ψ_i^* is the Laplace transform of $\psi_i(x)$.

9 General two dimensional Cramer type asymptotics.

We investigate the asymptotics in the case that the initial reserves tend to infinity along a ray $x_1/x_2 = a$ with $a < 1$. While this will be achieved here by classical **one-dimensional** results (Arfwedson-Hoglund), our example is also an interesting illustration of **first-passage two-dimensional large deviations theory**.

We assume throughout that the Cramér assumption hold true for X_1 and X_2, that is, there exist $\gamma_1, \gamma_2 > 0$ with $(-\gamma_1, 0)$ and $(0, -\gamma_2) \in \partial\mathcal{C}$ i.e.

$$\boxed{\kappa(-\gamma_1, 0) = \kappa(0, -\gamma_2) = 0}. \tag{19}$$

In order to guarantee that the asymptotic constants are positive we require that the following inequality is satisfied (coordinatewise)

$$\nabla\kappa(\theta) > -\infty \qquad \text{for } \theta = (-\gamma_1, 0), (0, -\gamma_2) \tag{20}$$

where $\nabla\kappa = (\partial_1\kappa, \partial_2\kappa)$ denotes the gradient of κ. In view of the strict convexity of κ, it follows that the set $\mathcal{C}$ is strictly convex, which by the supporting hyperplane theorem implies that, for fixed $\theta' \in \mathcal{C}$, it holds that

$$[\theta - \theta'] \cdot \nabla\kappa(\theta') \leq 0 \quad \text{for all } \theta \in \mathcal{C}, \tag{21}$$

where $\cdot$ denotes the inner-product.

Example 3 If $S(t)$ is a Lévy process and $X_i(t) = p_i t - S(t)$ (with the cumulant generating functions κ_i), then the joint cumulant generating function κ of (X_1, X_2) is related to κ_1 and κ_2 by

$$\kappa(\theta_1, \theta_2) = \kappa_1(\theta_1 + \theta_2) - \theta_2(p_1 - p_2) = \kappa_2(\theta_1 + \theta_2) + \theta_1(p_1 - p_2).$$

The two-dimensional degenerate Lévy process $X = (X_1, X_2)$ satisfies then the Cramér-conditions if there exist constants $\gamma_i > 0$ $(i = 1, 2)$ such that

$$\kappa_i(-\gamma_i) = 0. \tag{22}$$

Proposition 2 *Suppose that the Cramér assumptions hold, that is, there exist $\gamma_1, \gamma_2 > 0$ such that*

$$0 = \kappa(-\gamma_1, 0) = \kappa(0, -\gamma_2),$$
$$\frac{\partial \kappa}{\partial u}(u, v)\bigg|_{(u,v)=(-\gamma_1,0)} + \frac{\partial \kappa}{\partial v}(u, v)\bigg|_{(u,v)=(0,-\gamma_2)} > -\infty$$

and let $a > 0$. Then, as $t \to \infty$,

$$\psi_{\text{and}}(x_1, x_2) = \mathrm{o}(C_2 \mathrm{e}^{-\gamma_2 x_2} + C_1 \mathrm{e}^{-\gamma_1 x_1}) \tag{23}$$
$$\psi_{\text{or}}(x_1, x_2) \sim C_2 \mathrm{e}^{-\gamma_2 x_2} + C_1 \mathrm{e}^{-\gamma_1 x_1}. \tag{24}$$

9.1 Large deviations heuristic.

Heuristically, large deviations for a process X uses two principles:

a) viewed from far away, the paths along which rare events are realized concentrate typically along a finite number of **locally dominating paths**, which are the solutions of a deterministic variational problem.

b) the velocities along the "escape paths" may be viewed as expected drifts of measures belonging to the exponential family of the process.

Informally, we say that rare events happen due to a change of measure to one of the (most likely) measures in the exponential family; these are computed using the Cramer set, and may be useful also for deriving asymptotic correction terms: **precise large deviations**.

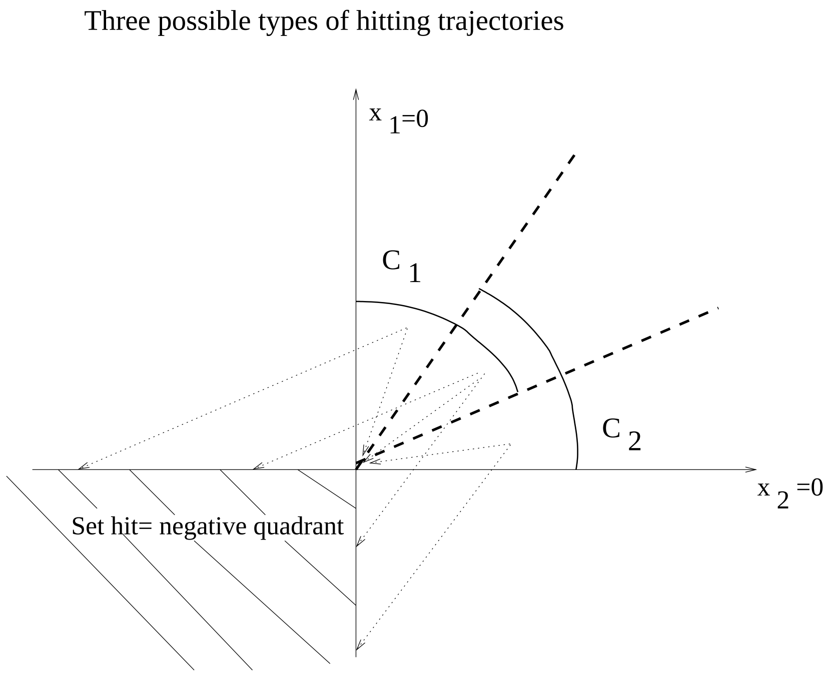

Figure 5: Typical paths hitting the negative quadrant.

In the intersection cone $\mathcal{C}_1 \cap \mathcal{C}_2$, three types of ruin are possible, suggesting maybe:

$$\psi_{or}(x_1, x_2) \approx \psi_1(x_1) + \psi_2(x_2) + \psi_0(x_1, x_2)$$

Definition 1 The Cramer set is defined as the set of all $\boldsymbol{\theta}$ satisfying

$$\kappa(\boldsymbol{\theta}) \leq 0$$

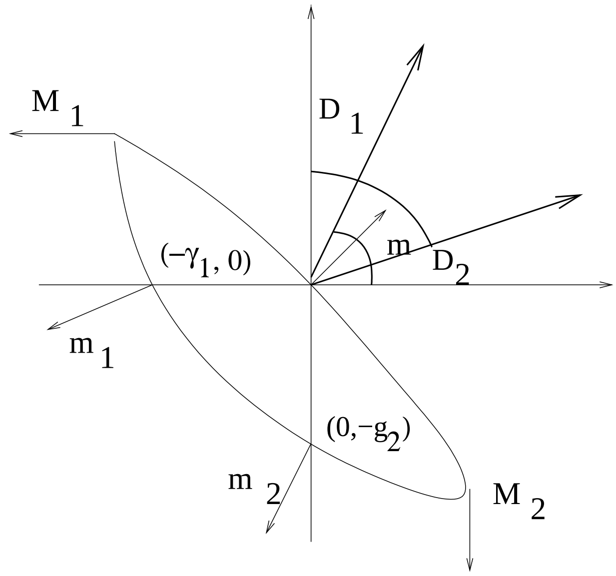

Figure 6: The Cramer set, the direct twists and the dominant points for hitting the negative quadrant

For the quadrant hitting problem, the most important shifts are the intersections different from the origin of the Cramer set with the axes, $\boldsymbol{\gamma}^{(2)} = (0, -\gamma_2)$ and $\boldsymbol{\gamma}^{(1)} = (-\gamma_1, 0)$.

Definition 2 a) The "boundary influence" cones for "or" ruin are the subsets $\mathcal{C}_i$ of the nonnegative quadrant generated by the axis $x_i = 0$ and by the negatives of the "dominating directions" $\kappa'(\gamma^{(i)})$, providing these intersect the quadrant. Thus,

$$\mathcal{C}_1 = \{x : x_1/x_2 \le s_1\} \qquad \mathcal{C}_2 = \{x : x_1/x_2 \ge s_2\}$$

where

$$s_1 := \frac{\kappa_1'(-\gamma_1)}{\kappa_2'(-\gamma_1)} \qquad s_2 := \frac{\kappa_1'(-\gamma_2)_+}{\kappa_2'(-\gamma_2)} \tag{25}$$

b) The "boundary influence" cones for "sim" ruin are the subsets

$$\mathcal{D}_1 = \mathcal{C}_1^c, \mathcal{D}_2 = \mathcal{C}_2^c.$$

10 Cramer type two terms large deviations results.

One key point of the degenerate setup is an equivalent description of the cones partition $\mathcal{D}_1 \cup \mathcal{D}_0 \cup \mathcal{D}_2$ in terms of comparisons with the time T.

Lemma 1 *The partition $\mathcal{C}^c = \mathcal{D}_1 \cap \mathcal{D}_0 \cap \mathcal{D}_2$ is equivalent to the partition:*

$$\mathcal{C}^c = \begin{cases} \{T_1 < T_2 < T\} = & \{0 < a(x) < s_2\} = \mathcal{D}_2 \\ \{T_1 < T < T_2\} = & \{s_2 < a(x) < s_1\} = \mathcal{D}_0 \\ \{T < T_2 < T_1\} = & \{s_1 < a(x) < 1\} = \mathcal{D}_1 \end{cases}$$

Note: By this result precise asymptotics will be obtained directly from Arfwedson's one-dimensional result, in the case of exponential ultimate ruin probabilities.

The leading term asymptotics of the two-dimensional ruin probabilities will be expressed in terms of the usual "one dimensional large deviations cast": the adjustment coefficients γ_i of X_i, and $\gamma(a)$ given for $0 < a < 1$ by

$$\gamma(a) = \kappa_2^*(-v_a)/v_a \quad \text{where} \quad v_a := (p_1 - p_2)/(1 - a).$$

We write $f \approx g+h$ if $\lim_{x\to\infty}(f-g)/h(x) = \lim_{x\to\infty}(f-h)/g(x) = 1$.

Theorem 3 *Let S be a compound Poisson process. Assume that (22) holds with $\kappa_1'(-\gamma_1) > -\infty$ and that there exist $\theta_v, \theta_v' \in \Theta_2^o$ with $\theta_v < \theta_v'$, $\kappa_2'(\theta_v) = -v = -v_a$ and $\kappa_2(\theta_v) = \kappa_2(\theta_v')$. Then it holds as $K \to \infty$,*

$$\psi_{\text{sim}}(aK, K) \sim \begin{cases} C_1 e^{-\gamma_1 aK} & \text{if } (aK, K) \in \mathcal{D} \\ (D^{\#}(v_a) + D'(v_a))K^{-1/2} e^{-\gamma(a)K} & \text{if } (aK, K) \in \mathcal{D} \\ C_2 e^{-\gamma_2 K} & \text{if } (aK, K) \in \mathcal{D} \end{cases}$$

where C_1 and C_2 are given the Cramer-Lundberg con-

stants (5) and

$$D'(w) = \frac{\theta'_w - \theta_w}{|\theta_w \theta'_w|} \frac{\sqrt{w}}{\sqrt{2\pi\kappa''_1(\theta_w)}}, \tag{26}$$

$$D^{\#}(w) = \left[\frac{1}{\theta_w} - \frac{1}{\theta'_w} + \frac{\kappa'_1(0)}{\kappa_1(\theta'_w)} - \frac{\kappa'_1(0)}{\kappa_1(\theta_w)}\right] \frac{\sqrt{w}}{\sqrt{2\pi\kappa''_1(\theta_w)}} \tag{27}$$

Next we turn to the ruin probability ψ_{and}. The asymptotics for ψ_{and} will be formulated in terms of the cone

$$\widehat{\mathcal{D}}_2 = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 < x_2 \widehat{s}_2\} \quad \widehat{\mathcal{D}}_0 = \mathbb{R}^2_+ \backslash [\overline{\mathcal{D}}_1 \cup \overline{\widehat{\mathcal{D}}}_2],$$

where $\widehat{s}_2 = \kappa'_1(-\widehat{\gamma}_2)/\kappa'_2(-\widehat{\gamma}_2)$ with $\widehat{\gamma}_2$ the largest root of $\kappa_1(-s) = \kappa_1(-\gamma_2)$. In view of the definition of $\widehat{s}_2$ it follows that if $\kappa'_1(-\gamma_2) > 0$ then $\widehat{\mathcal{D}}_2 \neq \emptyset = \mathcal{D}_2$ and otherwise the cones $\mathcal{D}_2$ and $\widehat{\mathcal{D}}_2$ coincide. In the next result it will be shown that if (aK, K) is contained in either $\mathcal{D}_1$ or $\mathcal{D}_2$, $\psi_{\text{sim}}(aK, K)$ and $\psi_{\text{and}}(aK, K)$ are of the same order.

Proposition 3 *If $x_1, x_2 \to \infty$ such that $x_2/T(x_1, x_2) = v$, $v \neq -\kappa'_2(-\widehat{\gamma}_2)$, it holds that*

$$\psi_{\text{and}}(x_1, x_2) \approx \psi_1(x_1) - \psi_1(x_1, T) + \{\overline{C}_1(v) e^{-\gamma_2 x_2} \psi_2^{(-\gamma_2)}(x_2, T) + \overline{C}_2(v) e^{-\gamma_2 x_2 - \widetilde{\gamma} x_1} \psi_1^{(-\widehat{\gamma}_2)}(x_1, T)\}, \tag{28}$$

where $\widetilde{\gamma} = \widehat{\gamma}_2 - \gamma_2$ *and*

$$\overline{C}_1(v) \;=\; \begin{cases} 0 & \text{if } 0 < v < -\kappa_2'(-\widehat{\gamma}_2) \\ |c_1(v,\gamma_2)|^{-1} \;\cdot\; \overline{\psi}_2^*(\theta_v') & \text{if } v > -\kappa_2'(-\widehat{\gamma}_2), \end{cases}$$

$$\overline{C}_2(v) \;=\; \begin{cases} C_2\frac{\kappa_1'(-\gamma_2)}{\kappa_1'(-\widehat{\gamma}_2)} & \text{if } 0 < v < -\kappa_2'(-\widehat{\gamma}_2) \\ |c_2(v,\widehat{\gamma}_2)|^{-1} \;\cdot\; [\psi_2^*(\theta_v^\star) - \theta_v^{-1}] & \text{if } v > -\kappa_2'(-\widehat{\gamma}_2). \end{cases}$$

Theorem 4 *Suppose that the assumptions of Theorem 3 hold true and that there exists a* $\theta_v^\star > \theta_v$ *such that* $\kappa_1(\theta_v) = \kappa_1(\theta_v^\star)$. *Then it holds that, as* $K \to \infty$,

$$\psi_{\text{and}}(aK,K) \;\sim\; \begin{cases} C_1 \mathrm{e}^{-\gamma_1 aK} & \text{if } (aK,K) \in \mathcal{D} \\ (D^\star(v_a) - D^\dagger(v_a))K^{-1/2}\mathrm{e}^{-\gamma(a)K} & \text{if } (aK,K) \in \widehat{\mathcal{D}} \\ \widehat{C}_2 \mathrm{e}^{-(a\widehat{\gamma}_2 + (1-a)\gamma_2)K} & \text{if } (aK,K) \in \widehat{\mathcal{D}} \end{cases}$$

where

$$\widehat{C}_2 = C_2\kappa_1'(-\gamma_2)/\kappa_1'(-\widehat{\gamma}_2),$$

and $D^\star(w)$ *and* $D^\dagger(w)$ *are respectively given by (26) and (27) with* θ_w' *replaced by* $\theta_w^\star$ *and* κ_1 *replaced by* κ_2.

11 Examples

We now develop two explicit examples that illustrate the results shown in the previous sections.

11.1 Cramér-Lundberg model with exponential jumps

Let X be a drift p minus a compound Poisson process with rate λ and exponential jump sizes with mean μ starting at x. Then, the characteristic function of X reads as $\kappa(\theta) = p\theta - \lambda\theta/(\mu+\theta)$ and, if $p > \frac{\lambda}{\mu}$, the ultimate ruin probability is equal to $\psi(x) = C\mathrm{e}^{-\gamma x}$, where the adjustment coefficient is $\gamma = \mu - \lambda/p$ and $C = \lambda/(\mu p)$. More generally, it was shown by Asmussen (1984), Knessl and Peters (1994) (with $p = 1$) and Pervozvansky (1998) that the finite time ruin probability $\psi(x,t)$ is given by

$$\overline{\psi}(x,t) = 1-\psi(x,t) = [1-C\mathrm{e}^{-\gamma x}]I_{(\gamma>0)}+w(x,t), \tag{29}$$

where

$$w(x,t) = \frac{1}{\pi}\sqrt{\frac{\lambda}{\mu p}}\int_{s_-}^{s_+} \mathrm{e}^{a(q)x-qt}\sin(b(q)x-\phi(q))\frac{\mathrm{d}q}{q} \tag{30}$$

with $s_\pm = (\sqrt{\lambda}\pm\sqrt{\mu p})^2$, $\phi(q) = \arccos\left(\frac{p\mu+\lambda-q}{2\sqrt{\lambda\mu p}}\right)$ and

$$a(q) = \frac{\lambda-\mu p-q}{2p}, \quad b(q) = \frac{\sqrt{4pq\mu-(\lambda-\mu p-q)^2}}{2p} \tag{31}$$

Further, we note that, under $P^{(c)}$, X is still a drift p minus a compound Poisson process with exponential jumps with the changed rates $\lambda_c = \lambda\frac{\mu}{\mu+c}$ and $\mu_c = \mu + c$. In particular, $\lambda_{-\gamma} = \mu\, p$ and $\mu_{-\gamma} = \lambda/p$ are the parameters under $P^{(-\gamma)}$.

In view of the previous paragraph, we see that, under $P^{(-\gamma_1)}$, the drift of X_2 is always negative, $\kappa_2^{(-\gamma_1)\prime}(0) = \kappa_2'(-\gamma_1) < 0$. Also, under $P^{(-\gamma_2)}$, the adjustment parameter of X_1 is positive if and only if $\rho > \rho^* := p_2^2/p_1$ and is then equal to

$$\widetilde{\gamma} = \widehat{\gamma}_2 - \gamma_2 = \frac{\mu}{p_2}\left(\rho - \frac{p_2^2}{p_1}\right), \tag{32}$$

and the asymptotic constant $\widetilde{C}$ satisfies $\widehat{C}_2 = \widetilde{C}C_2 = \frac{p_2}{p_1}$. Inserting the expressions (29) – (31) (with the proper choices of parameters) into Corollary **??** leads then to explicit expressions for $\psi_{\text{and}}, \psi_{\text{sim}}$ and ψ_{or}.

It is a matter of calculus to verify that

$$s_1 = \frac{\frac{p_1^2}{\rho} - p_1}{\frac{p_1^2}{\rho} - p_2}, \quad s_2 = \frac{(\frac{p_2^2}{\rho} - p_1)_+}{\frac{p_2^2}{\rho} - p_2}, \text{ and, if } \rho > \rho^*,\ \widehat{s}_2 = \frac{\rho\frac{p_1^2}{p_2^2} - p_1}{\rho\frac{p_1^2}{p_2^2} - p_2}.$$

Also, by invoking Corollary **??** or by a direct calculation we see that

$$C_i(v) \equiv \frac{\lambda}{\mu p_i} = C_i, \quad i = 1, 2.$$

Inserting these quantities into Propositions 1 and 3 yields explicit asymptotics expansions for $\psi_{\text{and}}, \psi_{\text{sim}}$ and ψ_{or}.

11.2 Brownian motion with drift

If $X(t) = mt + B(t)$ where $B(t)$ is standard Brownian motion, then its characteristic exponent reads as $\kappa(\theta) = \frac{1}{2}\theta^2 + m\theta$. If $m > 0$, $\psi(x) = \mathrm{e}^{-\gamma x}$, where $\gamma = 2m$ is the adjustment coefficient. Further, under $P^{(c)}$, X is still Brownian motion, but the drift changes to $m + c$. The drift of the measure associated to $c = -\gamma$ is $-m$, i.e. the Brownian motion switches its drift. In view of the Corollary **??** and the well known first-passage distribution of Brownian motion with drift,

$$\overline{\psi}(x,t) = \Phi\left(\frac{x+mt}{\sqrt{t}}\right) - \mathrm{e}^{-2mx}\Phi\left(\frac{-x+mt}{\sqrt{t}}\right) \quad (33)$$

we find that if $x_2 > x_1$, then:

$$\begin{aligned}
\psi_{\mathrm{or}}(x_1,x_2) &= P(\tau_1(x_1) \le T) + \mathrm{e}^{-2p_2x_2}\ P^{(-2p_2)}(\tau_1(x_1) > T) \\
&= 1 - \Phi\left(a(x_1,p_1)\right) + \mathrm{e}^{-2p_1x_1}\Phi\left(a(-x_1,p_1)\right) \\
+\ \mathrm{e}^{-2p_2x_2} &\times \Big[\Phi\left(a(x_1,p_1-2p_2)\right) - \mathrm{e}^{-2x_1(p_1-2p_2)}\Phi\left(a(-x_1,p_1-2p_2 \right. \\
\psi_{\mathrm{sim}}(x_1,x_2) &= P(\tau_2(x_2) \le T) + \mathrm{e}^{-2p_1x_1}\ P^{(-2p_1)}(\tau_2(x_1) > T) \\
&= 1 - \Phi\left(a(x_2,p_2)\right) + \mathrm{e}^{-2p_2x_2}\Phi\left(a(-x_2,p_2)\right) \\
+\ \mathrm{e}^{-2p_1x_1} &\times \Big[\Phi\left(a(x_2,p_2-2p_1)\right) - \mathrm{e}^{-2x_2(p_2-2p_1)}\Phi\left(a(-x_2,p_2-2p_1 \right. \\
\psi_{\mathrm{and}}(x_1,x_2) &= \Phi\left(a(x_1,p_1)\right) - 1 + \mathrm{e}^{-2p_1x_1}\left(1 - \Phi\left(a(-x_1,p_1)\right)\right) \\
+\ \mathrm{e}^{-2p_2x_2} &\times \Big[\mathrm{e}^{-2x_1(p_1-2p_2)}\Phi\left(a(-x_1,p_1-2p_2)\right) + 1 - \Phi\left(a(x_1,p_1 - \right.
\end{aligned}$$

where $a(x,p) = [x+pT]/\sqrt{T}$ and Φ denotes the cumulative standard normal distribution function. In view of the facts that $\Phi(-x) = 1-\Phi(x)$ and $1-\Phi(x) \sim (2\pi)^{-1/2}x^{-1}\exp(-x^2/2)$ as $x\to\infty$, it follows from (35) and (36) that if x_1, x_2 tend to infinity with $x_1/x_2 = a$ then

$$\psi_{\text{and}}(x_1,x_2) \approx \begin{cases} \mathrm{e}^{-2p_2x_2-2(p_1-2p_2)^+x_1} + o(av_a) & \text{if } 0<a<\widehat{s}_2 \\ o(av_a) & \text{if } \widehat{s}_2 < a < s_1 \\ \mathrm{e}^{-2p_1x_1} + o(av_a) & \text{if } s_1<a<1, \end{cases} \tag{37}$$

$$\psi_{\text{sim}}(x_1,x_2) \approx \begin{cases} \mathrm{e}^{-2p_2x_2} + \delta(v_a) & \text{if } 0<a<s_2 \\ \delta(v_a) & \text{if } s_2 < a < s_1 \\ \mathrm{e}^{-2p_1x_1} + \delta(v_a) & \text{if } s_1<a<1, \end{cases} \tag{38}$$

where

$$o(v) = \left[\frac{2v}{p_1^2-v^2} + \frac{2v}{v^2-(p_1-2p_2)^2}\right]\frac{\sqrt{v}}{\sqrt{2\pi x_1}}\mathrm{e}^{-x_1(v+p_1)^2/[2v]},$$

$$\delta(v) = \left[\frac{2v}{v^2-p_2^2} + \frac{2v}{(p_2-2p_1)^2-v^2}\right]\frac{\sqrt{v}}{\sqrt{2\pi x_2}}\mathrm{e}^{-x_2(v+p_2)^2/[2v]}$$

and

$$s_1 = \frac{p_1}{2p_1-p_2}, \qquad s_2 = \frac{(2p_2-p_1)_+}{p_2}, \text{ and, if } p_1>2p_2, \widehat{s}_2 = \frac{p_1-2p_2}{2p_1-3p_2}$$

The asymptotics of ψ_{sim} agree with the asymptotics of the steady state distribution of a tandem queue calculated in Lieshout and Mandjes [**?**].

By straightforward calculations it can be verified that, if S is a Brownian motion, then $C_i = 1$ and $\theta_v = -v - p_2$, $\theta'_v = v - p_2$, $\theta^\star_v = v + p_2 - 2p_1$ and

$$\widetilde{C}_i(v) = 1, \quad \kappa_i^*(-v) = \frac{(v+p_i)^2}{2}, \quad \kappa_1^*(-av_a) = \kappa_2^*(-v_a)$$

for $i = 1, 2$. Inserting these quantities in (18) — (28) and comparing with (37) and (38) it follows that Propositions 1 and 3 and Theorems 3 and 4 remain valid if S is a Brownian motion.

A Proof of Proposition 2

The proof of Proposition 2 is based on the following estimates:

Lemma 2 *The following hold true:*

(i) $\max\{\psi_1(x_1), \psi_2(x_2)\} \le \psi_{\mathrm{or}}(x_1, x_2) \le \psi_1(x_1) + \psi_2(x_2)$

(ii) $\psi_{\mathrm{or}}(x_1, x_2) = \psi_{1\le 2}(x_1, x_2) + \psi_{2\le 1}(x_1, x_2) - \psi_{1=2}(x_1, x_2)$ *where*

$$\psi_{i\le j}(x_1, x_2) := P(\tau_i(x_i) \le \tau_j(x_j), \tau_i(x_i) < \infty)$$

and

$$\psi_{1=2}(x_1, x_2) := P(\tau_1(x_1) = \tau_2(x_2) < \infty).$$

Lemma 3 *Suppose that (23) holds and write* $\gamma_{a,\beta} := \beta a\gamma_1 + (1-\beta)\gamma_2$.

(i) If $a\gamma_1 = \gamma_2$ *it holds that, as* $K \to \infty$,

$$\psi_{1<2}(aK, K) \sim C_1 \mathrm{e}^{-\gamma_1 aK}, \qquad \psi_{2<1}(aK, K) \sim C_2 \mathrm{e}^{-\gamma_2 K} \tag{39}$$

(ii) For $a > 0$ *and* $\beta \in (0,1)$, $\psi_{\mathrm{sim}}(aK, K) = \mathrm{o}(\mathrm{e}^{-\gamma_{a,\beta}K})$ $(K \to \infty)$.

We have made all preparations to complete the proof of Proposition 2:

Proof of Proposition 2: First note that, in view of the Cramér-Lundberg asymptotics (5) and equation (12) the asymptotics in (24) imply those in (23). The rest of the proof is therefore devoted to establishing (24).

In view of (5) and Lemma 2(i), it follows that, if $\gamma_1 a > \gamma_2$ [resp. $\gamma_1 a < \gamma_2$], the lower bound and upper bound in Lemma 2(i) are of the same order of magnitude, $C_2\mathrm{e}^{-\gamma_2 K}$ [resp. $C_1\mathrm{e}^{-\gamma_1 aK}$], as $K \to \infty$. Thus (24) is valid if $\gamma_1 a \neq \gamma_2$.

Next we turn to the case $\gamma_1 a = \gamma_2$. Since $\psi_{1=2}$ is dominated by ψ_{sim} and $\gamma_{a,\beta} = \gamma_2 = a\gamma_1$ if $a\gamma_1 = \gamma_2$, it follows, by appealing to Lemma 3(ii), that $\psi_{1=2}(aK, K) = \mathrm{o}(\mathrm{e}^{-\gamma_2 K}) = \mathrm{o}(\mathrm{e}^{-\gamma_1 aK})$ as $K \to \infty$. In view of Lemma 2(ii) and Lemma 3(i) it therefore follows that (24) is also valid if $a\gamma_1 = \gamma_2$. □

Proof of Lemma 2: The estimates follow in view of the observations that,

$$\{\tau_i(x_i) < \infty\} \subset \{\tau_{\rm or}(x_1, x_2) < \infty\} \subset \cup_{i=1}^2 \{\tau_i(x_i) < \infty\} \text{ for } i = 1, 2,$$

$$\{\tau_{\rm or}(x_1, x_2) < \infty\} = A_1 \cup A_2 \backslash [A_1 \cap A_2]$$

where $A_i = \{\tau_i(x_i) < \infty, \tau_i(x_i) \leq \tau_{3-i}(x_{3-i})\}$. □

Proof of Lemma 3: (i) The asymptotics of $\psi_{1<2}$ follow once we have shown that as $K \to \infty$ it holds that

$$e^{\gamma_1 aK} \psi_{1<2}(aK, K) = E^{(-\gamma_1,0)}(e^{-\gamma_1 X_1(\tau_1)} \mathbf{1}_{\{\tau_1 \leq \tau_2, \tau_1 < \infty\}}) \to C_1, \tag{40}$$

where $\tau_1 = \tau_1(aK)$ and $\tau_2 = \tau_2(K)$. To prove this claim we compare the asymptotic behaviour of τ_1 and τ_2 as $K \to \infty$, adapting the argument developed in Glasserman and Wang (1997) (Prop. 2) for random walk. If $E^{(-\gamma_1,0)}[X_2(1)] > 0$, then $P^{(-\gamma_1,0)}(\tau_2 = \infty) \to 1$ as $K \to \infty$ and, appealing to (5), the claim (40) follows. If $E^{(-\gamma_1,0)}[X_2(1)] \leq 0$, it follows in view of Lemma 4(a) that as $K \to \infty$

$$\tau_1(aK)/\tau_2(K) \to a\partial_2\kappa(-\gamma_1, 0)/\partial_1\kappa(-\gamma_1, 0) \;\; P^{(-\gamma_1,0)}\text{-a.s.} \tag{41}$$

Applying (21) with $\theta = (0, -\gamma_2)$ and $\theta' = (-\gamma_1, 0)$ we see that the right-hand side of (41) is bounded above by $a\gamma_1/\gamma_2$, which is equal to one if $\gamma_2 = a\gamma_1$. Therefore $\tau_2(K)$ dominates $\tau_1(aK)$ for all K large enough and

(40) follows as a consequence of the Cramér-Lundberg asymptotics (5). The asymptotics of $\psi_{1>2}$ can be treated similarly.

(ii) Choose $\beta \in (0,1)$ and write $\gamma^\beta = \beta(\gamma_1, 0) + (1-\beta)(0,\gamma_2)$. By strict convexity of $\mathcal{C}$ there exists a $-\gamma^* = -(\gamma_1^*, \gamma_2^*) \in \mathcal{C}^o$ such that $\gamma_i^* > \gamma_i^\beta$, $(i = 1,2)$. By changing the measure, we see that $\psi_{\text{sim}}(aK, K)$ is equal to

$$\mathrm{e}^{-(\gamma_1^* a + \gamma_2^*)K} E^{(-\gamma^*)}[\mathrm{e}^{\gamma_2^* X_2(\tau_{\text{sim}}) + \gamma_1^* X_1(\tau_{\text{sim}}) + \kappa(-\gamma_1^*, -\gamma_2^*)\tau_{\text{sim}}} \mathbf{1}_{\{\tau_{\text{sim}} < \infty\}}],$$

where $\tau_{\text{sim}} = \tau_{\text{sim}}(aK, K)$. Since $X_i(\tau_{\text{sim}}) < 0$ and $\kappa(-\gamma_1^*, -\gamma_2^*) \leq 0$, this expectation is bounded above by 1, and, as $a\gamma_1^* + \gamma_2^* > \gamma_{a,\beta}$, it thus follows that $\psi_{\text{sim}}(aK, K) = \mathrm{o}(\mathrm{e}^{-(\gamma_1^\beta a + \gamma_2^\beta)K}) = \mathrm{o}(\mathrm{e}^{-\gamma_{a,\beta} K})$ as $K \to \infty$. □

B Proof of Theorem 3

We include first for reference a one dimensional result concerning the behaviour of the time of ruin for large initial reserves:

Lemma 4 *Suppose that $E[|X(1)|] < \infty$ and $E[X(1)] \leq 0$. Then, as $x \to \infty$,*

(a) $\tau(x)/x \to -E[X(1)]^{-1}$ *P-a.s.* *and* *(b)* $E[\tau(x)]/x \to -E[X(1)$

Proof of Lemma 4:

(b) If $E[X(1)] = 0$, then the Lévy process X oscillates and the identity follows since then $E[\tau(x)] = +\infty$ for every x (see e.g. Bertoin [14, Ch. VI, Prop. 17(iii)]). Suppose now that $-\infty < E[X(1)] < 0$ (so that X drifts to $-\infty$) and first exclude that case that X is a compound Poisson process. Denoting by $L^{-1}(t) = \inf\{u \geq 0 : L(u) > t\}$ the inverse of the local time L of X and $T(x) = \inf\{t \geq 0 : H(t) > x\}$ the first passage time of the ladder height process $H(t) = X(L^{-1}(t))$ of X it is easily verified that $\tau(x) = L^{-1}(T(x))$. The pair (L^{-1}, H) forms a two-dimensional Lévy process and we denote its bivariate Laplace exponent by $\widehat{\kappa}$. The Laplace transform of $E[\tau(x)]$ can then be expressed as follows:

$$\int_0^\infty \mathrm{e}^{-\lambda x} E[\tau(x)]\mathrm{d}x = \frac{\partial_1 \widehat{\kappa}(0, 0^+)}{\lambda \widehat{\kappa}(0, \lambda)},$$

where ∂_i means a partial derivative with respect to ith variable (see e.g. Bertoin [14, Ch. VI, Prop. 17]). As $\widehat{\kappa}(0, 0) = 0$ and $\widehat{\kappa}(0, \cdot)$ is right-differentiable in zero, it follows in view of a Tauberian theorem that

$$E[\tau(x)] \sim x\, \partial_1 \widehat{\kappa}(0, 0^+)/\partial_2 \widehat{\kappa}(0, 0^+) \quad \text{as } x \to \infty.$$

The strong law of large numbers implies that the product $H(t)/L^{-1}(t) = [X(L^{-1}(t))/t] \times [t/L^{-1}(t)]$ converges to

$$E[X(1)] = E[X(L^{-1}(1))]E[L^{-1}(1)]^{-1} \tag{42}$$

(the corresponding identity for random walks is known as the famous *Wald identity*). Since $\partial_1\widehat{\kappa}(0^+,0) = E[L^{-1}(1)]$ and $\partial_2\widehat{\kappa}(0,0^+) = E[X(L^{-1}(1))]$, the claim follows. The case of a compound Poisson process follows by adding a small drift.

(a) The strong law of large numbers implies that, P-a.s.,

$$\tau(x)/x = L^{-1}(T(x))/T(x)\cdot T(x)/x \to E[L^{-1}(1)]/E[H(1)] = E[X(1)]$$

as $x \to \infty$, where we used the Wald-identity (42). $\square$

We will show now some properties of the cones $\mathcal{D}_0, \mathcal{D}_1$ or $\mathcal{D}_2$.

Lemma 5 *The following hold true:*

(i) The cones $\mathcal{D}_i, i = 0,1,2$ are disjoint and $\mathcal{D}_0, \mathcal{D}_1 \neq \emptyset$.

(ii) $\mathcal{D}_2 \neq \emptyset$ iff $\kappa_1'(-\gamma_2) < 0$ and $\widehat{\mathcal{D}}_2 \neq \emptyset$ iff $\kappa_1'(-\gamma_2) \neq 0$.

(iii) $\mathcal{D}_1 \subset \mathcal{U} := \{(x_1,x_2) \in \mathbb{R}_+^2 : x_2\gamma_2 < x_1\gamma_1\}$ and $\mathcal{D}_2 \subset \mathbb{R}_+^2 \backslash \mathcal{U}$, where $\mathbb{R}_+^2 = (0,\infty)^2$.

Proof Writing

$$\frac{\kappa_1'(s)}{\kappa_2'(s)} = \frac{\kappa_2'(s) + p_1 - p_2}{\kappa_2'(s)} = 1 + \frac{p_1 - p_2}{\kappa_2'(s)},$$

it follows that $s_1 < 1$, since $\kappa_2'(-\gamma_1) < 0$, and that $s_2 < s_1$, since $\gamma_1 > \gamma_2$ and, by the strict convexity of κ_2, κ_2 is strictly increasing on its domain. Next, in view of the definitions of s_2 and $\widehat{s}_2$, it follows that $\widehat{s}_2 = 0$ [resp. $s_2 = 0$] iff $\kappa_1'(-g_2) = 0$ [resp. $\kappa_1'(-\gamma_2) \leq 0$]. Subsequently, we note that on the ray $x_1/x_2 = \gamma_2/\gamma_1$ it holds that

$$\frac{x_2}{T(x_1,x_2)} = \frac{p_1 - p_2}{1 - \gamma_2/\gamma_1} = \frac{\kappa_2(-\gamma_1) - \kappa_1(-\gamma_1)}{\gamma_1 - \gamma_2} = \frac{\kappa_2(-\gamma_1) - \kappa_2(-\gamma_2)}{\gamma_1 - \gamma_2}.$$

The strict convexity of κ_2 thus implies that along the ray $x_1/x_2 = \gamma_2/\gamma_1$ it holds that $-\kappa_2'(-\gamma_2) < x_2/T(x_1,x_2) < -\kappa_2'(-\gamma_1)$. It is a matter of algebra to verify that these inequalities are equivalent to $s_2 < \gamma_2/\gamma_1 < s_1$ (see also Lemma 6 below). The assertions (i), (ii) and (iii) ollow then in view of the definitions of $\mathcal{D}_i, i = 0,1,2$ and $\widehat{\mathcal{D}}_2$. □

A key point in the analysis of the asymptotics of the degenerate risk processes is an equivalent description of the cones $\mathcal{D}_i$ in terms of comparisons with the time $T = T(x_1,x_2)$ defined in (13), which will enable us to translate the asymptotics in two-dimensional space into the 'space-time'-asymptotics of Arfwedson (1955) and Höglund (1990).

Lemma 6 *Writing* $T_i = x_i/[-\kappa_i'(-\gamma_i)]$ *and* $\widetilde{T}_i = x_i/[-\kappa_i'(-\gamma_{3-i})]$,

$i = 1, 2$, *the following hold true:*

$$\mathcal{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2_+ : T_1 > T(x_1, x_2)\} = \{(x_1, x_2) \in \mathbb{R}^2_+ : \widetilde{T}_2 > T(x_1$$
$$\mathcal{D}_2 = \{(x_1, x_2) \in \mathbb{R}^2_+ : T_2 < T(x_1, x_2)\} = \{(x_1, x_2) \in \mathbb{R}^2_+ : \widetilde{T}_1 < T(x_1$$

Remark. The cones $\mathcal{D}_1$ and $\mathcal{D}_2$ can equivalently be defined as the set of rays $a = x_1/x_2$ for which the expected time of ruin (under the measure $P^{(-\gamma_i)}$) is larger resptectively smaller than T, if x_1, x_2 are large enough. This observation follows by combining Lemmas 4 and 6.

Proof of Lemma 6: In view of the definition of T it is a matter of algebra to check that

$$x_1/x_2 = a \Leftrightarrow x_2/T(x_1, x_2) = (p_1 - p_2)/(1 - a) = v_a \tag{43}$$

The assertions (i) and (ii) follow by using that $\kappa_1'(s) = p_1 - p_2 + \kappa_2'(s)$ and applying (43) for $a = s_1$ and $a = s_2$ (that were defined in (25)), respectively. $\square$

Proof of Theorems 3 and 4:. In view of (18) the first step consists in applying Theorem 5 in order to obtain the asymptotics of $\psi_2(x_2, T)$ and $\overline{\psi}_2^{(-\gamma_1)}(x_2, T)$ as $x_1, x_2 \to \infty$ along a ray (note that $\overline{\psi}_2^{(-\gamma_1)}(x_2, T)$ is equal to $\psi_2^{(-\gamma_1)}(x_2) - \psi_2^{(-\gamma_1)}(x_2, T)$ since $\psi_2^{(-\gamma_1)}(x_2) = 1$ as $\kappa_2'(-\gamma_1) < 0$). Appealing to the characterisation of the cones in Lemma 6 and to (43) we then read off that, if $(x_1, x_2) \in \mathcal{D}_2$ and $(x_1, x_2) \in \overline{\mathcal{D}}_2^c := \mathbb{R}^2_+ \backslash \overline{\mathcal{D}}_2$, it holds respectively that

$$\psi_2(x_2,T) \sim C_2 \mathrm{e}^{-\gamma_2 x_2} \qquad \psi_2(x_2,T) \sim |D(v_a)|(v_a/x_2)^{1/2}\mathrm{e}^{-x_2\kappa_2^*(-v_a)/v_a} \tag{44}$$

Similarly, we see that, if $(x_1,x_2) \in \mathcal{D}_1$, $\overline{\psi}_2^{(-\gamma_1)}(x_2,T) \sim 1$ and if $(x_1,x_2) \in \overline{\mathcal{D}}_1^c$

$$\overline{\psi}_2^{(-\gamma_1)}(x_2,T) \sim |D^{(-\gamma_1)}(v_a)|(v_a/x_2)^{1/2}\mathrm{e}^{-x_2\kappa_2^{*(-\gamma_1)}(-v_a)/v_a}. \tag{45}$$

Here D and $D^{(-\gamma_1)}$ are specified in Theorem 5 with $\kappa = \kappa_2$ and $\kappa = \kappa_2^{(-\gamma_1)}$, respectively. Taking note of the facts that $\gamma(a) > \max\{a\gamma_1, \gamma_2\}$ for $a \neq s_1, s_2$ and that

$$\gamma(a) = a\gamma_1 + \kappa_2^{*(-\gamma_1)}(-v_a)/v_a = a\gamma_1 + \kappa_1^{*(-\gamma_1)}(-av_a)/v_a = \kappa_1^*(-av_a)/v_a$$

we deduce, in view of Proposition 1, (44) and (45) that $\psi_{\text{sim}}(x_1,x_2)$ is equivalent to $C_i \mathrm{e}^{-\gamma_i x_i}$ if x_1, x_2 tend to infinity along a ray with $(x_1,x_2) \in \mathcal{D}_i$, $i = 1,2$, respectively, and equivalent to $[D(v) + \widetilde{C}_1(v)D^{(-\gamma_1)}(v)](v/x_2)^{1/2}\mathrm{e}^{-\gamma(a)x_2}$ if (x_1,x_2) follows a ray in the interior of the complement of $\mathcal{D}_1 \cup \mathcal{D}_2$. Using that $\kappa_1'' = \kappa_2''$ and that, if $\kappa_i'(0^+) > 0$, it holds that $\psi_i^*(\theta) = \theta^{-1} - \kappa_i'(0^+)/\kappa_i(\theta)$ (cf. [14, Thm.VII.10]), it is a matter of algebra to verify the form of the constant $D^{\#}$.

□

Proof of Proposition 1: In view of (15), the Markov property and a change of measure it follows that $\psi_{\text{or}}(x_1,x_2) =$

$\psi_1(x_1,T) + Z_{12}(x_1,x_2)$, where

$$\begin{aligned} Z_{12}(x_1,x_2) &:= P(\tau_1(x_1) > T, \inf_{T\le s<\infty} X_2(s) < 0) \\ &= e^{-\gamma_2 x_2} E^{(-\gamma_2)}[h_2(X_1(T))\mathbf{1}_{\{\tau_1(x_1)>T\}}] \end{aligned} \quad (46)$$

where $h_2(x) = e^{\gamma_2 x}\psi_2(x)$ (using that $X_1(T) = X_2(T)$ and recalling that $X_i(0) = x_i$). If $x_1, x_2 \to \infty$ such that x_1/x_2 is constant and $x_1 + \kappa_1'(-\gamma_2)T > 0$ then the strong law of large numbers implies that $X_1(T) \to \infty$ $P^{(-\gamma_2)}$-a.s. and that $P^{(-\gamma_2)}(\tau_1(x_1) \le T)$, which is bounded above by $P^{(-\gamma_2)}(X_T \le 0)$, tends to zero. Appealing to the Cramér-Lundberg approximation and the bounded convergence theorem, it follows that $h_2(y) \to C_2$ as $y \to \infty$ and that the last factor in (46) converges to C_2.

In the case that $x_1 + \kappa_1'(-\gamma_2)T < 0$ it holds that Z_{12} is equal to

$$Z_{12}(x_1,x_2) = e^{-\gamma_2 x_2}\overline{\psi}_1^{(-\gamma_2)}(x_1,T)E^{(-\gamma_2)}[h_2(X_1(T))|\tau_1(x_1) > T]. \quad (47)$$

Denote by $\widetilde{\pi}_v$ the measure given by (7) with θ_v, θ_v' replaced by $\theta_v^{(-\gamma_2)}, \theta_v^{\star(-\gamma_2)}$ respectively (defined as the roots $\theta_v, \theta_v^\star$ under the measure $P^{(-\gamma_2)}$). With regard to Corollary 1 (applied to $X_1(T)$) and the observation that θ_v also satisfies $\kappa_1'(\theta_v) = -v'$ where $v' = x_1/T(x_1,x_2)$, we conclude that as $T \to \infty$ the conditonal expectation on the rhs of (47) converges to

$$\int_0^\infty h_2(x)\widetilde{\pi}_v(\mathrm{d}x) = \int_0^\infty \psi_2(x)\frac{(\theta_v+\gamma_2)(\theta_v^\star+\gamma_2)}{\theta_v^\star-\theta_v}[\mathrm{e}^{-\theta_v x} - \mathrm{e}^{-\theta_v^\star x}]\mathrm{d}x \tag{48}$$

as $T\to\infty$, where we used that $\theta_v^{(-\gamma_2)}=\theta_v+\gamma_2$. In view of the decomposition (15) the proof of the asymptotics of ψ_{or} is complete. The proof of the asymptotics of ψ_{sim} is similar and omitted. □

Proof of Proposition 3: Writing $\widetilde{\gamma} := \widehat{\gamma}_2-\gamma_2$ it follows by definition of $\widehat{\gamma}_2$ that $\kappa_1(-\gamma_2-\widetilde{\gamma})=\kappa_1(-\gamma_2)$ or, equivalently, $\kappa_1^{(-\gamma_2)}(-\widetilde{\gamma})=0$. In view of this observation and the form of $\kappa(u,v)$, derived in Example 3, it follows that $\kappa(-\widetilde{\gamma},-\gamma_2)=0$. Changing measure then with the martingale $\exp(-\widetilde{\gamma}X_1(t)-\gamma_2X_2(t))$ and applying the strong Markov property at $\tau_1(x_1)$ shows that

$$\begin{aligned}&\mathrm{e}^{\gamma_2x_2+\widetilde{\gamma}x_1}P(\tau_1(x_1)\le T,\tau_2(x_2)<\infty)\\ &\quad= E^{(-\widetilde{\gamma},-\gamma_2)}[\mathrm{e}^{\widetilde{\gamma}X_1(\tau_1(x_1))}h_2(X_2(\tau_1(x_1)))|\mathbf{1}_{\{\tau_1\le T\}}]\end{aligned} \tag{49}$$

Letting $x_1,x_2\to\infty$ such that x_1/x_2 is constant and $x_1+\kappa_1'(-\widehat{\gamma}_2)T<0$ it follows by the law of large numbers that $P^{(-\widehat{\gamma}_2)}(\tau_1\le T)$ tends to 1, being bounded below by $P^{(-\widehat{\gamma}_2)}(X_1(T)<0)$. Also, taking note of Lemma 4 and of the fact that, in view of the definition of T, it holds that $X_2(\tau_1(x_1))=X_1(\tau_1(x_1))+(p_1-p_2)[T-\tau_1(x_1)])$, it follows that $X_2(\tau_1(x_1)) \to \infty$ and $h_2(X_2(\tau_1(x_1))) \to C_2$, $P^{(-\widehat{\gamma}_2)}$-a.s. Therefore the bounded convergence theorem implies that the conditional expectation in (49) converges to $C_2\widetilde{C}$ (where the second factor denotes the asymptotic constant for ψ_1 under $P^{(-\gamma_2)}$).

In the opposite case that $x_1 + \kappa_1'(-\widehat{\gamma}_2)T > 0$ we note that

$$
\begin{aligned}
P(\tau_1 \le T, \tau_2 < \infty) &= P(\tau_1 \le T, \tau_2 \le T) + P(\tau_1 \le T, T < \tau_2 < \infty) \\
&= P(\tau_2 \le T) + E[\mathbf{1}_{\{\tau_1 \le T < \tau_2\}} P_{X_2(T)}(\tau_2 < \infty)] \\
&= E[\mathbf{1}_{\{\tau_2 \le T\}} \overline{\psi}_2(X_2(T))] + E[\mathbf{1}_{\{\tau_1 \le T\}} \psi_2(X_2(T))] \\
&= \mathrm{e}^{-\gamma_2 x_2} E^{(-\gamma_2)}[\mathrm{e}^{\gamma_2 X_2(T)} \overline{\psi}_2(X_2(T)) | \tau_2 \le T] \psi_2^{(-\gamma_2)} \\
+ \mathrm{e}^{-\gamma_2 x_2 - \widetilde{\gamma} x_1} &\times E^{(-\widehat{\gamma}_2)}[\mathrm{e}^{\widehat{\gamma}_2 X_1(T)} \psi_2(X_1(T)) | \tau_1 \le T] \psi_1^{(-\widehat{\gamma}_2)}(x_1, T
\end{aligned}
$$

where in the second line we applied the Markov property and used that $\{\tau_2 \le T\} \subset \{\tau_1 \le T\}$ and in the fourth line changed the measure. Invoking Corollary 1 yields the form of $\overline{C}_1(v)$ and $\overline{C}_2(v)$ and in view of (16) the proof is finished. □

C Extensions: optimal control.

Consider the optimal management of two insurance companies which split the amount they pay out of each claim in proportions δ_1 and δ_2 where $\delta_1 + \delta_2 = 1$, and receive premiums at rates c_1 and c_2, respectively.

Dividends are also paid to shareholders, above some barriers b_1, b_2, to be optimized. In addition, the companies may transfer funds between each other, when one of them arrives to negative holdings. However, jumping into a certain subset of $\mathbb{R}^2$, which includes the negative quadrant, as well as some triangle with a vertex at the origin, included in the positive quadrant and with intercepts a_1, a_2; this event, called ruin, will entail penalties. The optimal choice of the control levels defined above requires solving a complicated first-passage problem involving both reflecting and absorbing boundaries.

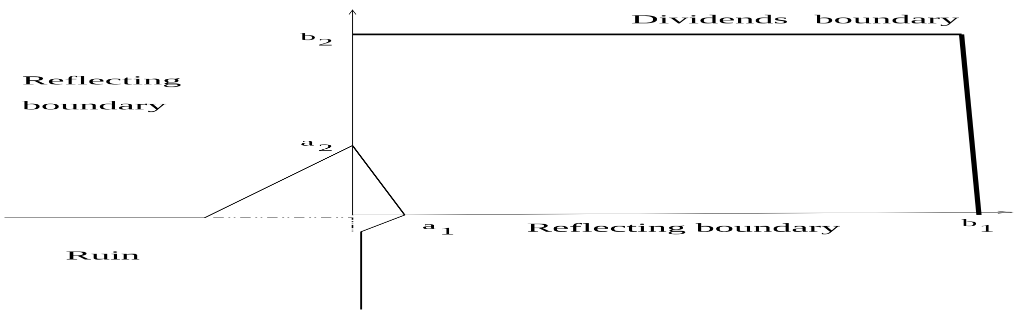

Figure 7: Control with absorbing and reflecting boundaries

While only possible numerically in general, this problem is nevertheless feasible analytically under our "proportional reinsurance model". The asymptotic behavior of the solution is of considerable interest as well for calibrating numerical solutions under more complicated models, as well as for being a particular case of analytically solvable two-dimensional sharp large deviations problem.

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D The Afwedson-Hoglund theorem

There are four conceptually distinct cases to consider for finite time ruin probabilities: that of processes with sure eventual ruin, with $\kappa'(0) = p - \rho < 0$, and the opposite case (in which the asymptotics depend on the shifted measure with parameter $-\gamma$, under which eventual ruin is sure since $\kappa'(-\gamma) < 0$). Then, there are the cases of ruin times which are shorter/longer than expected. Despite the different interpretations, all these cases have a common "large deviations" formulation, expressed in terms of the cumulant exponent $\kappa(\theta)$ of X!!

Theorem 5 *(Arfwedson-Hoglund) Let X_t be a Levy process satisfying either $\kappa'(0) < 0$ (D), or Cramer's condition $\exists\gamma \geq 0$ such that $\kappa(-\gamma) = 0, \kappa'(-\gamma) < 0$ (U).*

Let $\Theta = \{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}, \Theta^o$, denote the domain of the cumulant exponent $\kappa(\theta)$ of X and its interior, and let its convex conjugate be:

$$\kappa^*(v) = \sup_{\beta \in \mathbb{R}} [(-v)\beta - \kappa(\beta)] \tag{50}$$

Let $\psi(x) = C\mathrm{e}^{-x\gamma}$ where $C = -\kappa'(0)/\kappa'(-\gamma)$ in case (U) and $\psi(x) = 1$ in case (D).

If $x, t \to \infty$ such that $x/t = v$, then

$$\psi(t,x) \sim \tag{51}$$
$$\begin{cases} |D(v)|\frac{\mathrm{e}^{-t\kappa^*(-v)}}{t^{1/2}} & \text{if (D), } t < \frac{x}{-\kappa'(0)} \text{or if (U) } -\kappa'(-\gamma) < \frac{x}{t} \\ \psi(x) - w(t,x) & \text{if (D), } t > \frac{x}{-\kappa'(0)} \text{or if (U) } -\kappa'(-\gamma) > \frac{x}{t}, \end{cases}$$

$$w(t,x) \sim \tag{52}$$
$$\begin{cases} \psi(x) - \psi(t,x) & \text{if (D), } t < \frac{x}{-\kappa'(0)} \text{or if (U) } -\kappa'(-\gamma) < \frac{x}{t} \\ |D(v)|\frac{\mathrm{e}^{-t\kappa^*(-v)}}{t^{1/2}} & \text{if (D), } t > \frac{x}{-\kappa'(0)} \text{or if (U) } -\kappa'(-\gamma) > \frac{x}{t}, \end{cases}$$

where

$$D(v) = c(v) \cdot \frac{1}{\sqrt{2\pi\kappa''(\theta_v)}} \qquad \text{with} \qquad c(v) = \frac{\theta'_v - \theta_v}{\theta_v \theta'_v}. \tag{53}$$

Alternatively, the exponent may be written in terms of t, using $t\kappa^*(v) = x\gamma(v)$, where $\gamma(v) = \theta'_v - \theta_v$.

Note that all the "twists" θ with $\kappa'(\theta) < 0$ (for which ruin is sure under the shifted measure $P^{(\theta)}$) play a role in this result: the ones with positive values $\kappa(\theta) > 0$ yield the most likely way of achieving "quick ruin" in less than the expected time, while the others with negative values $\kappa(\theta) < 0$ serve to obtain estimates for "late ruin probabilities", for times larger than the expected ruin time – see [7], Theorem 4.8.