Hopf Algebra Duality in Relation with Graded Graph Duality

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This poster is related to [**NJ**].

Motivations and definitions

Motivations

1. Graph duality introduced by **S. Fomin** [**Fo94**, **Fo95**].
2. **Many examples** of dual Hopf algebras in which one can build **dual graded graphs**.

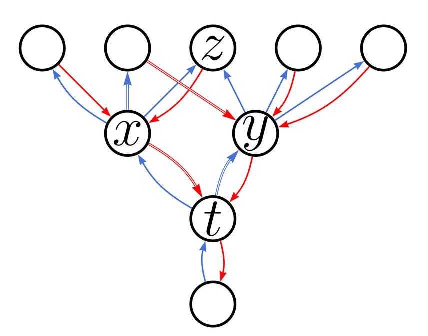
Main purpose

1. Connections between graph duality and Hopf algebra duality.
2. Construction of dual graded graphs from dual Hopf algebras.
3. Construction of dual Hopf algebras from dual graded graphs.

Graph duality (simple duality)

- 2 graded graphs (G_1,G_2), **same set of vertices, rank function**.
- An Up operator U and a Down operator D and the relation:

$$D_{n+1}U_n \;=\; U_{n-1}D_n \;+\; I_n$$

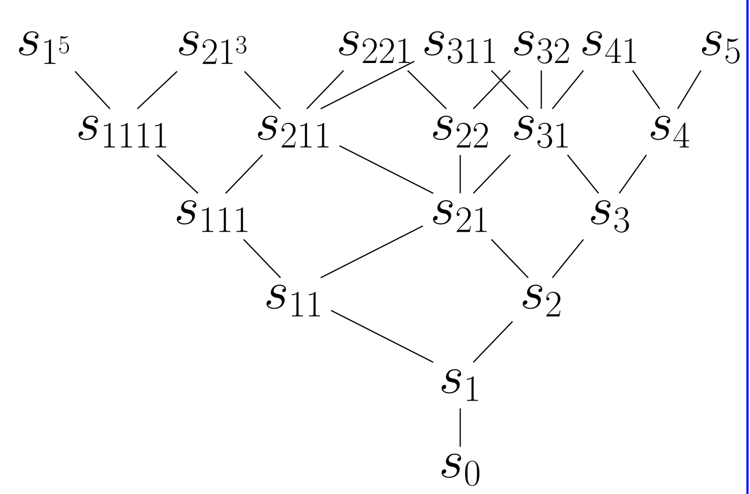


$$\mathrm{U}\,t \;=\; \sum_{v \text{ covers } t \text{ in } G_1} m_1(t,v)\; v$$

$$\mathrm{D}\,z \;=\; \sum_{z \text{ covers } v \text{ in } G_2} m_2(v,z)\; v$$

$m_i(a,b)$ = multiplicity of the edge (a,b) in G_i.

Young lattice, self-dual graph.



Products $s_\lambda s_1$ in the algebra of symmetric functions.

Hopf algebra duality

- A Hopf algebra A with a product $.$ and a coproduct Δ.
- A Hopf algebra A^* with a product $\star$ and a coproduct $\Delta^{'}$.

$$\langle x.y, z\rangle \;=\; \langle x \otimes y, \Delta^{'}(z)\rangle \quad \text{and} \quad \langle x{\star}y, z\rangle \;=\; \langle x \otimes y, \Delta(z)\rangle$$

General observation

- 2 dual Hopf algebras spanned by the basis $\{b_\alpha\}$ and $\{b^*_\alpha\}$.
- 2 graded graphs G_1 and G_2 describing the $(b_\alpha b_1)$ and the $(b^*_\alpha b^*_1)$.

The multiplication graphs G_1 and G_2 are generally in duality.

This is an experimental observation.

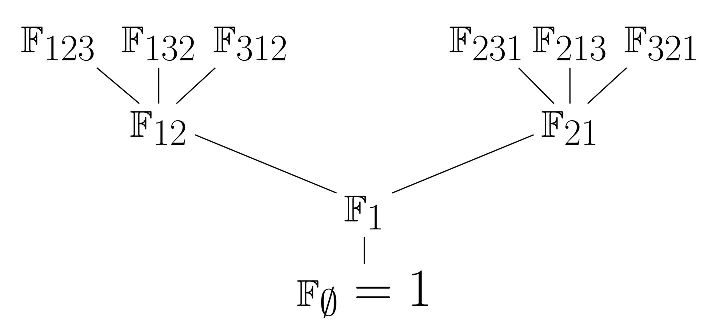
FQSym is spanned by $\{\mathbb{F}_\sigma\}$:

$$\mathbb{F}_\sigma \;=\; \sum_{w\,:\,std(w)\,=\,\sigma^{-1}} w$$

with the multiplication rule:

$$\mathbb{F}_\alpha\,\mathbb{F}_\beta \;=\; \sum_{\sigma\,\in\,\alpha\,\cup\!\!\cup\,\overrightarrow{\beta}} \mathbb{F}_\sigma$$

This graph describes $(\mathbb{F}_\sigma\mathbb{F}_1)$:



$12\,\cup\!\!\cup\,\overrightarrow{21} = 12\,\cup\!\!\cup\,\mathbf{43} = 1243{+}1423{+}1432 + 4123 + 4132 + 4312.$

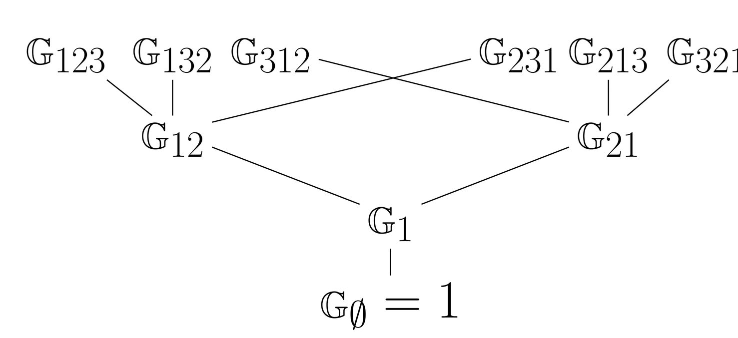
FQSym* is spanned by $\{\mathbb{G}_\sigma\}$:

$$\mathbb{G}_\sigma \;=\; \mathbb{F}_{\sigma^{-1}} \;=\; \sum_{w\,:\,std(w)\,=\,\sigma} w$$

with the multiplication rule:

$$\mathbb{G}_\alpha\,\mathbb{G}_\beta \;=\; \sum_{\sigma^{-1}\,\in\,\alpha^{-1}\,\cup\!\!\cup\,\overrightarrow{\beta^{-1}}} \mathbb{G}_\sigma$$

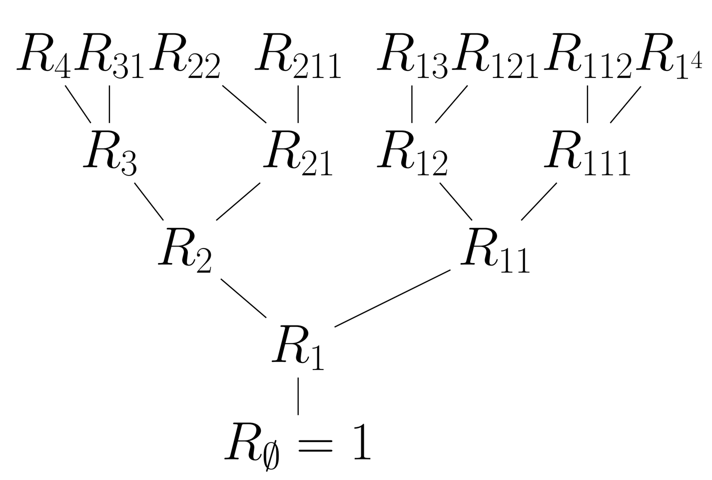
This graph describes $(\mathbb{G}_\sigma\mathbb{G}_1)$:



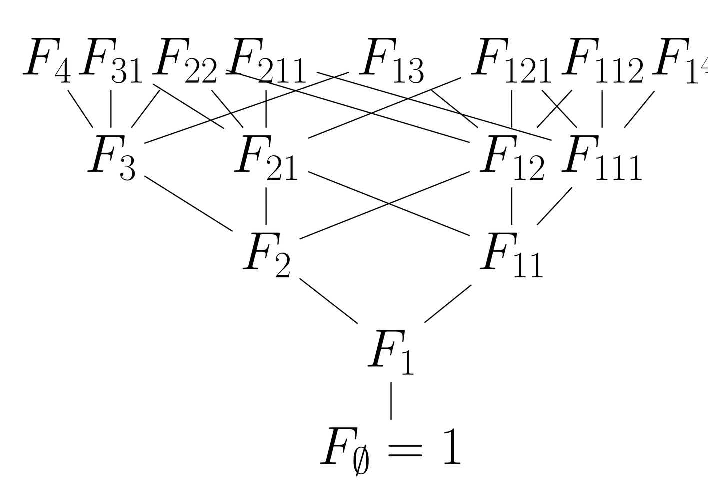
The two multiplication graphs are in duality.

This example can be used to explain the duality of graphs in quotients and subalgebras of FQSym.

The products $(R_I R_1)$ in **NCSF**:



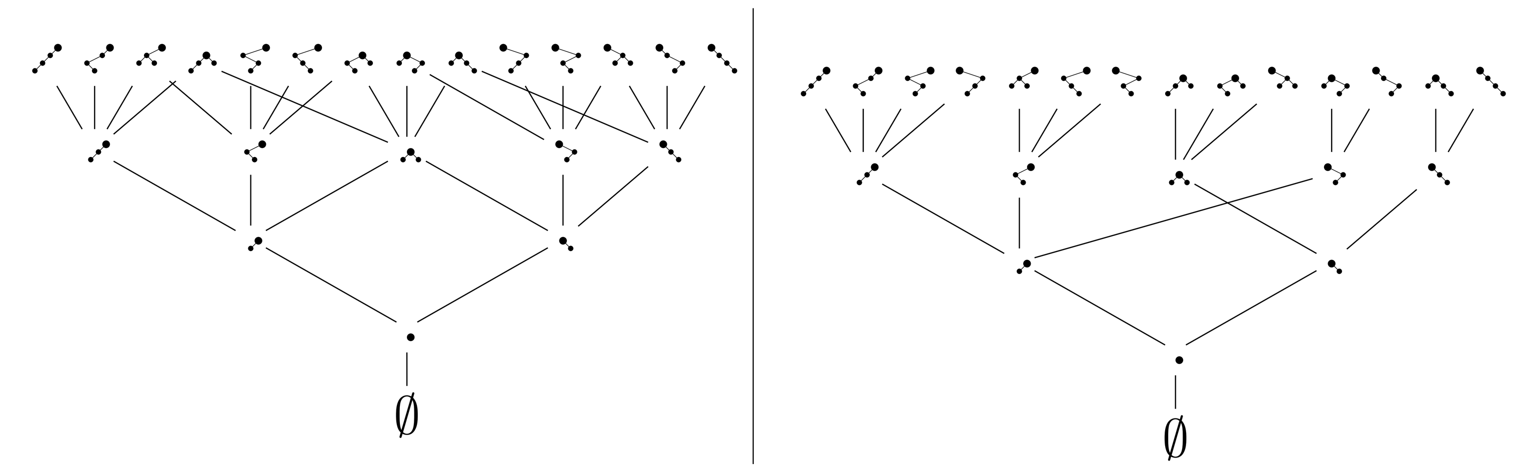
The products $(F_I F_1)$ in *QSym*:



The two multiplication graphs are in duality.

An example with Fomin's approach for RSK

Multiplication graphs in the algebra of **P**lanar **B**inary **T**rees.



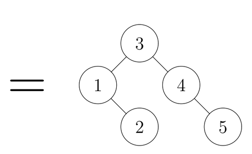
The two multiplication graphs are in duality.

Through Fomin's approach, one gets 2 paths which naturally convert into the insertion and the recording tree of σ.

The Schensted-like algorithm is the binary search tree insertion.

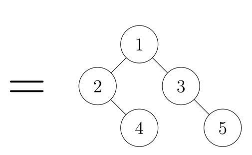
- Insertion tree:

$$P(\sigma) \;=\;$$



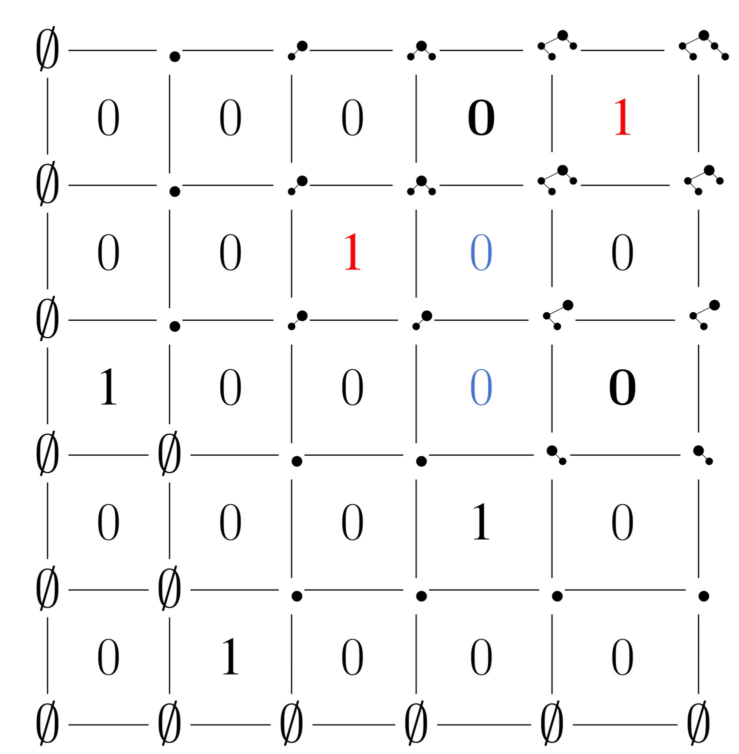
- Recording tree:

$$Q(\sigma) \;=\;$$



- **A similar process for the graphs in NCSF and *QSym*.**

Growth diagram for $\sigma = 31425$:



Computations are done with: MuPAD-Combinat, *http://mupad-combinat.sourceforge.net/*.

Conclusions

1. Examples suggest a strong connection between the two dualities.
2. A combinatorial (algebraic) interpretation of dual graded graphs.

References

[**NJ**] J. Nzeutchap, *Dual Graded Graphs and Fomin's r-correspondences associated to the Hopf Algebras of Planar Binary Trees, Quasi-symmetric Functions and Noncommutative Symmetric Functions*, **FPSAC'06**.

[**Fo94**] S. Fomin, *Duality of Graded Graphs*, J. Alg. Comb. **3** (1994), 357-404.

[**Fo95**] S. Fomin, *Schensted Algorithms for Dual Graded Graphs*, J. Alg. Comb. **4** (1995), 5-45.

18th International Conference on Formal Power Series & Algebraic Combinatorics, June 19-23, 2006 ; San Diego, California - USA.