Algorithmic Aspects of the Intersection and Overlap Numbers of a Graph

Stéphane Vialette
LIGM Université Paris-Est Marne-la-Vallée

Danny Hermelin
Max-Planck-Institut für Informatik

Romeo Rizzi
Università degli Studi di Udine

March, 16 2012

Outline

Basic definitions

Definition (Graph)

We consider undirected and simple graphs.

- $\mathbf{V}(G)$ is the set of vertices of G.
- $\mathbf{E}(G)$ is the set of edges of G.
- The degree of a vertex is the number of edges that connect to it, *i.e.*,

$$d_G(u) = |\{\{u, v\} : \{u, v\} \in \mathbf{E}(G)\}|$$

- $\Delta(G)$ is the maximum degree of a vertex of G, *i.e.*,

$$\Delta(G) = \max\{d_G(u) : u \in \mathbf{V}(G)\}$$

Intersection graphs

Definition (Intersection graph)

Let $F = (S_1, S_2, \ldots, S_n)$ be a family of sets (allowing sets in F to be repeated).

The intersection graph of F, denoted $\Omega(F)$, is an undirected graph that has a vertex for each member of F and an edge between each two members that have a nonempty intersection.

$$\mathbf{V}(\Omega(F)) = \{u_i : 1 \leq i \leq n\}$$
$$\mathbf{E}(\Omega(F)) = \{\{u_i, u_j\} : i \neq j \ \wedge \ S_i \cap S_j \neq \emptyset\}$$

Intersection graphs

Theorem (Szpilrajn-Marczewski, 1945)

Every graph is an intersection graph.

Classes of intersection graphs

- **interval graphs**: intersection of intervals on the real line,
- **circular arc graphs**: intersection of arcs on a circle,
- **circle graphs**: intersection of chords on a circle,
- **unit disk graphs**: intersection of unit disks in the plane,
- **string graphs**: intersection of curves on a plane,
- ...

The best general reference is [McKee, and McMorris, 1999].

Intersection number

Definition

The **intersection number** of a graph G, denoted $i(G)$, is the minimum total number of elements in any intersection representation of the graph.

Definition (Problem)

INTERSECTION NUMBER is the associated optimization problem.

Intersection number

Remark

- Not to be confused with the **interval number** which is also denoted $i(G)$ in the literature.
- The **interval number** of a graph G the smallest integer t such that G is the intersection graph of some family of sets $I_1, I_2, \ldots, I_n$, with every I_i being the union of at most t intervals.

Edge-clique cover

Definition (Edge-clique cover)

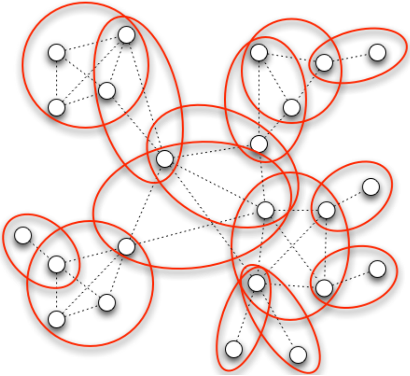
An **edge-clique cover** of a graph G is any family $\mathcal{E} = \{Q_1, Q_2, \ldots, Q_k\}$ of complete subgraphs of G such that every edge of G is in at least one of $Q_1, Q_2, \ldots, Q_k$.

The minimum cardinality of an edge-clique cover of G is denoted $\theta(G)$.

Definition (Problem)

EDGE-CLIQUE COVER is the associated optimization problem.

Edge-clique cover



Intersection number and Edge-clique cover

Theorem (Erdös, Goodman, and Pósa.1966)

For every graph G, $i(G) = \theta(G)$.

Remarks

- The equivalence between the two directions is straightforward to prove.
- A graph with m edges has intersection number at most m.
- Every graph with n vertices has intersection number at most $n^2/4$.

Edge-clique cover

Classical complexity and optimization

Computing $\theta(G)$ – EDGE-CLIQUE COVER

- **NP**-hard for **planar graphs** and **graphs with maximum degree** 6 [Kou, and Stockmeyer.1978; Orlin.1977].
- Polynomial-time solvable for for **chordal graphs** [Ma, Wallis, and Wu.1989], **graphs with maximum degree** 5 [Hoover.1992], **line graphs** [Orlin.1977], and **circular-arc graphs** [Hsu, and Tsai.1991].
- Not approximable to within ratio n^{ε} for some $\varepsilon > 0$ [Lund, and Yannakakis.1994].
- Approximable to within ratio $O(n^2 \frac{(\log\log n)^2}{(\log n)^3})$ [Ausiello, Crescenzi, Gambosi, Kann, Marchetti, Spaccamela, and Protasi.1999].

Edge-clique cover

Parameterized complexity

Computing $\theta(G)$ – EDGE-CLIQUE COVER

- EDGE-CLIQUE COVER is **fixed-parameter tractable** (standard parameterization) [Gramm, Guo, Hüffner, and Niedermeier.2008].
- EDGE-CLIQUE-COVER has a **size-2^k kernel** [Gramm, Guo, Hüffner, and Niedermeier.2008].
- EDGE-CLIQUE COVER does not have a **polynomial kernel** [Cygan, Kratsch, Pilipczuk, Pilipczuk, and Wahlström.2011].

Overlap graphs

Definition (Overlap graph)

Let $F = (S_1, S_2, \ldots, S_n)$ be a family of sets (allowing sets in F to be repeated).

The **overlap graph** of F, denoted $O(F)$, is an undirected graph that has a vertex for each member of F and an edge between each two members that overlap.

$$\mathbf{V}(O(F)) = \{u_i : 1 \leq i \leq n\}$$
$$\mathbf{E}(O(F)) = \{\{u_i, u_j\} : S_i \cap S_j \neq \emptyset \ \wedge \ S_i \setminus S_j \neq \emptyset \ \wedge \ S_j \setminus S_i \neq \emptyset\}$$

Overlap graphs

Theorem

Every graph is an overlap graph.

Classes of intersection graphs

- **interval overlap graphs**: overlap of intervals on the real line,
- **overlap circular arc graphs**: overlap of arcs on a circle,
- **overlap rectangle graphs**: overlap of rectangles in the plane,
- ...

The most well-known overlap graph

Interval overlap graph

- There is an $O(n^2)$ time algorithm that tests whether a given n-vertex undirected graph is a circle graph and, if it is, constructs a set of chords that represents it [Spinrad.1994].
- Polynomial-time solvable combinatorial problems: TREEWIDTH [Kloks.1996], FILL-IN [Kloks, Kratsch, and Wong.1998], CLIQUE, INDEPENDENT SET, ...
- **NP**-complete combinatorial problems: DOMINATING SET, CONNECTED DOMINATING SET [Keil.1993], ...

Overlap number

Definition

The **overlap number** of a graph G, denoted $\varphi(G)$, is the minimum total number of elements in any overlap representation of the graph.

Definition (Problem)

OVERLAP NUMBER is the associated optimization problem.

Overlap number

Computing $\varphi(G)$ – Overlap Number

- Complexity unknown so far.
- The following upper bounds for a n-vertex graph are known: $n+1$ for **trees**, $2n$ for **chordal graphs**, $\frac{10}{3}n-6$ for **planar graphs**, and $\lfloor n^2/4 \rfloor + n$ for **general graphs** [Rosgen.2005;Rosgen, and Stewart.2010].
- The overlap number of K_n is the minimum ℓ such that a ℓ-set contains n pairwise incomparable sets
- $\varphi(C_n) = n-1$.

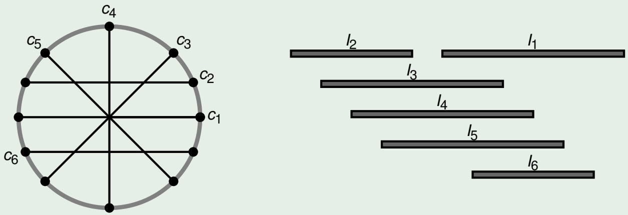
Intersection and overlap representations

Remark

Some graph classes can play it both ways:

A graph is an intersection graph of chords in a circle (*i.e.* circle graph) if and only if it is has an overlap representation using intervals on a line.

Example



Our results

Proposition

There exists a constant $c > 1$ such that computing the **overlap number** of a graph is hard to approximate to within c.

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$?" is linear-time solvable.

Outline

Overlap number

Proposition

There exists a constant $c > 1$ such that computing the **overlap number** of a graph is hard to approximate to within c.

A detour through EDGE-CLIQUE COVER

EDGE-CLIQUE COVER

Input: A graph G.

Solution: A clique cover for G, *i.e.*, a collection $\mathcal{E} = \{Q_1, Q_2, \ldots, Q_k$ of subsets of $\mathbf{V}(G)$ such that

- each Q_i induces a complete subgraph of G, and
- for each edge $e = \{u, v\} \in \mathbf{E}(G)$ there is some Q_i that contains both u and v.

Measure: Cardinality of the clique cover, *i.e.*, the number of subsets Q_i.

A detour through EDGE-CLIQUE COVER

Proposition

EDGE-CLIQUE COVER is **APX**-hard for biconnected graphs with maximum degree 7.

Key elements

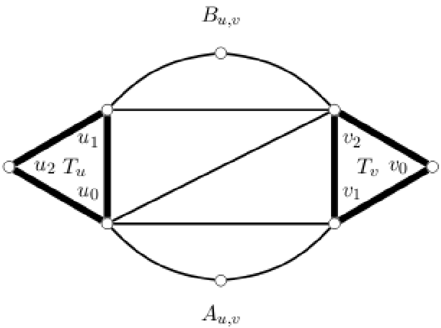
- $\theta(G) = i(G)$, and hence the same result applies for INTERSECTION NUMBER.
- Reduction from VERTEX COVER for cubic graphs which is known to be **APX**-hard [Alimonti, and Kann.2000;Papadimitriou, and Yannakakis.1991].

A detour through EDGE-CLIQUE COVER

EDGE-CLIQUE COVER is APX-hard for biconnected graphs with maximum degree 7

- Let G be a n-vertex cubic graph.
- We represent each vertex $u \in \mathbf{V}(G)$ by a triangle T_u with vertices u_0, u_1 and u_2 in the new graph H.
- These n triangles are all vertex disjoint in H, and each of them can offer a different edge for three connections.

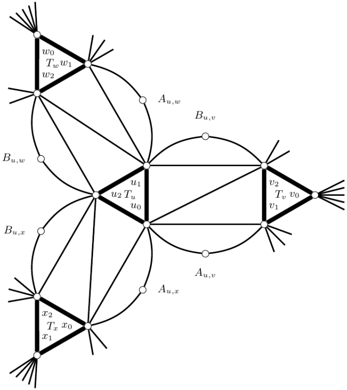
A detour through EDGE-CLIQUE COVER



A detour through EDGE-CLIQUE COVER

Claim

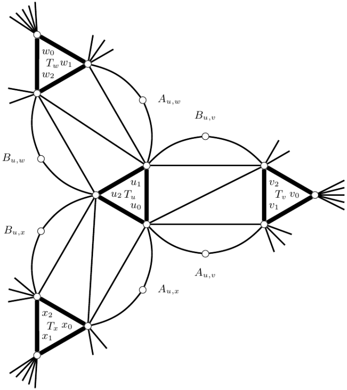
G has a vertex cover of size k is and only $\theta(H) \leq 3m + k$.



A detour through EDGE-CLIQUE COVER

Claim

G has a vertex cover of size k is and only $\theta(H) \leq 3m + k$.



Cartesian product

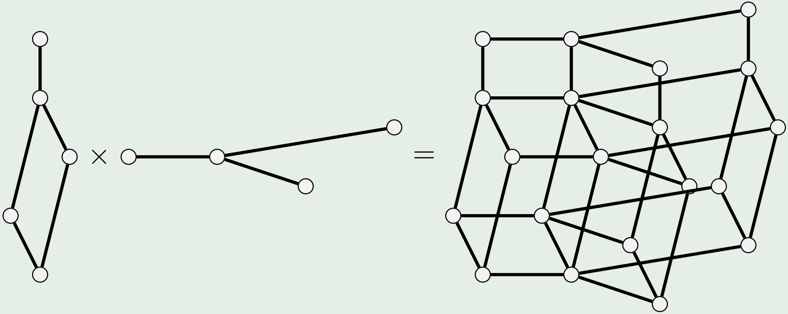
Definition

The Cartesian product $G \times H$ of graphs G and H is the graph such that

- the vertex set of $G \times H$ is the Cartesian product $\mathbf{V}(G) \times \mathbf{V}(H)$, and
- any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G.

Cartesian product

Example



Cartesian product

Remarks on $G \times H$

- Each row induces a copy of H.
- Each column induces copy of G
- This terminology is consistent with a representation of $G \times H$ by the points of the $|\mathbf{V}(G)| \times |\mathbf{V}(H)|$ grid.

$G \times H$

$\mathbf{V}(G) = \{u_1, u_2, u_3\}$

$\mathbf{V}(H) = \{v_1, v_2, v_3, v_4\}$

	G	G	G	G
H	(u_1, v_1)	(u_1, v_2)	(u_1, v_3)	(u_1, v_4)
H	(u_2, v_1)	(u_2, v_2)	(u_2, v_3)	(u_2, v_4)
H	(u_3, v_1)	(u_3, v_2)	(u_3, v_3)	(u_3, v_4)

OVERLAP NUMBER

Proposition

OVERLAP NUMBER is **APX**-hard.

Key elements

- EDGE-CLIQUE COVER is **APX**-hard for biconnected graphs with maximum degree 7.
- Cartesian product of graphs.

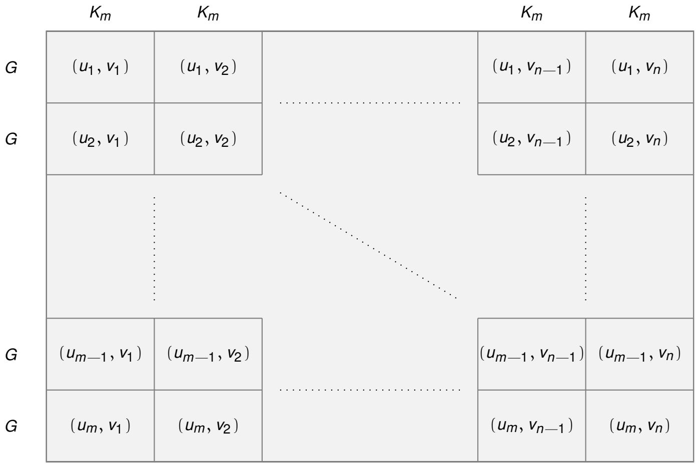
Overlap Number is APX-hard

Construction

- Let G be a biconnected graph with maximum degree 7 (without isolated vertices).

  For simplicity, write $\mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$.
- Let m be a constant (to be precisely defined later).
- Let K_m be the complete graph with m vertices, and write $\mathbf{V}(K_m) = \{u_1, u_2, \ldots, u_m\}$.
- Construct $H = K_m \times G$.

OVERLAP NUMBER is APX-hard



OVERLAP NUMBER is APX-hard

Lemma

$\varphi(H) \leq n + m\,\theta(G)$.

Proof

OVERLAP NUMBER is APX-hard

Lemma

$\varphi(H) \leq n + m\theta(G)$.

Proof

- Let $k = \theta(G)$.
- Let $\mathcal{E} = \{Q_1, Q_2, \ldots, Q_k\}$ be a size-k edge-clique cover of G.
- To every vertex $(u_i, v_j) \in \mathbf{V}(H)$, we associate the set $S_{(u_i,v_j)}$ defined as follows:

$$S_{(u_i,v_j)} = \{v_j\} \cup \{(u_i, p) : v_j \in Q_p\}.$$

OVERLAP NUMBER is APX-hard

Lemma

$\varphi(H) \leq n + m\theta(G)$.

Proof

- $\mathcal{F} = \{S_{(u_i,v_j)} : (u_i, v_j) \in \mathbf{V}(H)\}$ defined over the ground set

$$X = \bigcup_{(u_i,v_j)\in\mathbf{V}(H)} S_{(u_i,v_j)} = \mathbf{V}(G) \cup (\mathbf{V}(K_m) \times [k]).$$

- $|X| = n + km$.
- The lemma reduces to proving that $O(\mathcal{F})$ and H are isomorphic graphs.

OVERLAP NUMBER is APX-hard

Lemma

$\varphi(H) \leq n + m\theta(G)$.

Proof

Let $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ be two subsets of $\mathcal{F}$.

- If $u_i \neq u_r$ and $v_j \neq v_s$, then (u_i, v_j) and (u_r, v_s) are not adjacent vertices in H.

  It is easily verified that $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ are disjoint subsets, and hence $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ are not adjacent vertices in $O(\mathcal{F})$.

OVERLAP NUMBER is APX-hard

Lemma

$\varphi(H) \leq n + m\theta(G)$.

Proof

Let $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ be two subsets of $\mathcal{F}$.

- If $u_i \neq u_r$ and $v_j = v_s$, then (u_i, v_j) and (u_r, v_s) are adjacent vertices in H since K_m is a clique.

  $v_j \in S_{(u_i,v_j)}$ and $v_j \in S_{(u_r,v_s)}$, and hence $S_{(u_i,v_j)} \cap S_{(u_r,v_s)} \neq \emptyset$.
  $v_j \in S_{(u_i,v_j)} \setminus S_{(u_r,v_s)}$ and $v_s \in S_{(u_r,v_s)} \setminus S_{(u_i,v_j)}$ are non-empty (since $u_i \neq u_r$ and v_j is not an isolated vertex of G).

  Therefore, $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ overlap, and hence $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ are adjacent vertices in $O(\mathcal{F})$.

Overlap Number is APX-hard

Lemma

$\varphi(H) \leq n + m\,\theta(G)$.

Proof

Let $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ be two subsets of $\mathcal{F}$.

- If $u_i = u_r$ and $v_j \neq v_s$, then (u_i, v_j) and (u_r, v_s) are adjacent vertices in H if and only if $\{v_i, v_j\} \in \mathbf{E}(G)$.

  $v_j \in S_{(u_i,v_j)} \setminus S_{(u_r,v_s)}$ and $v_s \in S_{(u_r,v_s)} \setminus S_{(u_i,v_j)}$.
  Therefore, the two sets overlap if and only if v_j and v_j belong to a same Q_p for some $1 \leq p \leq k$.

  Hence, $S_{(u_i,v_j)}$ and $S_{(u_r,v_s)}$ are adjacent vertices in $O(\mathcal{F})$ if and only if $\{v_i, v_j\} \in \mathbf{E}(G)$.

Overlap Number is APX-hard

Remarks

- For the reverse direction, we need to be careful about inclusion of sets.
- Fortunately, $H = K_m \times G$ behaves nicely enough w.r.t. inclusion of sets.

Lemma

Let $(\mathcal{F} = \{S_{(u_i,v_j)} : (u_i, v_j) \in \mathbf{V}(H)\}, X)$ be an overlap representation of $H = K_m \times G$.

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

OVERLAP NUMBER is APX-hard

Remarks

- For the reverse direction, we need to be careful about inclusion of sets.
- Fortunately, $H = K_m \times G$ behaves nicely enough w.r.t. inclusion of sets.

Lemma

Let $(\mathcal{F} = \{S_{(u_i,v_j)} : (u_i, v_j) \in \mathbf{V}(H)\}, X)$ be an overlap representation of $H = K_m \times G$.

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

OVERLAP NUMBER is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 1)

OVERLAP NUMBER is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 1)

- If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ then vertices (u_r, v_s) and (u_i, v_j) are not adjacent in H.
- Let (u_p, v_q) be any vertex of H distinct from (u_r, v_s) that is not adjacent to (u_i, v_j).
- Let H' be the graph obtained from H by deleting every vertex in the close neighborhood of vertex (u_i, v_j).

OVERLAP NUMBER is APX-hard

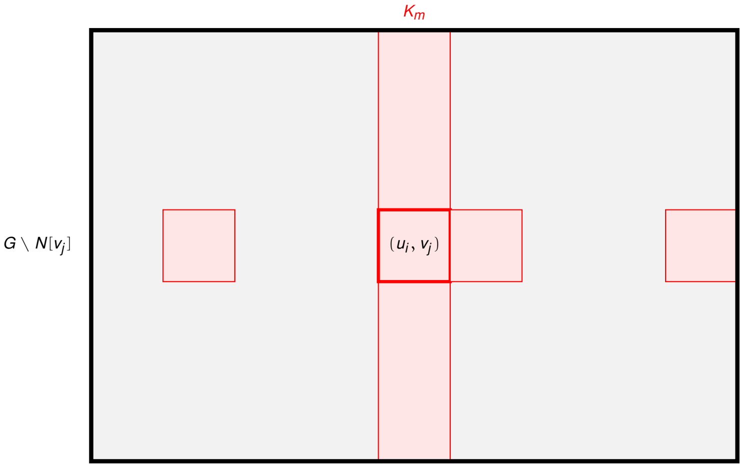
Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 1)

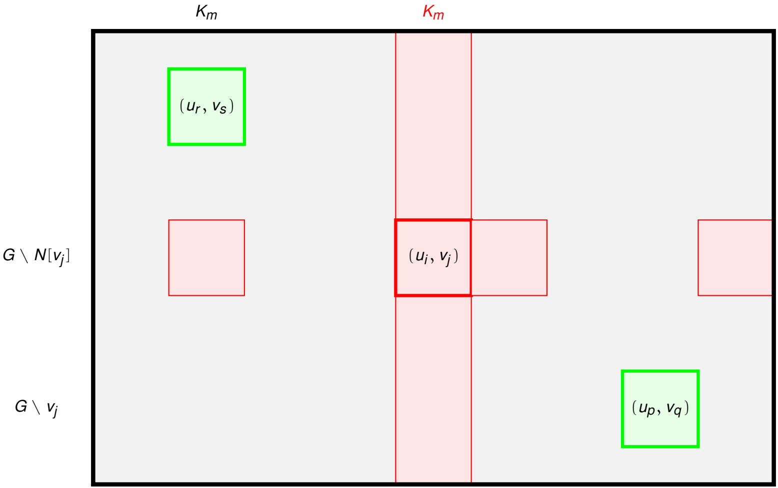
- Since (u_r, v_s) and (u_p, v_q) are not adjacent to (u_i, v_j) in H, they are both vertices of H'.
- **Claim**: there exists a path between vertices (u_r, v_s) and (u_p, v_q) in H'.

Overlap Number is APX-hard

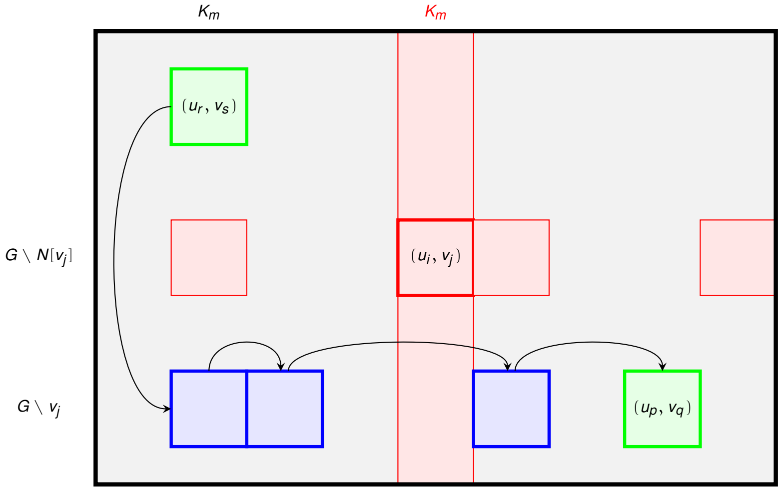


$H' = (K_m \times G) \setminus (u_i, v_j)$

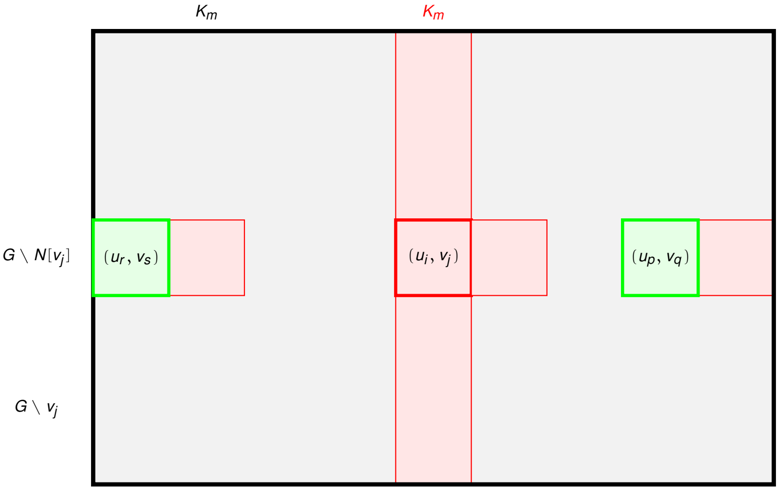
Overlap Number is APX-hard



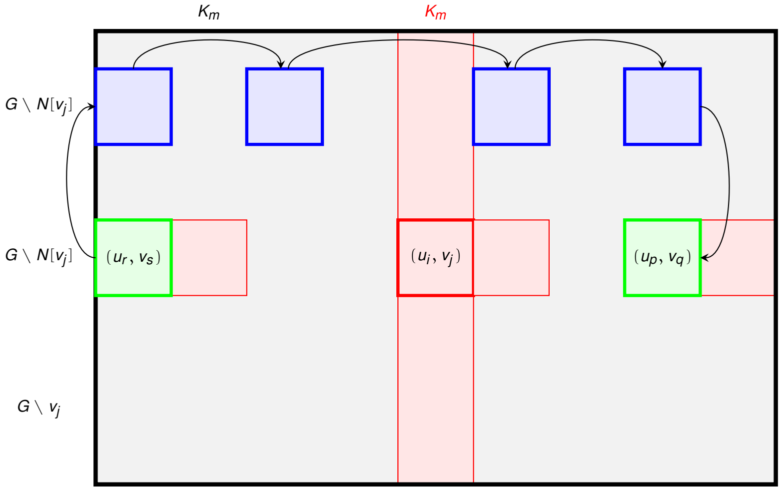
Overlap Number is APX-hard



OVERLAP NUMBER is APX-hard



Overlap Number is APX-hard



Overlap Number is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 2)

- What is left is to prove that $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for any vertex (u_p, v_q) of H that is adjacent to (u_r, v_s) but not to (u_i, v_j).
- Easy contradiction.

Overlap Number is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 2)

- Suppose $S_{(u_p,v_q)} \not\subset S_{(u_i,v_j)}$.
- Since $S_{(u_p,v_q)} \neq \emptyset$ (H does not contain any isolated vertex), then there exists $x \in X$ such that $x \in S_{(u_p,v_q)}$ and $x \notin S_{(u_i,v_j)}$.
- Therefore, $S_{(u_p,v_q)} \setminus S_{(u_i,v_j)} \neq \emptyset$.

Overlap Number is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 2)

- (u_p, v_q) and (u_r, v_s) are adjacent vertices in H
- $S_{(u_p,v_q)}$ and $S_{(u_r,v_s)}$ have to overlap, and hence there exist $x', x'' \in X$ such that
  - $x' \in S_{(u_p,v_q)}$ and $x' \in S_{(u_r,v_s)}$, and
  - $x'' \notin S_{(u_p,v_q)}$ and $x'' \in S_{(u_r,v_s)}$.

OVERLAP NUMBER is APX-hard

Lemma

If $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$ for some vertices (u_i, v_j) and (u_r, v_s) of H, then $S_{(u_p,v_q)} \subset S_{(u_i,v_j)}$ for every vertex (u_p, v_q) of H which is not adjacent to vertex (u_i, v_j).

Proof (Part 2)

- But $S_{(u_r,v_s)} \subset S_{(u_i,v_j)}$, and hence $x' \in S_{(u_i,v_j)}$ and $x'' \in S_{(u_i,v_j)}$.
- Therefore, $S_{(u_i,v_j)} \setminus S_{(u_p,v_q)} \neq \emptyset$ and $S_{(u_p,v_q)} \cap S_{(u_i,v_j)} \neq \emptyset$.
- Hence, $S_{(u_p,v_q)}$ and $S_{(u_i,v_j)}$ overlap, a contradiction.

Overlap Number is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

OVERLAP NUMBER is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- Let $(\mathcal{F} = \{S_{(u_i,v_j)} : (u_i, v_j) \in \mathbf{V}(H)\}, X)$ be an overlap representation of H.
- We focus on the most annoying situation when some containment does occur in $\mathcal{F}$.

OVERLAP NUMBER is APX-hard

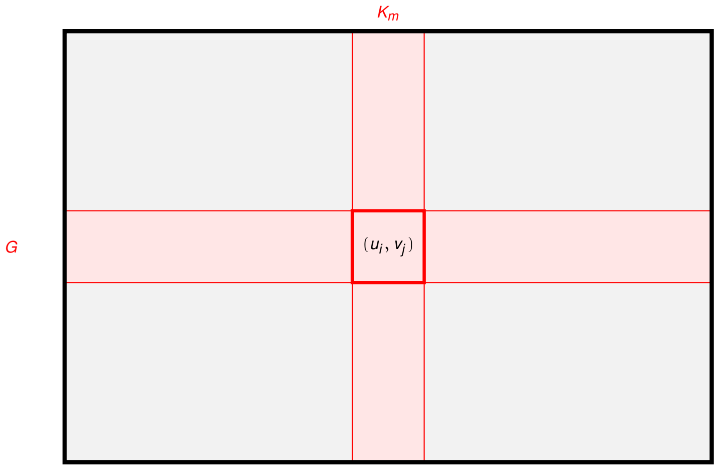
Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- Suppose that there exists some subset $S_{(u_i,v_j)} \in \mathcal{F}$ that strictly contains at least one set of $\mathcal{F}$.
- According to the previous lemma, $S_{(u_i,v_j)}$ contains all subsets $S_{(u_r,v_s)} \in \mathcal{F}$ such that $u_i \neq u_r$ and $v_j \neq v_s$.
- In other words, $S_{(u_i,v_j)}$ contains all those subsets of $\mathcal{F}$ that are associated to vertices of H that are not in the same row nor column of vertex (u_i, v_j)).

OVERLAP NUMBER is APX-hard



$$H' = (K_m \times G) \setminus (u_i, v_j)$$

OVERLAP NUMBER is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- if there exist subsets $S_{(u_r,v_s)}, S_{(u_p,v_q)} \in \mathcal{F}$ distinct from $S_{(u_i,v_j)}$ such that $S_{(u_r,v_s)} \subset S_{(u_p,v_q)}$, then $u_i = u_p$ or $v_j = v_q$
- In other words, vertex (u_p, v_q) is on the same row or on the same column of vertex (u_i, v_j)

OVERLAP NUMBER is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- Let H' be the graph obtained from H by deleting all vertices (u_r, v_s) such that $u_r = u_i$ or $v_s = v_j$.
- In other words, H' is the graph obtained from H by deleting all vertices that are in the same row or column of vertex (u_i, v_j).

OVERLAP NUMBER is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- Let $\mathcal{F}' \subseteq \mathcal{F}$ be those subsets of $\mathcal{F}$ that correspond to vertices of H'
- Let $X' \subseteq X$ be the union of the subsets in $\mathcal{F}'$
- $\mathcal{F}'$ is an overlap representation of H' where no subset being a subset of another.
- $|X'| \leq |S_{(u_i,v_j)}|$.

OVERLAP NUMBER is APX-hard

Lemma

$$\theta(G) \leq \frac{\varphi(H) - n - 1}{m - 1} + 7.$$

Proof

- Let G' be the graph obtained from G by deleting vertex v_j.
- $H' = K_{m-1} \times G'$.
- **Claim**: $\theta(G') \leq \frac{|X|-n-1}{m-1}$.
- "*Edge-multi-coloring*" procedure of H' (details omitted).
- The lemma now follows from $\theta(G) \leq \theta(G') + \Delta(G)$.

OVERLAP NUMBER is APX-hard

Proposition

OVERLAP NUMBER is **APX**-hard.

Proof (sketch)

- There exists a constant $c > 1$ such that $\theta(G)$ cannot be approximated to within c (unless $\mathbf{P} = \mathbf{NP}$).
- Let m be the smallest integer such that $m \geq n$ and $c > \frac{m}{m-1}$.

OVERLAP NUMBER is APX-hard

Proposition

OVERLAP NUMBER is **APX**-hard.

Proof (sketch)

- Suppose that there exists a $\sqrt{c}$-approximation algorithm B for computing $\varphi(H)$.
- In other words, $\mathrm{B}(H) \leq \sqrt{c}\,\varphi(H)$.
- $\mathrm{B}(H) \leq \sqrt{c}\,\varphi(H) \leq \sqrt{c}\,(n + m\,\theta(G))$.

Overlap Number is APX-hard

Proposition

Overlap Number is **APX**-hard.

Proof (sketch)

We apply the constructive proof of the previous lemma to obtain an approximate $\mathrm{A}(G)$ of $\theta(G)$:

$$\begin{aligned}\mathrm{A}(G) &\leq \frac{\mathrm{B}(H) - n - 1}{m - 1} + 7 \\ &= \frac{\mathrm{B}(H)}{m-1} - \frac{n+1}{m-1} + 7 \\ &= c\,\theta(G) + O(1).\end{aligned}$$

Outline

Recognizing graphs

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$?" is linear-time solvable.

Well quasi order

Definition (Well quasi order)

A **quasi order** (*i.e.*, a binary reflexive transitive relation) is a **well quasi order** (or **wqo** for short) if it does not contain infinitely descending sequences nor infinite antichains.

Examples

- Graph minor Theorem [Robertson, and Seymour. 2004].
- Trees are wqo by the topological minor order [Kruskal. 1960].
- Graphs of treewidth at most 2 are wqo by induced minors [Thomas.1985].
- Cographs are wqo by induced subgraphs [Damaschke.1990].

Well quasi order

Definition

For two vectors $\overrightarrow{x} \in \mathbb{N}^{K_1}$ and $\overrightarrow{y} \in \mathbb{N}^{K_2}$, we write $\overrightarrow{x} \leq \overrightarrow{y}$ if

- $K_1 \leq K_2$, and
- $x_i \leq y_i$ for all $i \in \{1, \ldots, K_1\}$.

Lemma (Higman's Lemma)

The set $\mathbb{N}^{\leq K}$ is wqo by $\leq$ for any fixed $K \in \mathbb{N}$.

Characteristic vector

Definition (Characteristic vector)

A **characteristic vector** of a graph G is a vector $\overrightarrow{c} \in \mathbb{N}^K$ such that there exists a partitioning $\{V_1, V_2, \ldots, V_k\}$ of $\mathbf{V}(G)$ satisfying the two following properties for each $i \in \{1, 2, \ldots, K\}$:

1. $|V_i| = c_i$, and
2. $N[u] = N[v]$ for all $u, v \in V_i$.

Define the **dimension** of a graph G to be the minimum number K such that G has a characteristic vector of dimension K.

Characteristic vector

Lemma (Bounded dimension)

A graph G with $i(G) \leq k$ has dimension at most $K = 2^k$.

Proof

- If $i(G) \leq k$, then G has an intersection representation F with $|\bigcup_{S \in F} S| \leq k$.
- Therefore, there are at most 2^k distinct sets in F.
- Vertices of G that have identical sets in F have identical neighborhoods.

Characteristic vector

Lemma (Bounded dimension)

A graph G with $i(G) \leq k$ has dimension at most $K = 2^k$.

Proof

- If $i(G) \leq k$, then G has an intersection representation F with $|\bigcup_{S \in F} S| \leq k$.
- Therefore, there are at most 2^k distinct sets in F.
- Vertices of G that have identical sets in F have identical neighborhoods.

Characteristic vector

Definition (Isomorphism)

Two characteristic vectors $\overrightarrow{c}, \overrightarrow{d} \in \mathbb{N}^K$ are **isomorphic** if there is a permutation $\pi \in S_K$ such that $c_i = d_{\pi(i)}$ for all $i \in \{1, 2, \ldots, K\}$.

Lemma

Two graphs are isomorphic if and only if they both have the same dimension K, and any pair of characteristic vectors of dimensions K for these graphs are isomorphic.

Recognizing graphs

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$?" is linear-time solvable.

Lemma

Let $k \in \mathbb{N}$. The set of all graphs G with $i(G) \leq k$ is **wqo** by the induced subgraph order.

Lemma

Let $k \in \mathbb{N}$. For any two graphs G and H with intersection number at most k, there is a linear-time algorithm for deciding whether H is an induced subgraph of G.

Recognizing graphs

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$?" is linear-time solvable.

Proof (sketch)

- Let $\mathcal{G}$ be any intersection graph class.
- Let $\overline{\mathcal{G}}$ be the set of all finite graphs not in $\mathcal{G}$.
- Let $\mathcal{H}$ denote the set of all minimal graphs in $\overline{\mathcal{G}}$ w.r.t. the induced subgraph order, *i.e.*,

$$\mathcal{H} = \{H \in \overline{\mathcal{G}} : \nexists H' \in \overline{\mathcal{G}} \text{ s.t. } H' \text{ is an induced subgraph of } H\}$$

Recognizing graphs

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$**?**" is linear-time solvable.

Proof (sketch)

- $\mathcal{G}$ is closed under induced subgraphs, *i.e.*, $H \in \mathcal{G}$ whenever H is an induced subgraph of G for some $G \in \mathcal{G}$.
- Therefore, $G \in \mathcal{G}$ if and only if no graph $H \in \mathcal{H}$ is an induced subgraph of G.
- According to the previous lemma $\mathcal{H}$ is finite and its size depends only $\mathcal{G}$.

Recognizing graphs

Proposition

Let $\mathcal{G}$ be any intersection graph class. For every graph G with **fixed intersection number** (or **fixed overlap number**), deciding "$G \in \mathcal{G}$?" is linear-time solvable.

Proof (sketch)

- Given an input graph G, our algorithm simply checks whether any $H \in \mathcal{H}$ is an induced subgraph of G, determining that $G \notin \mathcal{G}$ if and only if any of these checks turns out positive.
- Since the number and sizes of graphs in $\mathcal{H}$ is constant w.r.t. the size of G and according to the previous lemma, the running-time of this algorithm is linear.