

Quasi-Linear Time Computation of the Abelian Periods of a Word

Gabriele Fici¹ Thierry Lecroq² Arnaud Lefebvre² Elise
Prieur-Gaston² William F. Smyth³

¹ I3S, CNRS & Université Nice Sophia Antipolis, France

² University of Rouen, LITIS EA 4108, France

³ McMaster University, Canada and University of Western Australia, Australia

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Outline

1 Introduction

2 No Head No Tail

3 No Head One Tail

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Notation and definitions

Given a word $w = w[1 \dots n]$ of length n over alphabet $\Sigma = \{a_1, \dots, a_\sigma\}$ of cardinality σ we denote by:

- $w[i]$ its i -th symbol
- $w[i \dots j]$ the factor from the i -th to the j -th symbols
- $|w|_a$ the number of occurrences of symbol a in w
- $\mathcal{P}w = (|w|_{a_1}, \dots, |w|_{a_\sigma})$ its Parikh vector

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Remarks on Parikh vectors

Consider $\mathcal{P}w = (|w|_{a_1}, \dots, |w|_{a_\sigma})$ then

- $\mathcal{P}w[i] = |w|_{a_i}$
- $|\mathcal{P}w| = \sum_{i=1}^{\sigma} \mathcal{P}w[i] = |w|$
- $\mathcal{P}w \in \mathcal{Q}$ iff $\mathcal{P}w[i] \leq \mathcal{Q}[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}w| < |\mathcal{Q}|$

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- $\mathcal{P}_w \in \mathcal{Q}$ iff $\mathcal{P}_w[i] \leq Q[i]$ for every $1 \leq i \leq \sigma$ and $|\mathcal{P}_w| < |\mathcal{Q}|$

Example

\mathcal{P}_{marne}

Remarks on Parikh vectors

Consider $\mathcal{P}_w = (|w|_{a_1}, \dots, |w|_{a_\sigma})$ then

- $\mathcal{P}_w[i] = |w|_{a_i}$
- $|\mathcal{P}_w| = \sum_{i=1}^{\sigma} \mathcal{P}_w[i] = |w|$
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Example

$$\mathcal{P}_{marne} \subset \mathcal{P}_{romane}$$

Remarks on Parikh vectors

Consider $\mathcal{P}_w = (|w|_{a_1}, \dots, |w|_{a_\sigma})$ then

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Example

$\mathcal{P}_{marne} \subset \mathcal{P}_{romane} \subset \mathcal{P}_{harmonie}$

Abelian periods

[Constantinescu and Ilie, 2006] introduced the notion of Abelian period.

Definition

A word w has Abelian period (h, p) iff $w = u_0u_1 \cdots u_{k-1}u_k$ such that:

- $\mathcal{P}_{u_0} \subset \mathcal{P}_{u_1} = \cdots = \mathcal{P}_{u_{k-1}} \supset \mathcal{P}_{u_k}$
- $|\mathcal{P}_{u_0}| = h, |\mathcal{P}_{u_1}| = p$

u_0 is called the *head* and u_k is called the *tail*.

P_w will denote the set of Abelian periods of w .

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a b a b a b a b b b a a$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a a b b | b b a a a b | a b a b a b | b a b b a a}$

$$P_w = \{(0, 6)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a a b b b b a a | a b a b a b a b b a b b a a}$

$$P_w = \{(0, 6), (0, 10)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a b a a b b b b a a a b a b a b a b a b b a a}$

$$P_w = \{(0, 6), (0, 10), (0, 12)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a a b b b b a a a b a b a b a b a b b a a}$

$$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a|b\ a\ a\ b\ b\ b\ b\ a\ a|a\ b\ a\ b\ a\ b\ a\ b\ b\ b|a\ b\ b\ a\ a}$

$$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24), (1, 9)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a|b\ a\ a\ b\ b\ b\ b\ a\ a\ a\ b|a\ b\ a\ b\ a\ b\ a\ b\ b\ a\ a}$

$$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24), \\ (1, 9), (1, 11)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ a \ b \ b \ b \ b \ a \ a} | a \ b \ a \ b \ a \ b \ a \ b | b \ a \ b \ b \ a \ a$

$$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24), \\ (1, 9), (1, 11), \\ (2, 8)\}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ | \ a \ b \ \boxed{b} \ \boxed{b} \ \boxed{b} \ a \ a \ a \ b \ | \ a \ b \ a \ \boxed{b} \ \boxed{a} \ \boxed{b} \ \boxed{b} \ a \ b \ \boxed{b} \ a \ a}$

$$\begin{aligned} P_w = & \{(0, 6), (0, 10), (0, 12), (0, 24), \\ & (1, 9), (1, 11), \\ & (2, 8), \\ & (3, 9)\} \end{aligned}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ a | b \ b \ b \ b \ a \ a \ a | b \ a \ b \ a \ b \ a \ b | b \ a \ b \ b \ a \ a}$

$$\begin{aligned} P_w = & \{(0, 6), (0, 10), (0, 12), (0, 24), \\ & (1, 9), (1, 11), \\ & (2, 8), \\ & (3, 9), \\ & (4, 7)\} \end{aligned}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ a \ b | b \ b \ b \ a \ a \ a \ b | a \ b \ a \ b \ a \ b \ b | a \ b \ b \ a \ a}$

$$\begin{aligned} P_w = & \{(0, 6), (0, 10), (0, 12), (0, 24), \\ & (1, 9), (1, 11), \\ & (2, 8), \\ & (3, 9), \\ & (4, 7), \\ & (5, 7)\} \end{aligned}$$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ a \ b | b \ b \ b \ a \ a \ a \ b \ a \ b \ a \ b \ a \ b \ a \ b \ a \ a}$

$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24),$
 $(1, 9), (1, 11),$
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 $(3, 9),$
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 $(5, 7), (5, 9)\}$

Abelian periods

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = \boxed{a \ b \ a \ a \ b | b \ b \ b \ a \ a \ a \ b \ a \ b \ a \ b \ a \ b \ a \ b \ a \ a}$

$P_w = \{(0, 6), (0, 10), (0, 12), (0, 24), \leftarrow \text{Abelian powers (weak Ap)}$
 $(1, 9), (1, 11),$
 $(2, 8),$
 $(3, 9),$
 $(4, 7),$
 $(5, 7), (5, 9)\}$

Abelian periods

Remark

a^n has n^2 Abelian periods.

Motivations (Christodoulakis & Christou, 2012)

Bioinformatics

- finding CpG islands
- finding clusters of genes
- proteomics: mass spectrometry

Other fields

- approximate pattern matching
- games (letters)

Computing all the Abelian periods

In [Fici *et al*, 2011], algorithms computing all the Abelian periods of a given word are presented: $O(n^2 \times \sigma)$.

Computing all the Abelian periods

In [Fici *et al*, 2011], algorithms computing all the Abelian periods of a given word are presented: $O(n^2 \times \sigma)$.

In [Christou *et al*, 2012], an optimal algorithm, running in $O(n^2)$, is given.

Computing all the Abelian periods

Definition (Christou et al, 2012)

Let w be a word of length n . Then the mapping $pr : \Sigma \rightarrow A$, where A is the set of the first σ prime numbers, is defined by:

$$pr(\sigma_i) = i\text{-th prime number}.$$

The P-signature of w is defined by:

$$P\text{-signature}(w) = \prod_{i=1}^n pr(w[i]).$$

Computing all the Abelian periods

For a word w of length n the array Pr of n elements is defined by

$$Pr[i] = \prod_{j=1}^i pr(w[j]),$$

then

$$P\text{-signature}(w[k .. \ell]) = \begin{cases} Pr[\ell]/Pr[k-1] & \text{if } k \neq 0 \\ Pr[\ell] & \text{otherwise.} \end{cases}$$

Computing all the Abelian periods

Lemma (Christou et al, 2012)

Two words have the same Parikh vector iff they share the same P-signature.

Computing all the Abelian periods

$w = abaab$

i	1	2	3	4	5
$w[i]$	a	b	a	a	b
$pr(w[i])$	2	3	2	2	3
$Pr[i]$	2	6	12	24	72

Computing all the Abelian periods

$w = \text{abaab}$

i	1	2	3	4	5
$w[i]$	a	b	a	a	b
$pr(w[i])$	2	3	2	2	3
$Pr[i]$	2	6	12	24	72

$$P\text{-signature}(w[3..5]) = P\text{-signature}(\text{aab}) = Pr[5]/Pr[2] = 72/6 = 12.$$

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Abelian periods with neither head nor tail

A first solution

Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$

Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$
 $P = \begin{matrix} 1 & 2 & 3 & 4 & & 6 & & 8 & & & & 12 & & & & & & & & & & & & & & & \end{matrix}$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a b a b a b a b b b a b b a a$

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Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a b a b a b a b b b a b b a a$

$P = \underline{2} \quad 3 \quad 4 \quad \quad 6 \quad \quad 8 \quad \quad \quad \quad 12$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a a b a b a b a b b b a b b a a$

$P = \underline{2} \quad 3 \quad 4 \quad \quad 6 \quad \quad 8 \quad \quad \quad \quad 12$

Abelian periods with neither head nor tail

↓ ↓
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a} b \boxed{a a} b b b b a a a a b a b a b a b b b a a$
 $P = \begin{matrix} 2 & 3 & 4 & & 6 & & 8 & & & 12 \end{matrix}$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$
 $P = \quad \textcolor{blue}{3} \quad 4 \quad 6 \quad 8 \quad \dots \quad 12$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a a b a b a b a b b b a b b a a$

$P = \quad 3 \quad 4 \quad 6 \quad 8 \quad \quad \quad 12$

Abelian periods with neither head nor tail

↓ ↓
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a} \boxed{a b b} b b a a a b a b a b a b b b a b a a$
 $P = \quad \begin{matrix} 3 \\ 4 \\ 6 \end{matrix} \quad \begin{matrix} 8 \\ \dots \\ 12 \end{matrix}$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a a b a b a b a b b b a b b a a$

$P = \quad \underline{4} \quad \quad 6 \quad \quad 8 \quad \quad \quad \quad 12$

Abelian periods with neither head nor tail

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
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 $P = \quad \quad \quad \boxed{6} \quad \quad \quad 8 \quad \quad \quad 12$

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↓ ↓
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 $w = \boxed{a b a a b b} b b a a a a b a b a b a b b b a b b a a$
 $P = \begin{matrix} & & 8 & & 12 \\ & & & & \\ & & & & 6 \end{matrix}$

Abelian periods with neither head nor tail



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$w = a b a a b b b b a a a b a b a b a b b b a b b a a$

$P = \quad \quad \quad 8 \quad \quad \quad 12$

6

Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$
 $P = \begin{matrix} & 8 \\ & 12 \\ & 6 \end{matrix}$

Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a a b b b b} \boxed{a a a b a b a b} a b b a b b a a$
 $P = \quad \quad \quad 8 \quad \quad \quad 12 \quad \quad \quad 6$

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a a b a b a b a b b b a b b a a$

$P = 12$

6

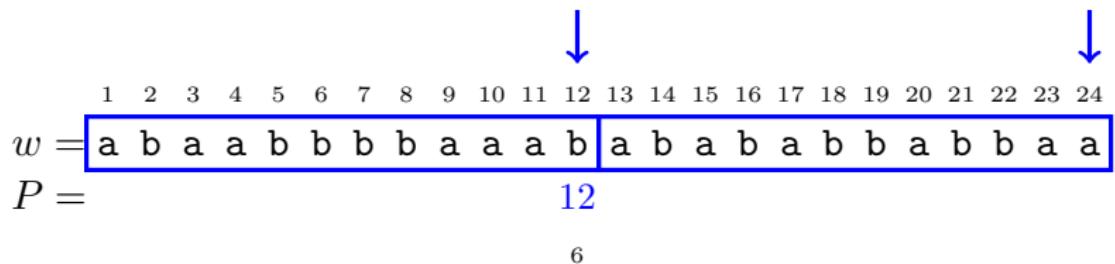
Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a a b a b a b a b b b a a$
 $P = \begin{matrix} 12 \\ 6 \end{matrix}$

Abelian periods with neither head nor tail

$w = \boxed{a \ b \ a \ a \ b \ b \ b \ b \ a \ a \ a \ b} \ a \ b \ a \ b \ a \ b \ b \ b \ a \ b \ b \ a \ a$

$P = \begin{matrix} 12 \\ 6 \end{matrix}$



Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b a a b b b b a a a b} a b a b a b b b a b b a a$
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6

Abelian periods with neither head nor tail



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

$w = a b a a b b b b a a a b a b a b a b b b a b b a a$

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12

6

Abelian periods with neither head nor tail

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 $w = \boxed{a b a a b b} b b a a a b \boxed{a b a b a b} b a b b a a$
 $P = 6$

Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
w = $\boxed{a \ b \ a \ a \ b \ b} \ b \ b \ a \ a \ a \ b \ \boxed{a \ b \ a \ b \ a \ b}$ $\boxed{b \ a \ b \ b \ a \ a}$
P = $\begin{matrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{matrix}$ 6 12



Abelian periods with neither head nor tail

↓
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$
 $P = \underline{6} \quad 12$

Abelian periods with neither head nor tail

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 $w = a b a a b b b b a a a b a b a b a b b b a a$
 $P = \quad \quad \quad 6 \quad \quad \quad 12$



Abelian periods with neither head nor tail

$w = \boxed{a b a a b b} b b a a a b a b a b a b \boxed{b a b b a a}$

$P = \begin{matrix} & & & & & & & & & & & & & & & 6 \\ & & & & & & & & & & & & & & & 12 \end{matrix}$

↓ ↓

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

Abelian periods with neither head nor tail

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 $w = \boxed{a b a a b b} b b a a a b a b a b a b \boxed{b a b b a a}$
 $P =$ 12
6



Abelian periods with neither head nor tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a b b a a$
 $P = \begin{matrix} 12 \\ 6 \end{matrix}$

Complexity

Theorem 1

The first algorithm computes all the Abelian periods with neither head nor tail of a word w of length n in time $O(n \log \log n)$ and space $O(n)$.

Abelian periods with neither head nor tail

A second solution

Abelian periods with neither head nor tail

Definition

$$e = \gcd(\mathcal{P}_w[1], \mathcal{P}_w[2], \dots, \mathcal{P}_w[\sigma])$$

Complexity: $O(\sigma + \log n/\sigma)$

Abelian periods with neither head nor tail

Definition

$$e = \gcd(\mathcal{P}_w[1], \mathcal{P}_w[2], \dots, \mathcal{P}_w[\sigma])$$

Complexity: $O(\sigma + \log n/\sigma)$

Definition

D : stack of all divisors of e in ascending order.

Complexity: $O(\sqrt{e})$.

Abelian periods with neither head nor tail

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D : stack of all divisors of e in ascending order.

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Definition

$$s = n/e$$

Abelian periods with neither head nor tail

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$$e = \gcd(\mathcal{P}_w[1], \mathcal{P}_w[2], \dots, \mathcal{P}_w[\sigma])$$

Complexity: $O(\sigma + \log n/\sigma)$

Definition

D : stack of all divisors of e in ascending order.

Complexity: $O(\sqrt{e})$.

Definition

$$s = n/e$$

Observation

The only possible Abelian periods p of w are of the form $p = d \times s$, where d is an entry in D .

Thus the smallest possible period is s .

Abelian periods with neither head nor tail

Definition

Recall that $s = n/e$. A factor $w[i..j]$ is a **segment** of w if:

- ① $i = k \times s + 1$ with $k \geq 0$;
- ② $j - i + 1 = t \times s$ with $t \geq 1$;
- ③ $\mathcal{P}_{w[i..j]}[k]/(j - i + 1) = \mathcal{P}_w[k]/|w|$ for every letter $\sigma_k \in \Sigma$;
- ④ there does not exist a $j' < j$ such that $j' - i + 1 = t' \times s$ and
 $\mathcal{P}_{w[i..j']}[k]/(j' - i + 1) = \mathcal{P}_w[k]/|w|$ for every letter $\sigma_k \in \Sigma$.

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- ② $j - i + 1 = t \times s$ with $t \geq 1$;
- ③ $\mathcal{P}_{w[i..j]}[k]/(j - i + 1) = \mathcal{P}_w[k]/|w|$ for every letter $\sigma_k \in \Sigma$;
- ④ there does not exist a $j' < j$ such that $j' - i + 1 = t' \times s$ and $\mathcal{P}_{w[i..j']}[k]/(j' - i + 1) = \mathcal{P}_w[k]/|w|$ for every letter $\sigma_k \in \Sigma$.

In other terms, the definition of segments corresponds to a factorization into factors of minimal size having the same proportion of each alphabet letter as the whole word.

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b a b b a b a a$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

	1
	2
	3
	4
	6
D	12

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b a a a b a b a b a b b b a b b a a$
 $L = 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = ?$$

	1
	2
	3
	4
	6
D	12

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a} b a a b b b b a a a b a b a b a b b b a b a a$
 $L = 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

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$$T = ?$$

	1
	2
	3
	4
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D	12

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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a\ b} a\ a\ b\ b\ b\ b\ a\ a\ a\ b\ a\ b\ a\ b\ a\ b\ b\ b\ a\ b\ a\ a$
 $L = 1\ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 1$$

	1
	2
	3
	4
	6
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$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a} b \boxed{a} a b b b b a a a a b a b a b a b b b a b a a$
 $L = 1 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

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	1
	2
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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a} b \boxed{a} a b b b b a a a a b a b a b a b b b a b a a$
 $L = 1 0$

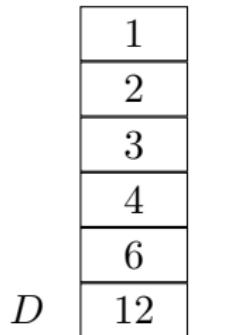
$$n = 24$$

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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b} \ a \ a \ b \ b \ b \ b \ a \ a \ a \ a \ b \ a \ b \ a \ b \ a \ b \ b \ b \ a \ b \ b \ a \ a$
 $L = 1 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 1$$

	1
	2
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Example

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 $w = \boxed{a \ b} \ a \ a \ b \ b \ b \ b \ a \ a \ a \ b \ a \ b \ a \ b \ a \ b \ b \ b \ a \ b \ b \ a \ a$
 $L = 1 \ 0 \ 1 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

	1
	2
	3
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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a} \boxed{b} \boxed{a} \boxed{a} \boxed{b} \boxed{b} \boxed{b} b a a a a b a b a b a b a b b a b a b b a a$
 $L = 1 0 1 0$

$$n = 24$$

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Example

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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a} \ a \ a \ b \ a \ b \ a \ b \ a \ b \ b \ a \ b \ a \ b \ a \ a$
 $L = 1 \ 0 \ 1 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

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 $L = 1 0 1 0 0 0 1 0$

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 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0$

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 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

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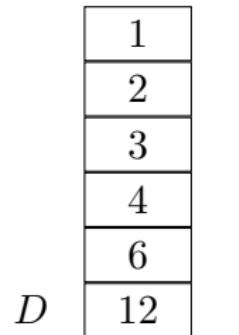
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Example

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 $w = \boxed{a} \boxed{b} \boxed{a} \boxed{a} \boxed{b} \boxed{b} \boxed{b} \boxed{b} \boxed{a} \boxed{a} \boxed{a} \boxed{b} \boxed{a} \boxed{b} \boxed{a} \boxed{b} \boxed{a} \boxed{b} \boxed{b} \boxed{b} \boxed{a} \boxed{b} \boxed{b} \boxed{a} \boxed{a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

	1
	2
	3
	4
	6
D	12

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0 0 0 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

	1
	2
	3
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$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b a b a b | b a b b a} a$
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	2
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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

	1
	2
	3
	4
	6
D	12

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

	1
	2
	3
	4
	6
D	12

$$d = ?$$

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b | a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$

D	2
	3
	4
	6
	12

$$d = 1$$

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b | a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$

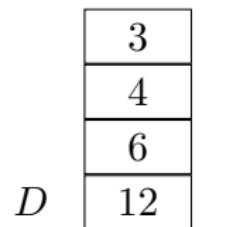
$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$



$$d = 2$$

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ a \ b \ | \ a \ b \ a \ b \ | \ b \ a \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$
↑ ↑

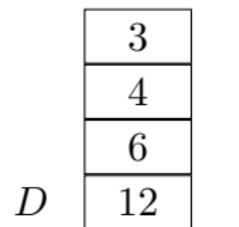
$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$



$$d = 2$$

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b | a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$

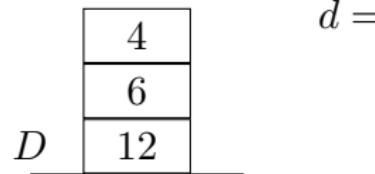
$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$



$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$
↑ ↑ ↑ ↑

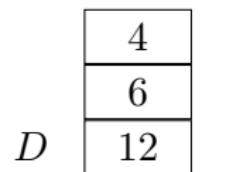
$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$



$$d = 3$$

$$R = \{\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$
↑ ↑ ↑ ↑

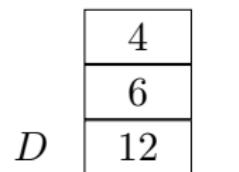
$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$s = 2$$

$$T = 2$$



$$d = 3$$

$$R = \{6\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$

$$n = 24$$

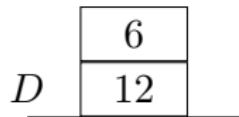
$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$d = 4$$

$$s = 2$$

$$T = 2$$



$$R = \{6\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b | a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$
↑ ↑

$$n = 24$$

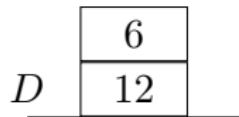
$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b a b a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$

$$n = 24$$

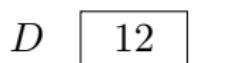
$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$d = 6$$

$$s = 2$$

$$T = 2$$



$$R = \{6\}$$

Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a \ b \ | \ a \ a \ b \ b \ | \ b \ b \ a \ a \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ a \ b \ | \ b \ a \ | \ b \ b \ a \ a}$
 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$



$$n = 24$$

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$$e = 12$$

$$d = 6$$

$$s = 2$$

$$T = 2$$

$$D \quad \boxed{12}$$

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 $L = 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0$



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Example

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = \boxed{a b | a a b b | b b a a | a b | a b a b | a b a b | b a b b a a}$
 $L = 1 0 1 0 0 0 1 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0$

$$n = 24$$

$$\mathcal{P}_w = (12, 12)$$

$$e = 12$$

$$d = 12$$

$$s = 2$$

$$T = 2$$

$$\underline{\underline{D}}$$

$$R = \{6, 12\}$$

Complexity

Theorem 2

The second algorithm computes all the Abelian periods with empty head and empty tail of w in time $O(n \log \log n)$ and in space $O(n)$.

Outline

1 Introduction

2 No Head No Tail

3 No Head One Tail

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
2 3 4 5 6 7 8 9 10 11 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
3 4 5 6 7 8 9 10 11 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
4 5 6 7 8 9 10 11 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
5 6 7 8 9 10 11 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
6 7 8 9 10 11 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
7 8 9 10 11 12
6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
8 9 10 11 12
6

Abelian periods with no head and a possibly non-empty tail

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Abelian periods with no head and a possibly non-empty tail

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 $w = a b a a b b b b a a a b a b a b a b b b a a$
10 11 12
6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a \ b \ a \ a \ b \ b \ b \ b \ a \ a \ a \ b \ a \ b \ a \ b \ a \ b \ b \ b \ a \ a$

11 12 10

6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
12 10
6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$

10 12

6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$
6 10 12

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b a b b a a$

10 12
 6

Abelian periods with no head and a possibly non-empty tail

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
 $w = a b a a b b b b a a a b a b a b a b b b a a$

12

6

10

Complexity

Theorem 2

The third algorithm computes all the Abelian periods with empty head and a possibly non-empty tail of w in time $O(n \log n)$ and in space $O(n)$.

Conclusion

- We propose new algorithms for computing Abelian periods of words:
 - ▶ no head no tail
 - ▶ no head one tail

Future works

- improving complexities
- computing the Abelian local period array of a given word, as for the classical periods
- computing the Abelian border array of a given word, as for the classical periods