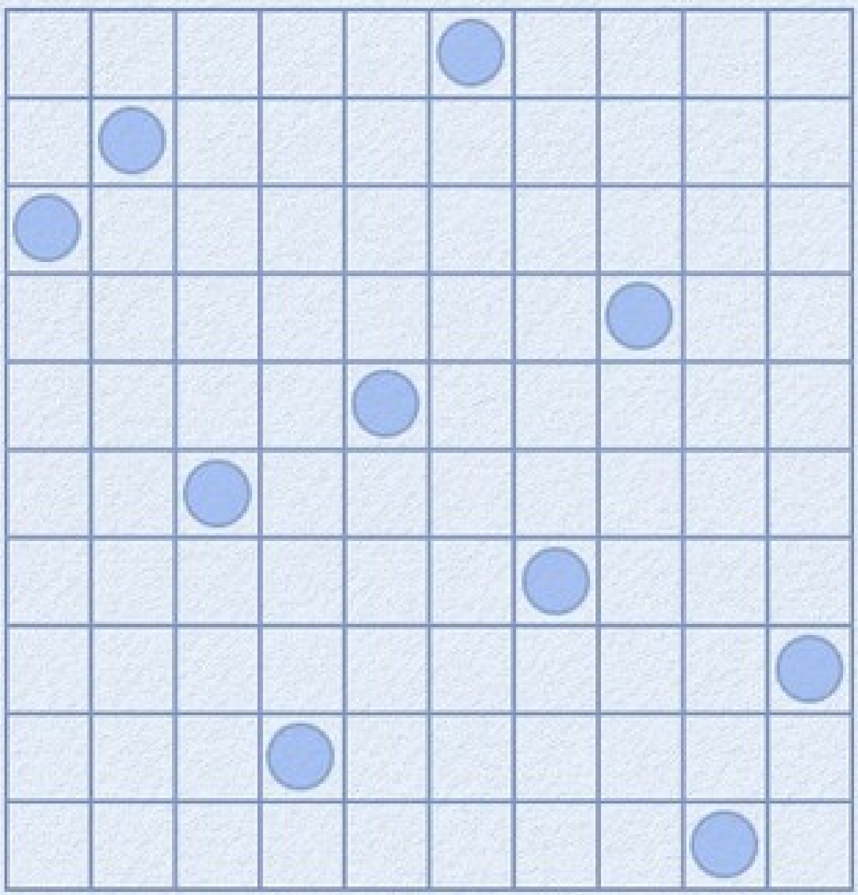
On Stanley-Wilf-Marcüs-Tardös Theorem

Dominique Rossin
LIX, CNRS, Ecole Polytechnique
Digiworlds, ParisTech, Digiteo, Université Paris Saclay

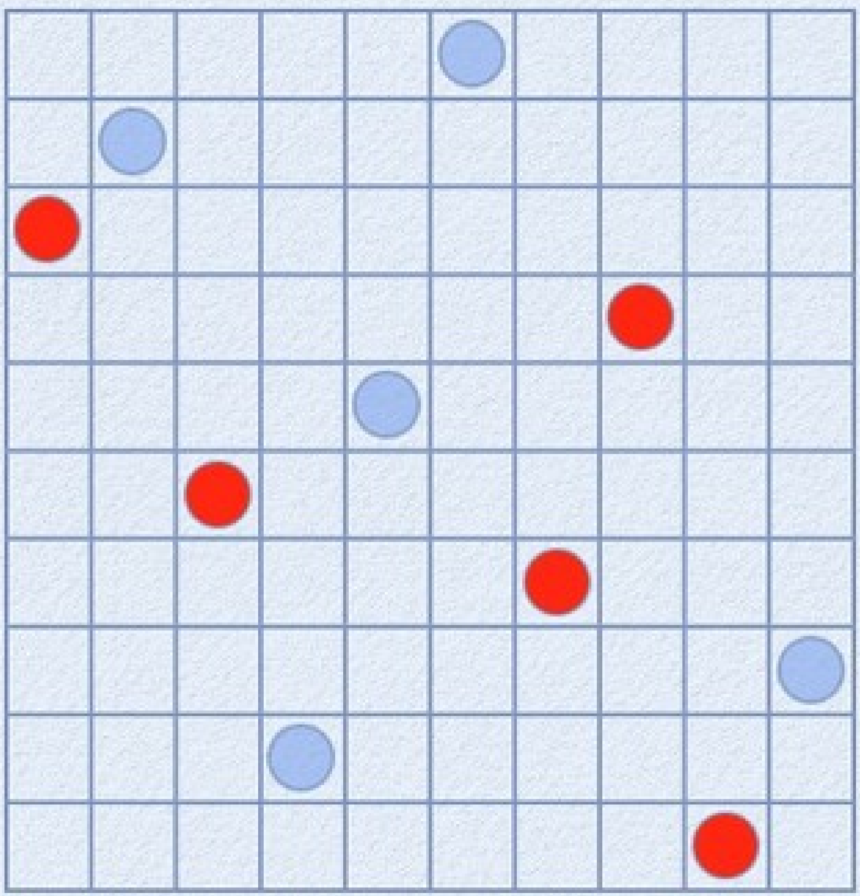
On Stanley-Wilf-Marcus-Tardös-Füredi-Hajnal-Klazar Theorem

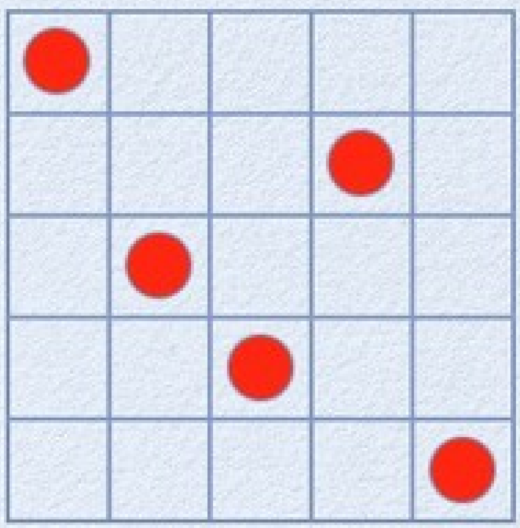
Patterns



8 9 5 2 6 10 4 7 1 3

PATTERNS





PERMUTATION CLASSES

We say that σ contains π

$\pi = 5\ 3\ 2\ 4\ 1 \quad < \quad \sigma = 8\ 9\ 5\ 2\ 6\ 10\ 4\ 7\ 1\ 3$

Otherwise σ avoids π

Permutations classes

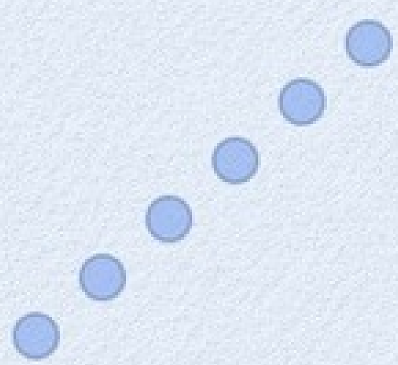
$\parallel\parallel$

Sets stable under $<$

$\sigma \in C$
$\pi < \sigma$ $\Rightarrow$ $\pi \in C$

21 avoiding permutations

- Which permutations avoid 21 ?
  - No decreasing sequences
  - Identity !



PERMUTATION CLASSES

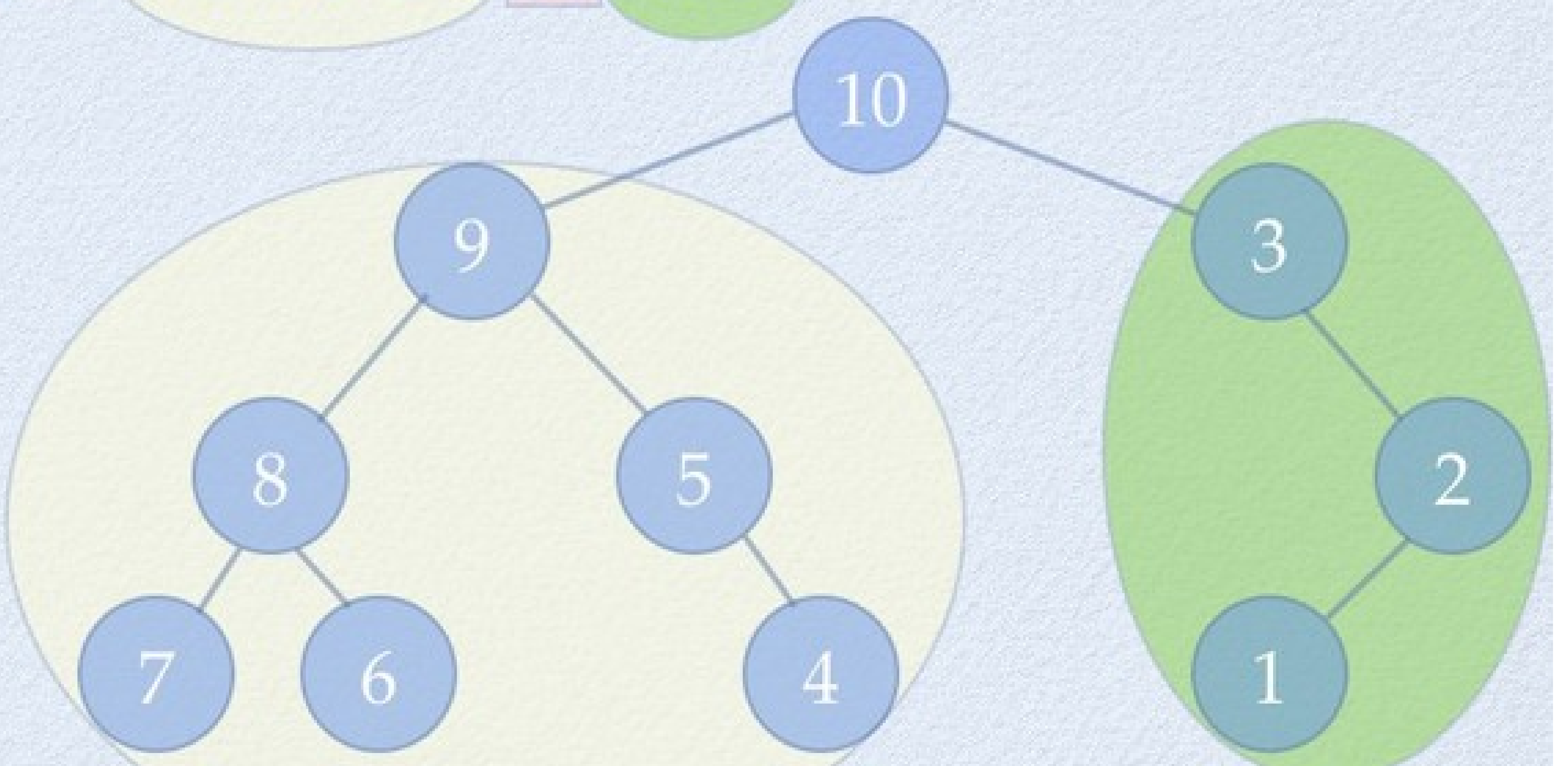
Permutations classes ≣ Sets stable under <

⫲ ⫳

Permutations that avoid a permutation in B

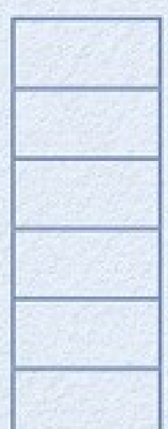
$\sigma = 7\ 8\ 6\ 9\ 5\ 4\ 10\ 3\ 1\ 2$

Example:
B = {132}
$\sigma \in Av(132)$



GOOD DESCRIPTION !

- Pattern-avoiding definition
  - C = Av(B), e.g. Av(231)
- Defined by a property
  - Permutations sortable by one stack ! (Knuth 73)

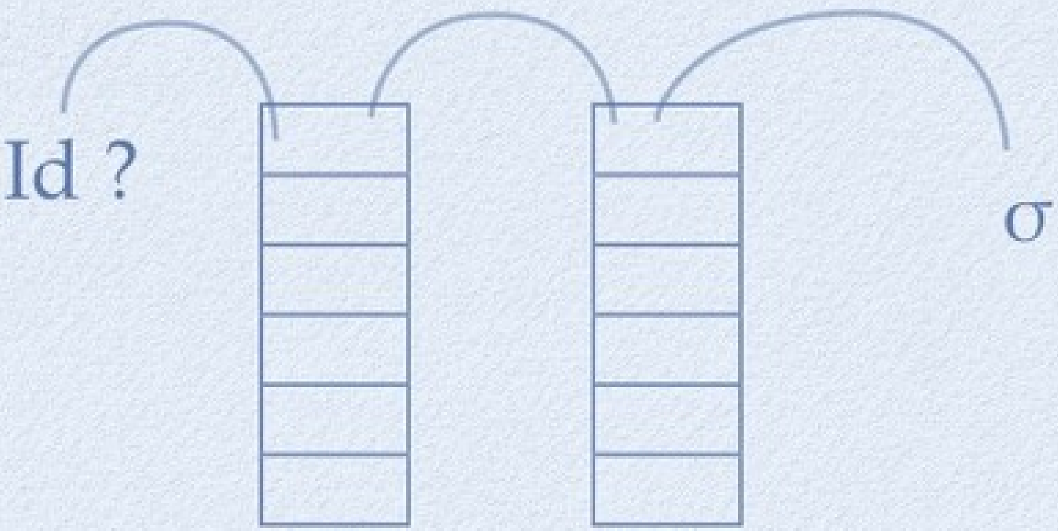


3 1 2 6 5 4

Av(231)

2-stacks sortable

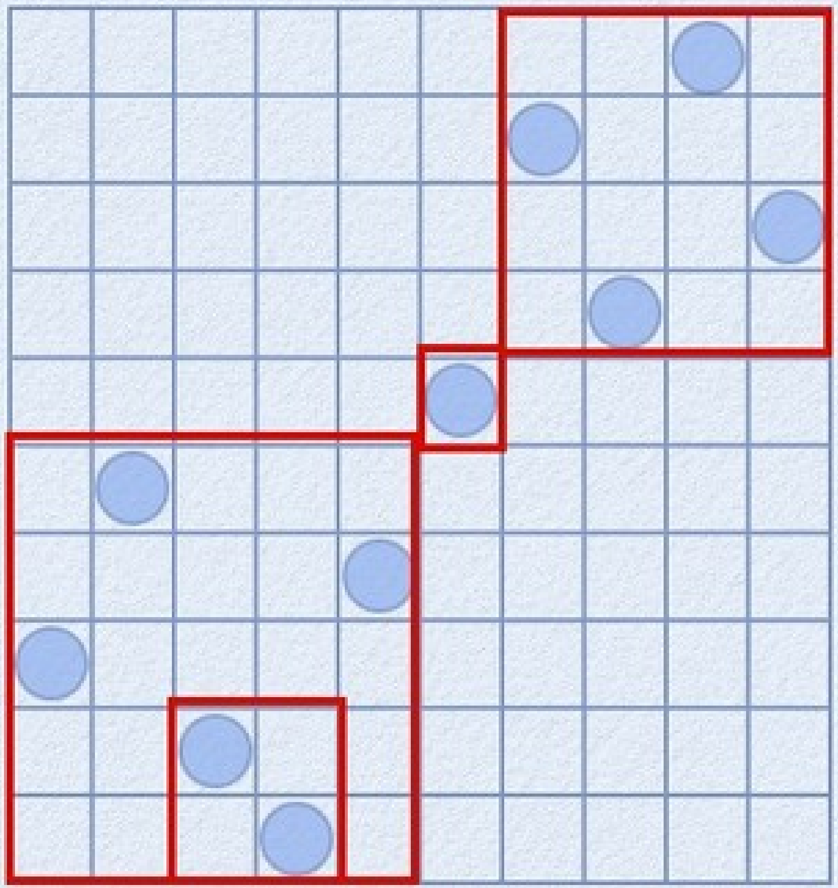
- Stable by pattern



Av(B) but B is infinite

DECOMPOSITION TREES

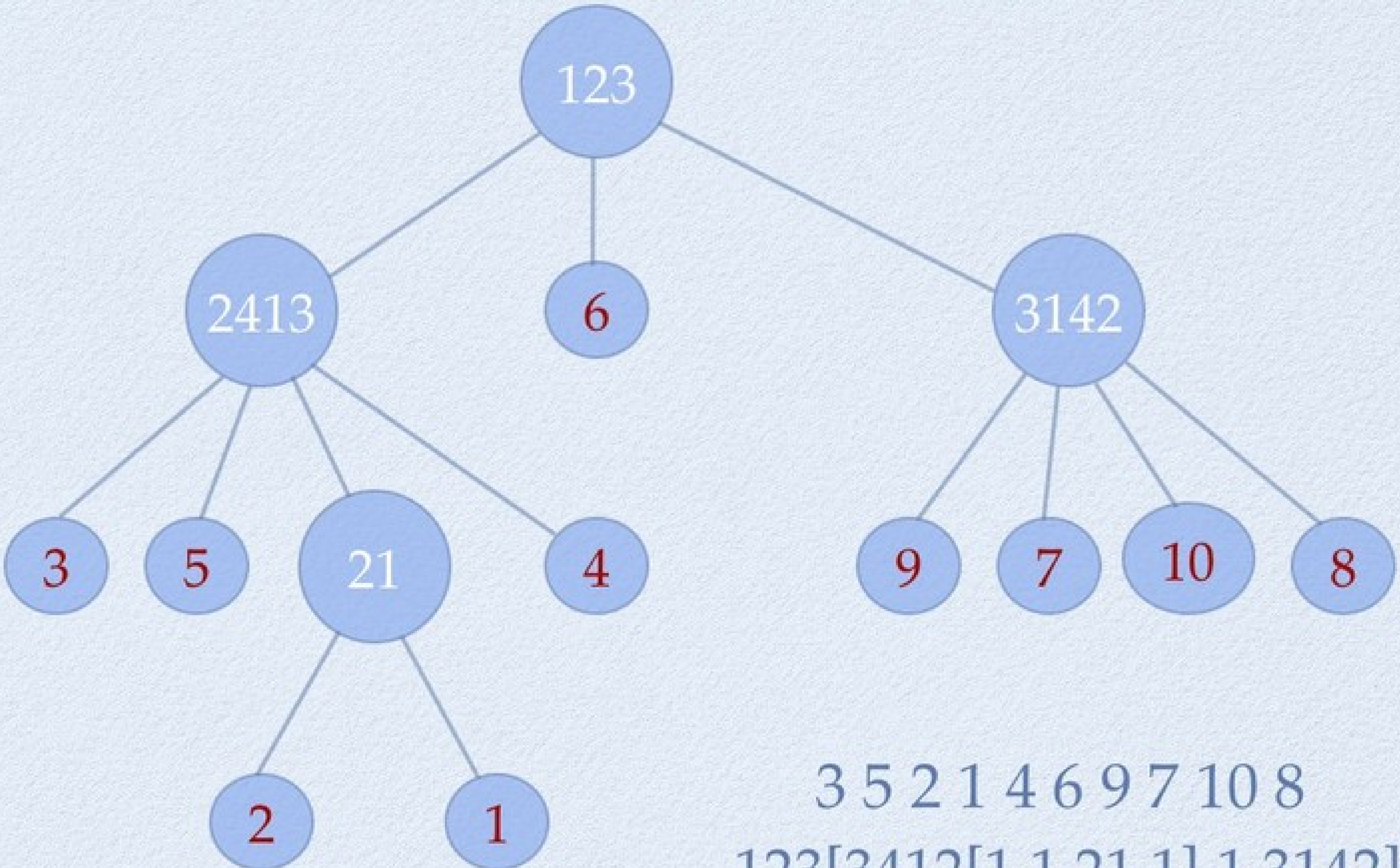
3 5 2 1 4 6 9 7 10 8



123[35214,1,3142]

123[3412[1,1,21,1],1,3142]

WREATH-CLOSED CLASSES



3 5 2 1 4 6 9 7 10 8

123[3412[1,1,21,1],1,3142]

Wreath-closed

- X a set of simple (prime) permutations
- P : Set of permutations whose decomposition tree have node labeled Id,-Id,X
- P is a wreath-closed class

  ↕

- P = Av(B) with B containing only simple permutations

ALGORITHMIC QUESTIONS

- Let π, σ permutations. Does $\pi < \sigma$?
  - NP-Hard problem
  - Naïve algorithm in $|\sigma|^{|\pi|}$
  - Open Problem (FPT):
    - Does there exist an algorithm of complexity $f(|\pi|)\, P(\sigma)$

QUESTIONS ?

$C = Av(B)$

- If $\sigma \in Sn$, decide if $\sigma \in C$.
  - For each $\pi \in B$, decide if $\pi < \sigma$?
- How many permutations s_n of size n in C ?
  - Stanley-Wilf (conjecture / theorem) 91-04
    - If $B=\{\pi\}$, $(s_n)^{1/n} = c_\pi$
    - In other terms the number of permutations of size n grows like c_π^n

Class Examples

- Patterns of size 3:
  - Av(132) : Binary trees with n nodes.
  - Catalan numbers. $\lim(s_n)^{1/n} = c_\pi = 4$
- Regev (81) $c_{12..k} = (k-1)^2$
- Conjecture (West 90): $c_\pi = f(|\pi|)$
- Disproved Bona 97 : False for $|\pi| = 4$

90-04:EXHAUSTIVE STUDY

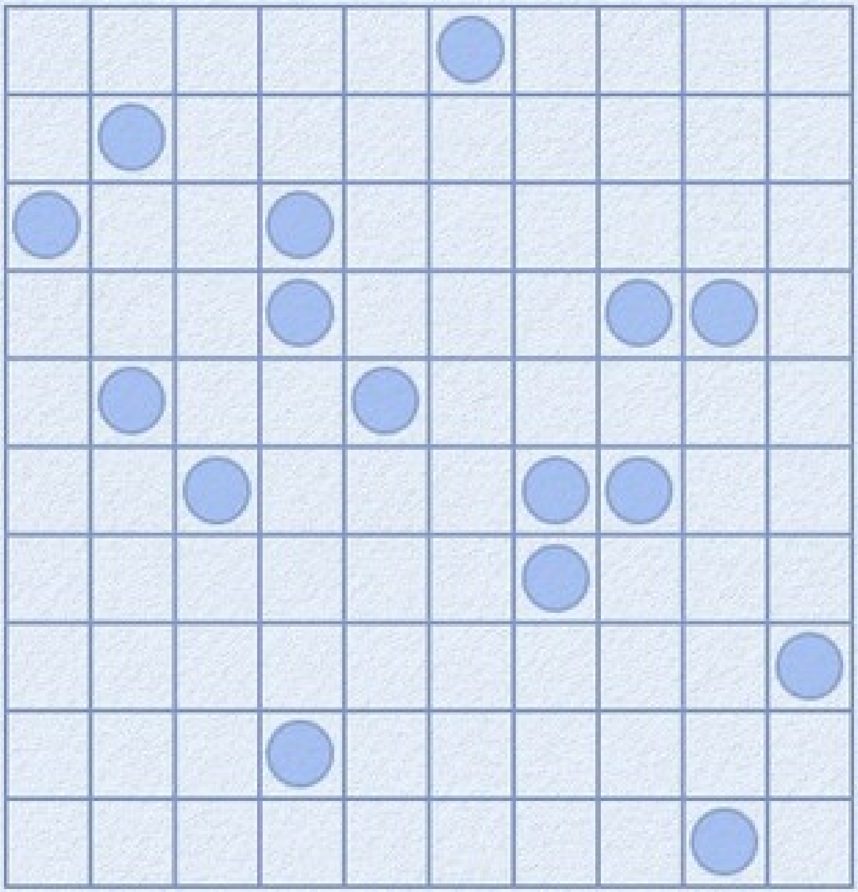
- Size 4 patterns (up to symetry and reverse)
  - 1234 (Gessel 90) ... $c_\pi=9$
  - 1342 (Bona 97). G.F :

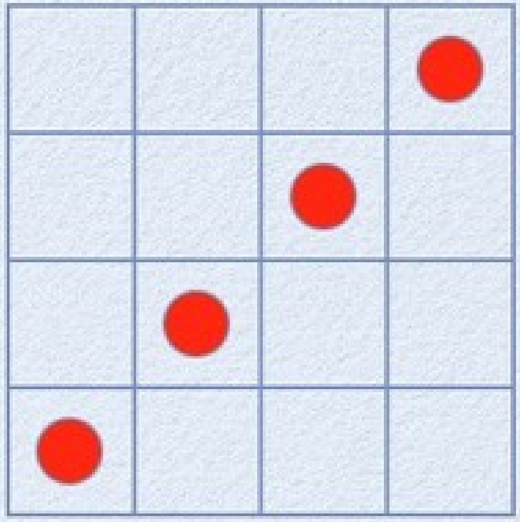
    $F(x) = 32x/(1+20x-8x^2-(1-8x)^{3/2})$
  - 1324 : Unknown .... wait the end

THE PROOF

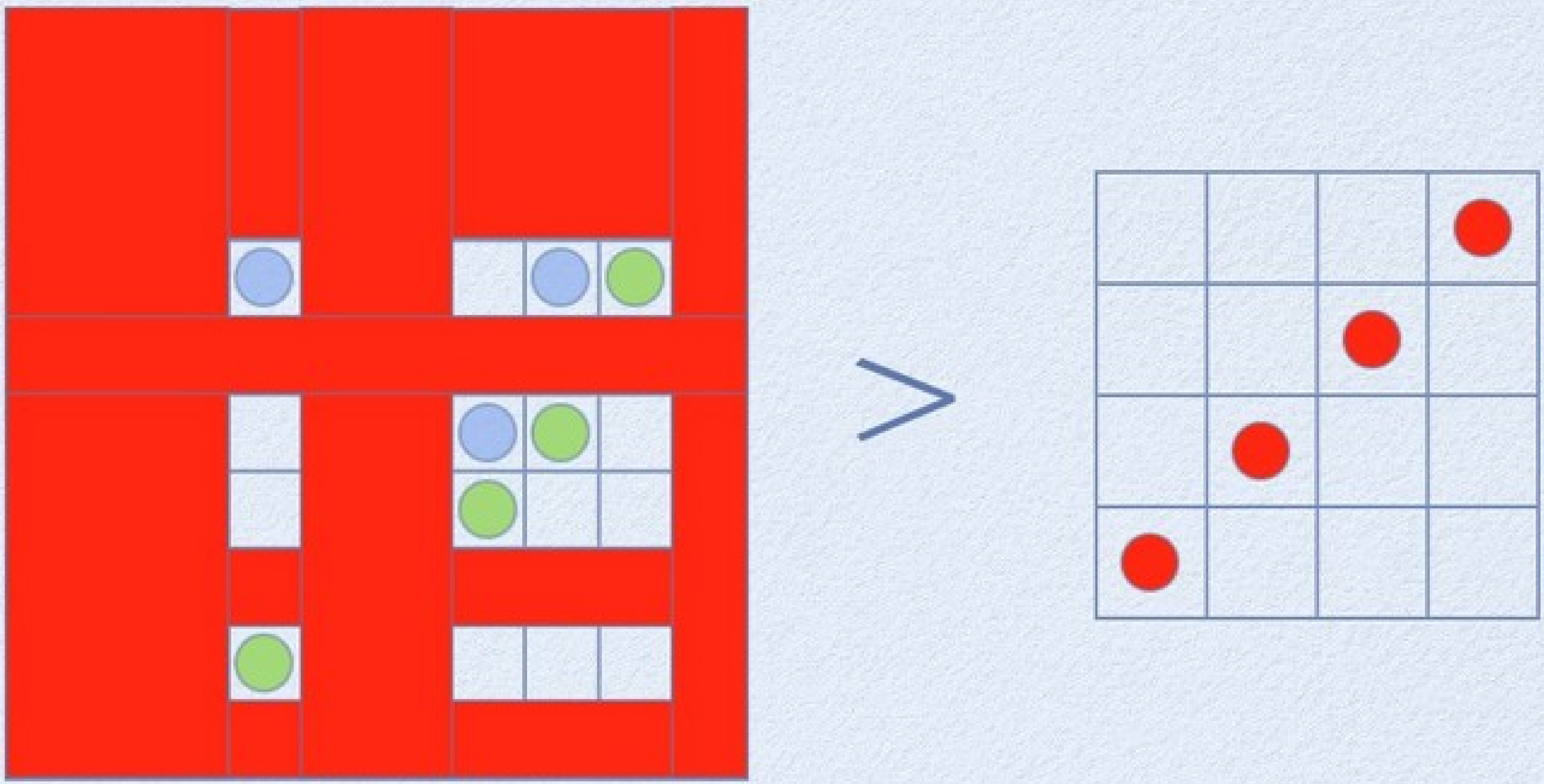
- Füredi-Hajnal
  - Stanley-Wilf equivalence on 0-1 matrices
- Stanley-Wilf

Füredi–Hajnal

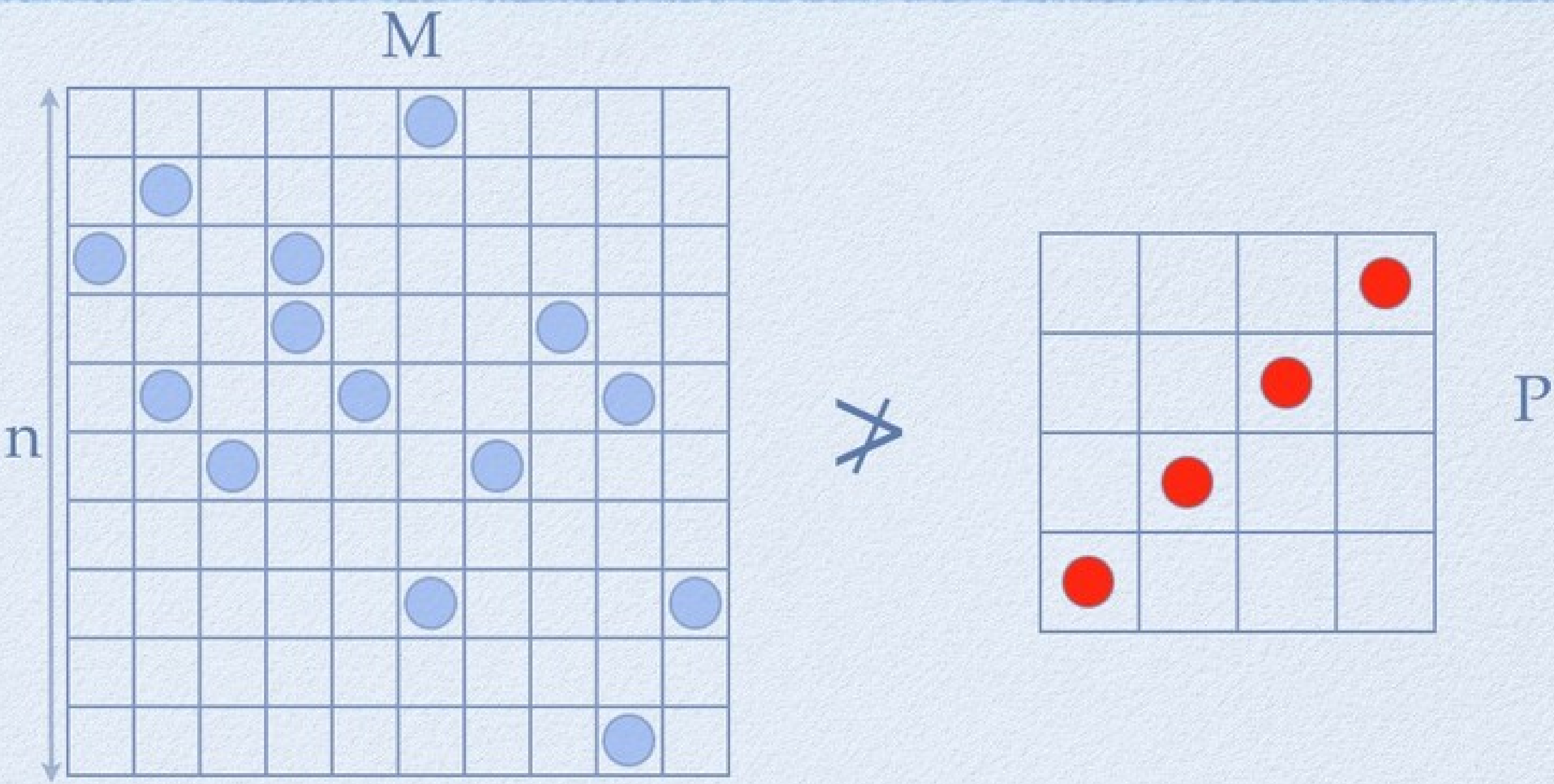




Füredi–Hajnal



FÜREDI–HAJNAL



Maximal number of points f(n,P) to avoid P

Füredi - Hajnal : f(n,P) < c n

FÜREDI–HAJNAL

Maximal number of points $f(n,P)$ to avoid P

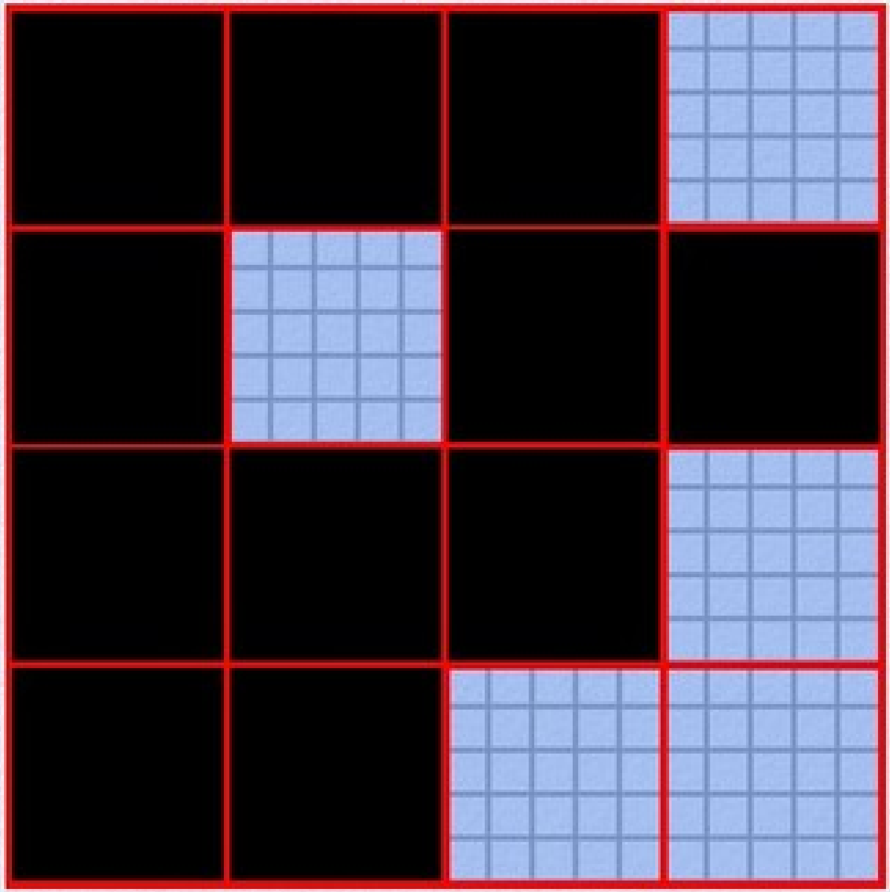
Füredi - Hajnal : $f(n,P) < c\,n$

$\Downarrow$ Klazar

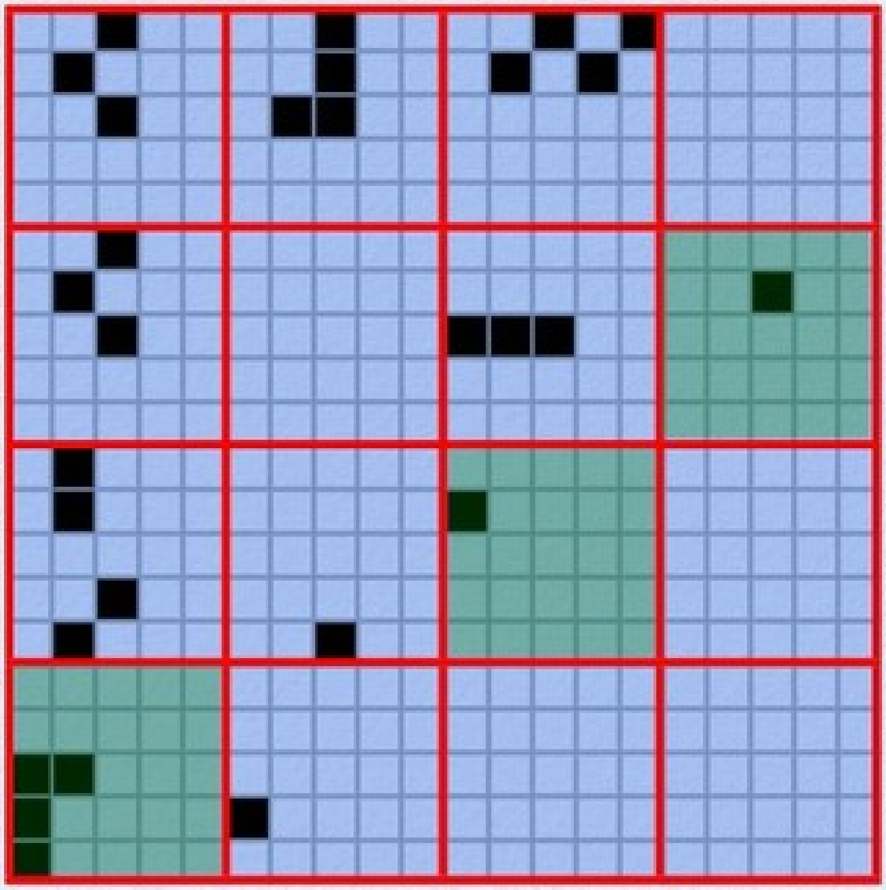
Stanley-Wilf (conjecture / theorem)

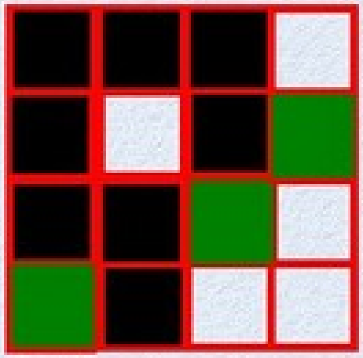
If $B=\{\pi\}$, $(s_n)^{1/n} = c_\pi$

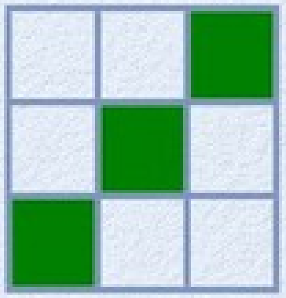
REDUCE



REDUCE

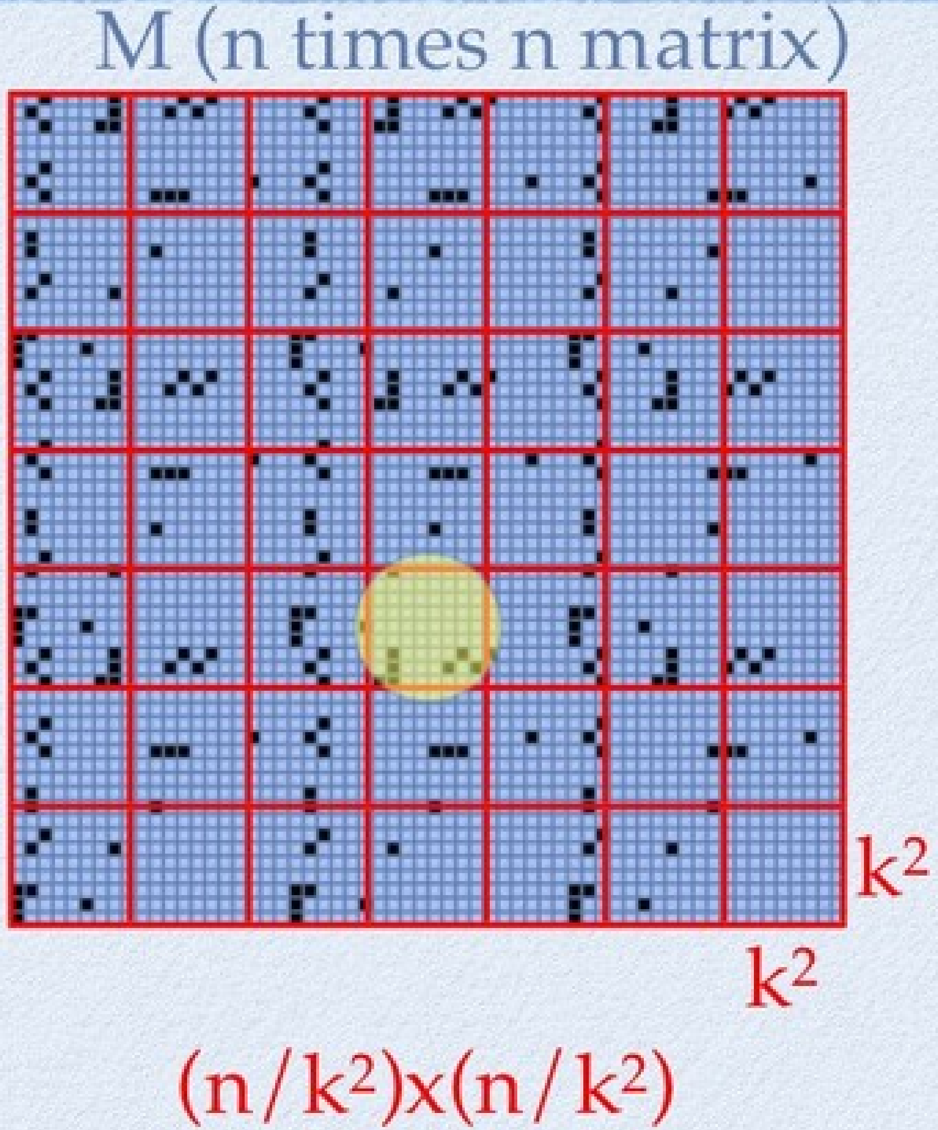


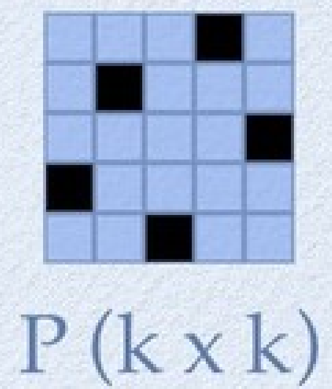




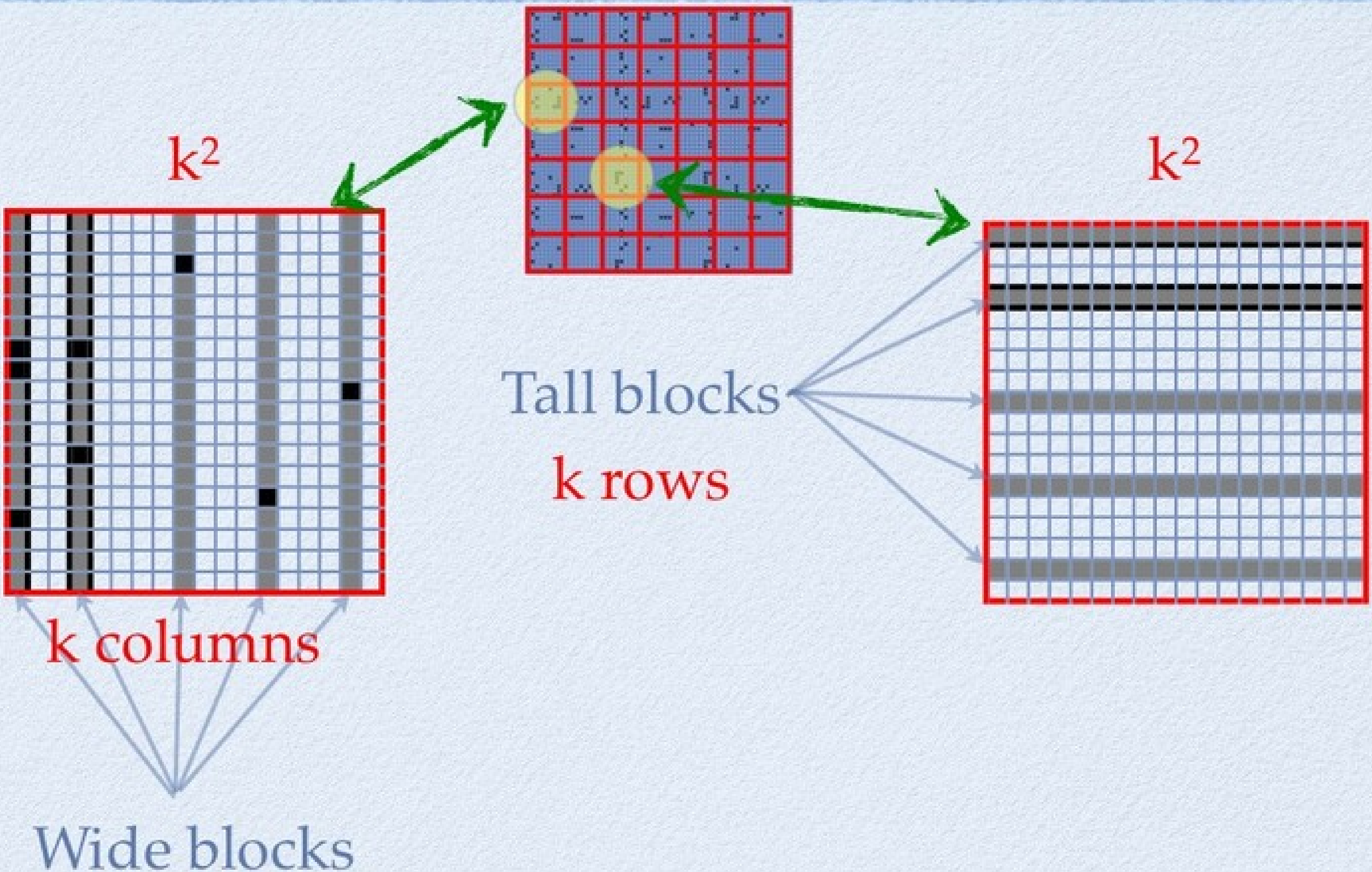
M avoids P ⇨ B avoids P

HOW MANY ONES ?

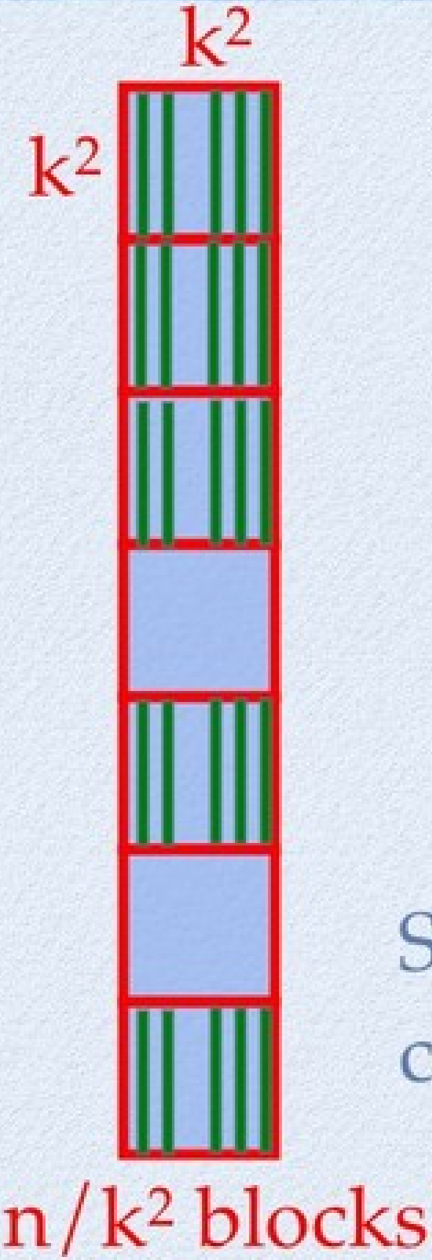


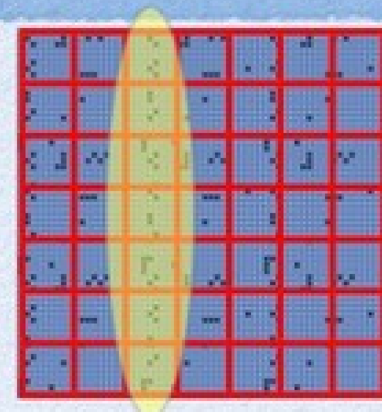


WIDE OR TALL ?

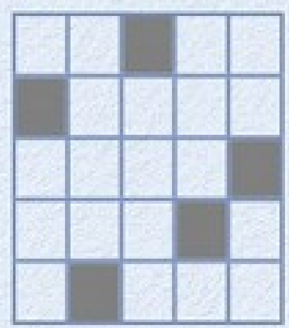


How many wide blocks in a column ?



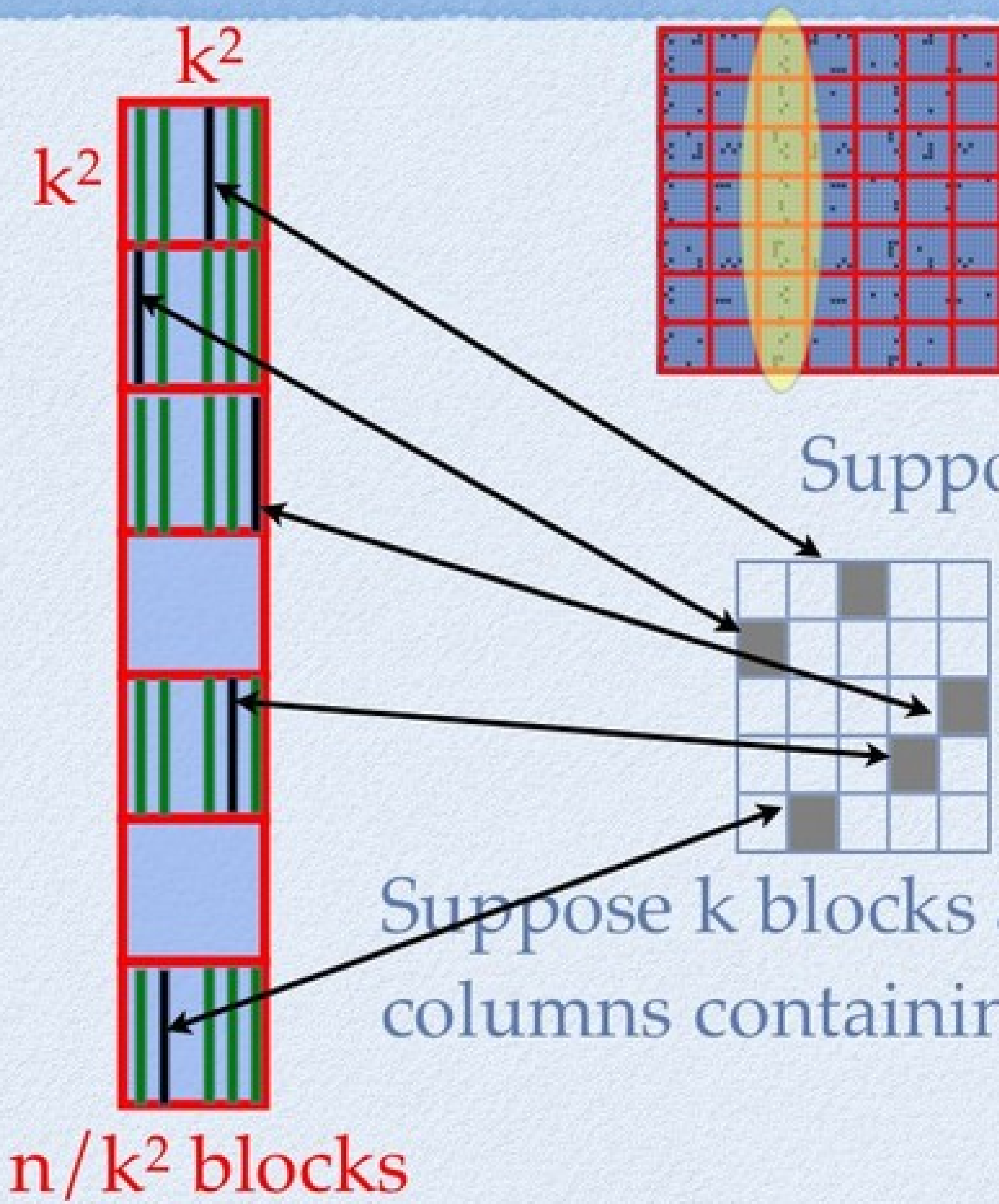


Suppose k=5 (pattern)



Suppose k blocks are wide with the same k columns containing a 1.

How many wide blocks in a column ?



Suppose k=5 (pattern)

Suppose k blocks are wide with the same k columns containing a 1.

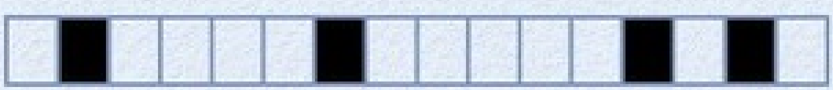
PIGEONHOLE PRINCIPLE

n pigeonholes

kn+1 pigeons. At least k+1 pigeons in one hole.

Find k blocks with the same k columns non-empty

k^2



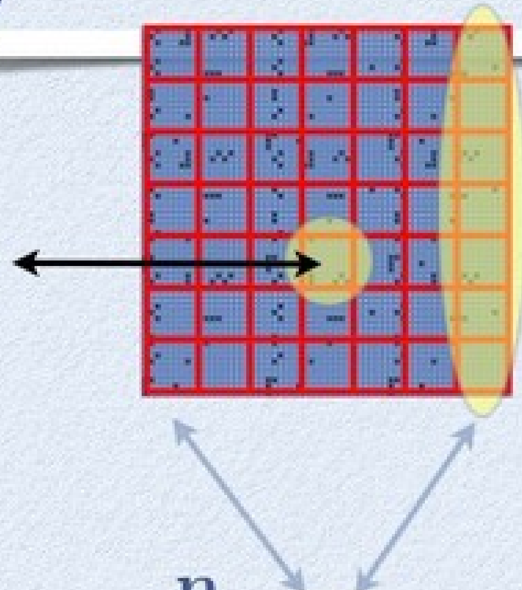
$\binom{k^2}{k}$ ways to choose k columns

$k\binom{k^2}{k}$ wide blocks in a column

WIDE BLOCKS

$\mathbf{k}\binom{\mathbf{k}^2}{\mathbf{k}}$ wide blocks in a column

$\mathbf{k}^4$ ones in a block

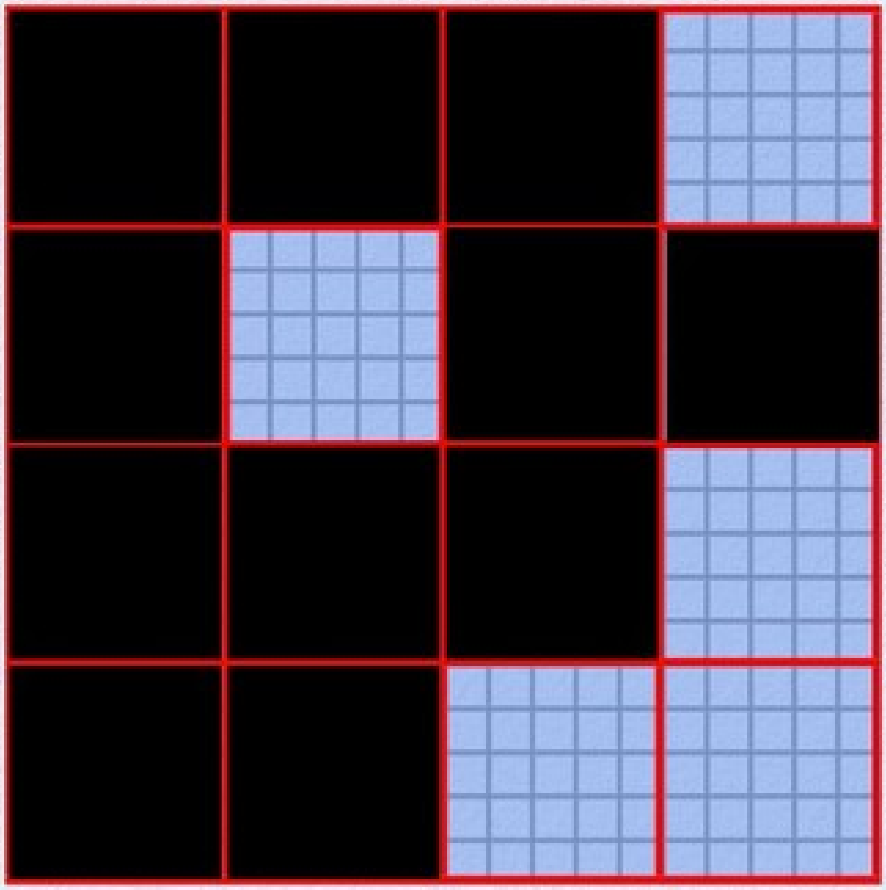


$\frac{\mathbf{n}}{\mathbf{k}^2}$ columns

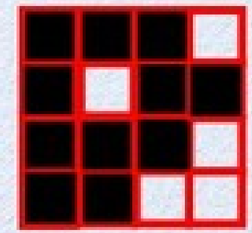
$\mathbf{k}^3\binom{\mathbf{k}^2}{\mathbf{k}}\mathbf{n}$ ones in wide blocks

Neither wide nor tall blocks

M



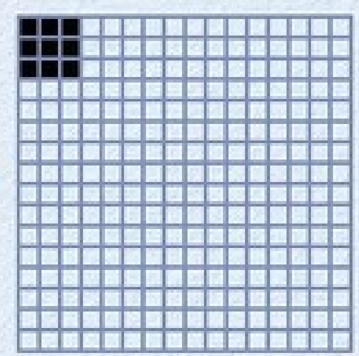
How many normal blocks ?



$\mathbf{f}(\frac{\mathbf{n}}{\mathbf{k^2}}, \mathbf{P})$ blocks

How many ones in a block ?

$k^2=16$



3 squares

$(\mathbf{k}-\mathbf{1})^{\mathbf{2}}$

- Count the ones in normal blocks

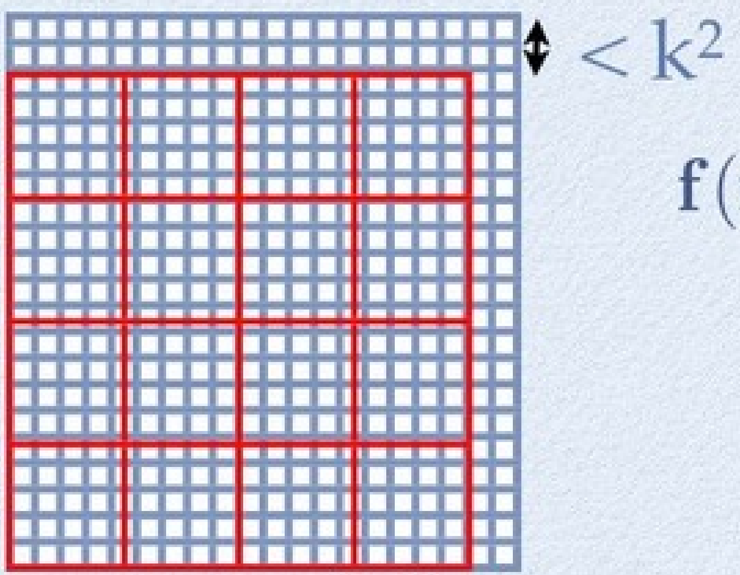
END OF PROOF (1)

$$\mathbf{f(n,P) \leq (k-1)^2 f(n/k^2, P) + 2k^3 \binom{k^2}{k} n}$$

Number of ones in a matrix that avoids P

Blocks neither wide nor tall

Ones in tall blocks
Ones in wide blocks

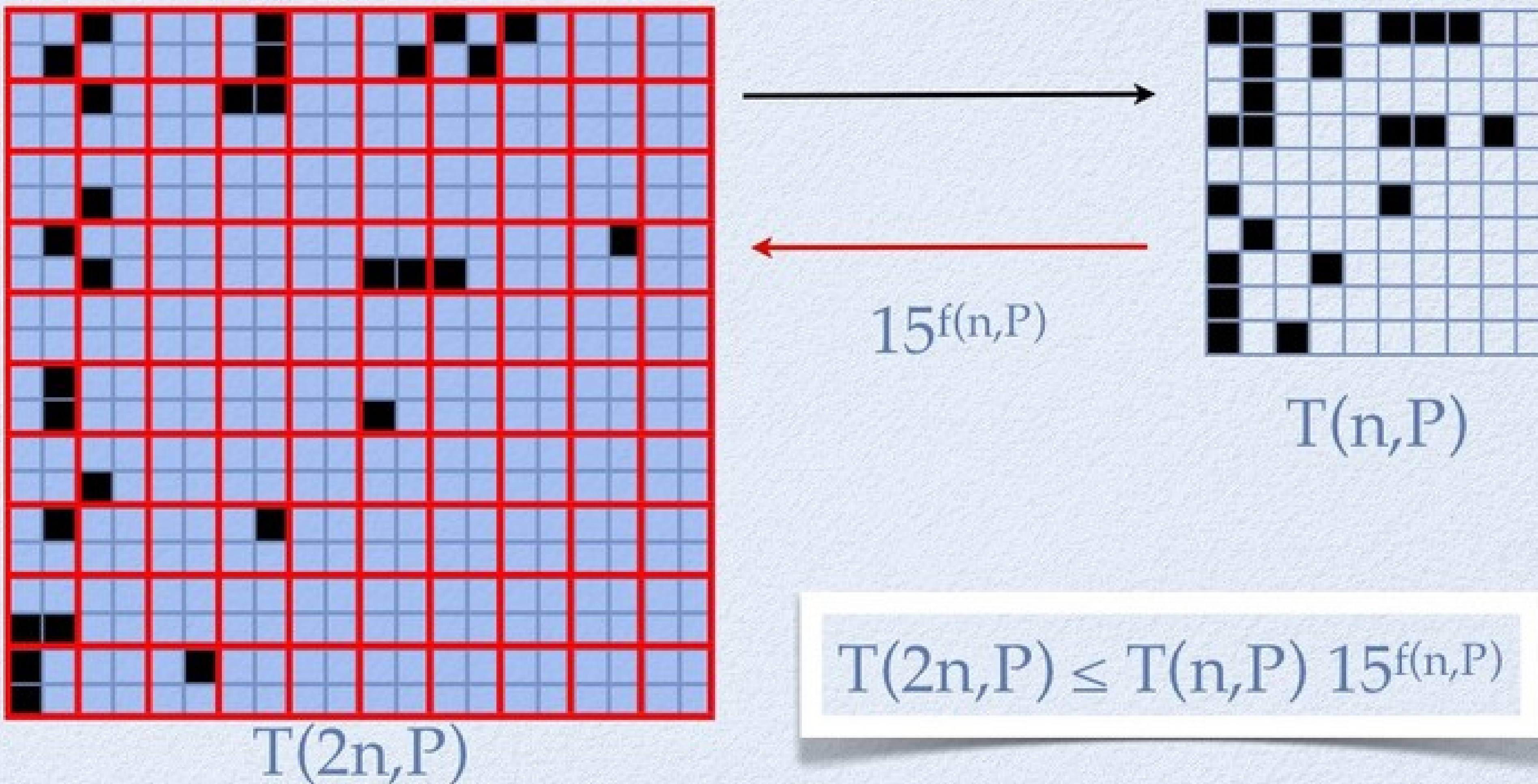


$$\mathbf{f(n,P) \leq f(k^2 n', P) + 2k^2 n}$$

$$\boxed{\mathbf{f(n,P) \leq 2k^4 \binom{k^2}{k} n}}$$

Füredi-Hajnal ⇒(Klazar) Stanley-Wilf

- Number of ones f(n,P)
- Number of matrices T(n,P) that avoid P



$$T(2n,P) \leq T(n,P)\, 15^{f(n,P)}$$

AND NOW ?

Füredi-Hajnal $\mathbf{f(n, P) \leq h_P n}$ with $\mathbf{h_P = 2k^4 \binom{k^2}{k} n}$

Stanley-Wilf $\mathbf{c_\pi \leq 15^{2k^4 \binom{k^2}{k}}}$

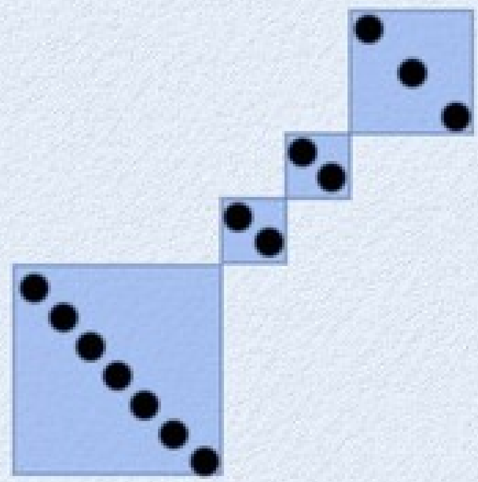
~~Arratia (1999) $\mathbf{c_\pi \leq (k-1)^2}$~~

Albert-Elder-Rechnitzer - Westcott - Zabrocki 2006 $\mathbf{c_{4231} \geq 9.47}$

Cibulka (2009) h_{P_π} and c_π are polynomially dependent

ACTUAL CONJECTURE

Conjecture (Bona 2005) Maximum for layered patterns



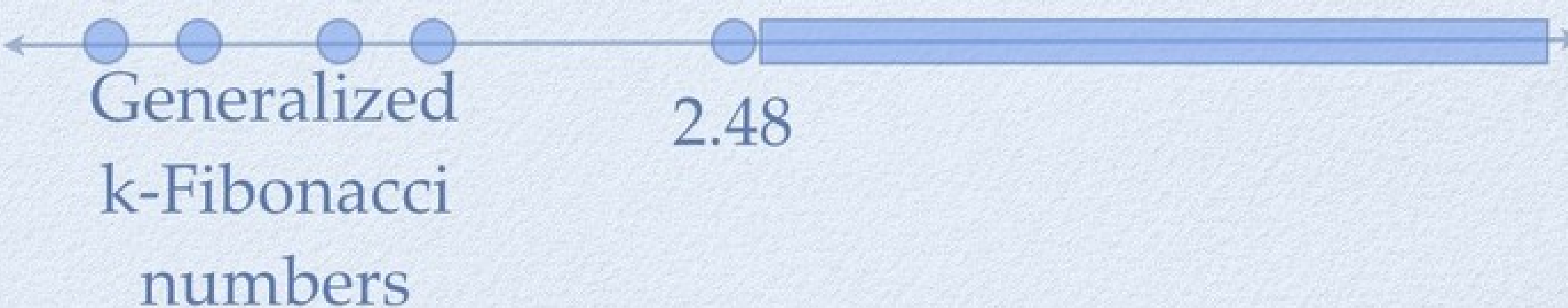
Claesson (2011) Layered pattern of size k: $c_\pi \leq 4k^2$

BEYOND PERMUTATIONS

- Füredi-Hajnal in higher dimension (Klazar Marcus 2007)
  - $f(n,P,d) \leq O(n^{d-1})$
- Graphs, hypergraphs

GROWTH RATES

- Av(B) with |B| ≥ 1
- Conjecture : lim $(s_n)^{1/n} = c_B$
- Vatter and al (05-) characterized the possible growth rates:



Thank you