Algorithms & Permutations 2012

Geometric permutations

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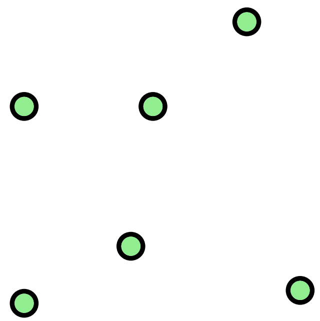
Geometric structures

To compute with geometric objects we often consider combinatorial structures they induce.



Point set P

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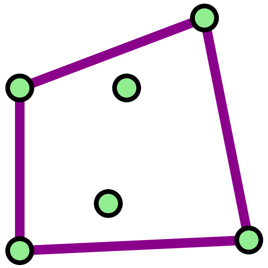
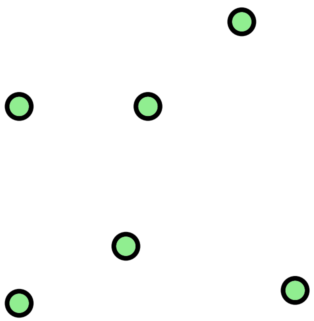


Convex hull



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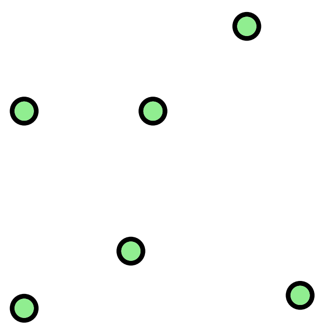

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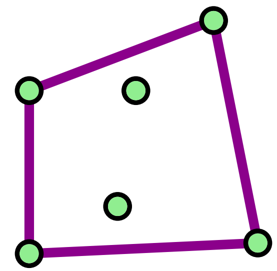

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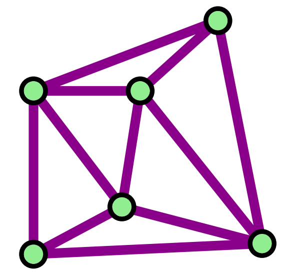

Delaunay triangulation

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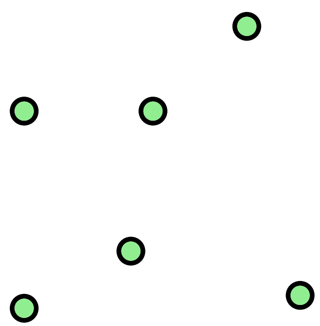

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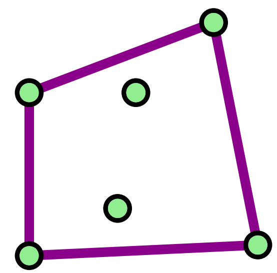

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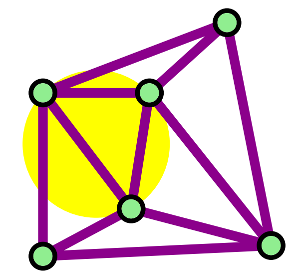

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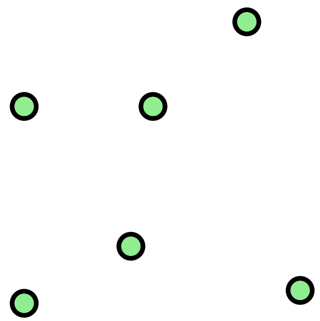

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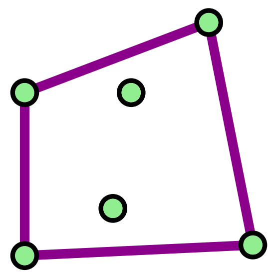

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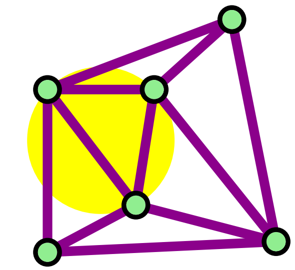

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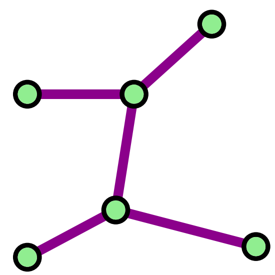

Minimum spanning tree

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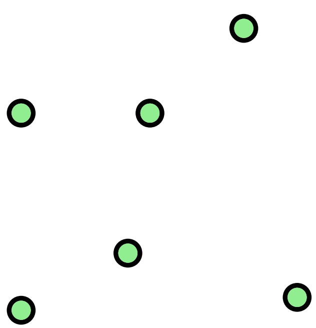

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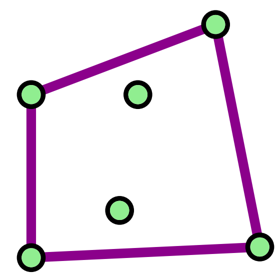

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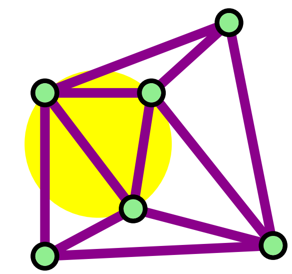

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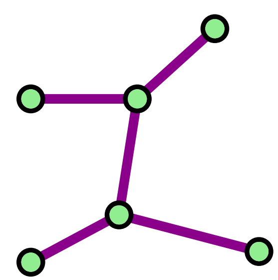

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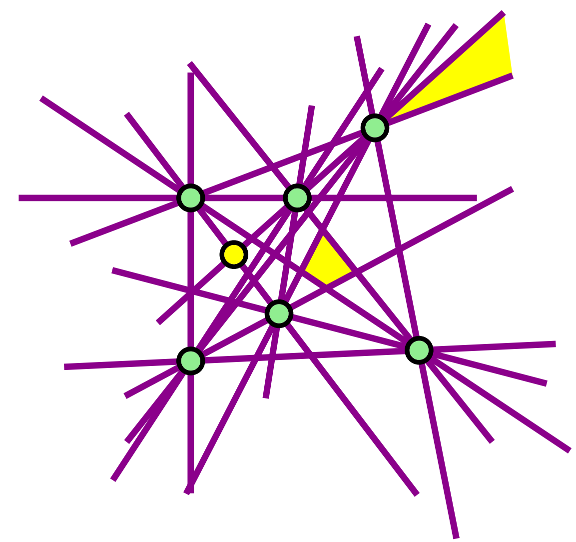

Arrangement of the lines spanned by P

Complexity of geometric structures

Predicting the size of a geometric structure is important (eg for complexity analysis).

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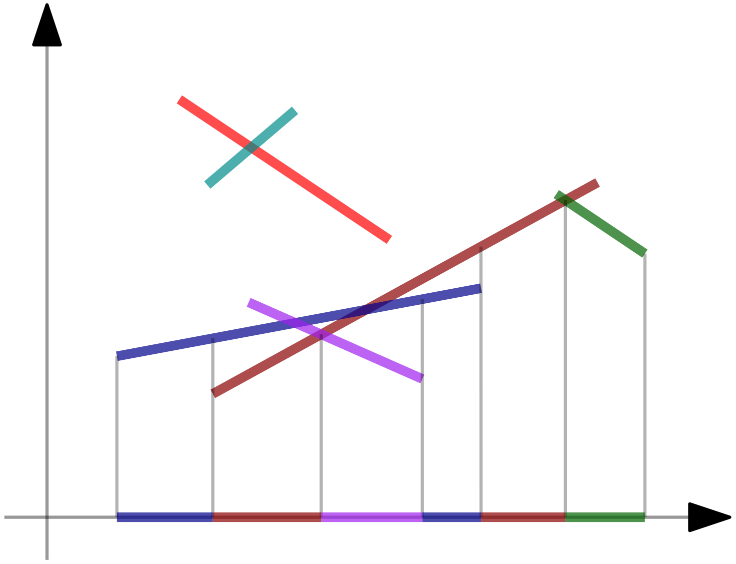
Sometimes intermediate combinatorial objects are useful...

Example: lower enveloppe of segments in $\mathbb{R}^2$



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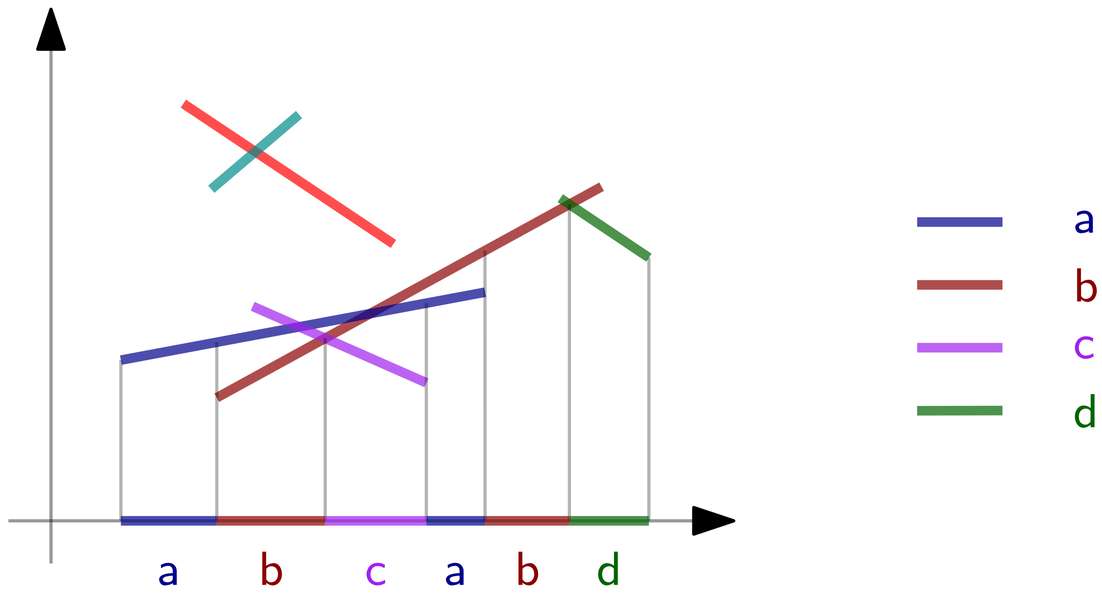

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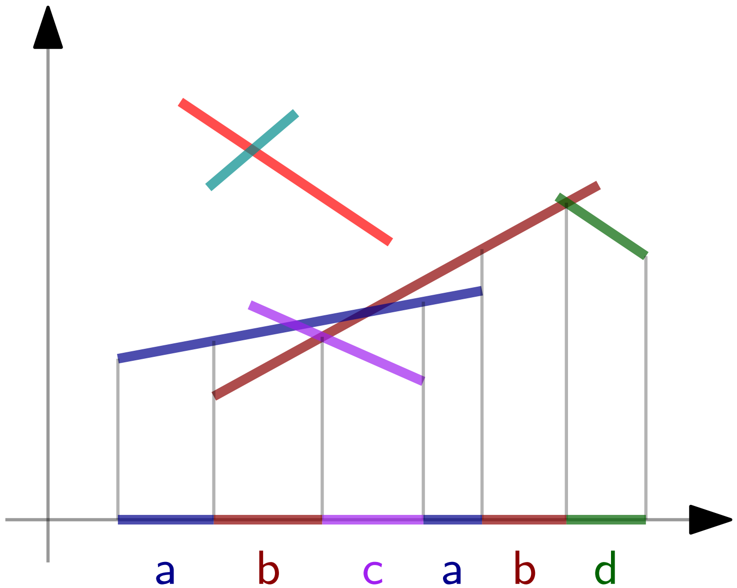

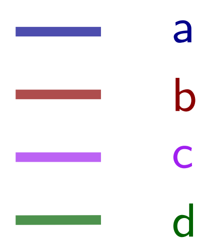

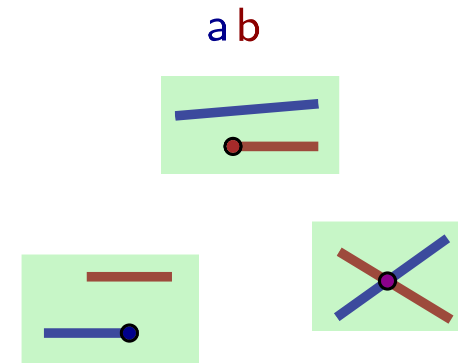

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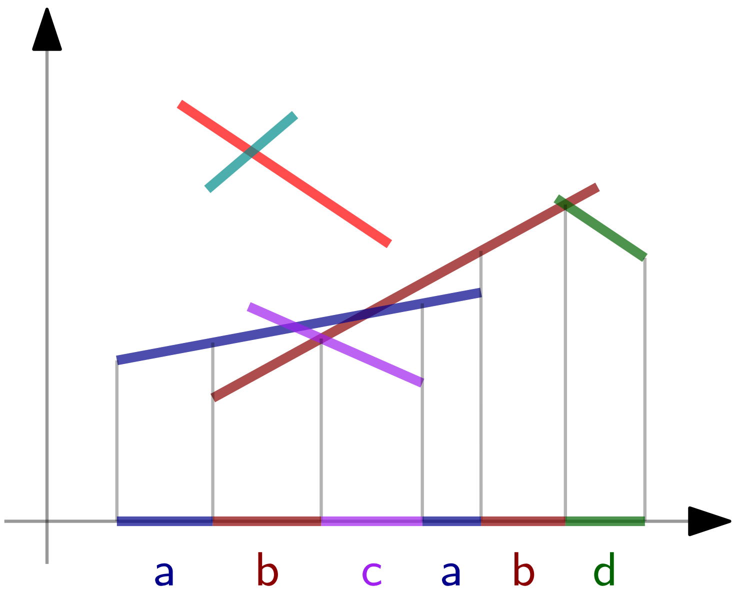

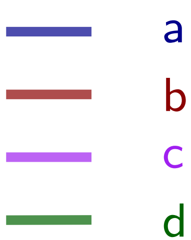

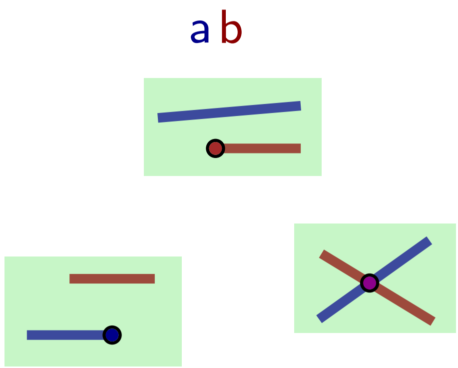

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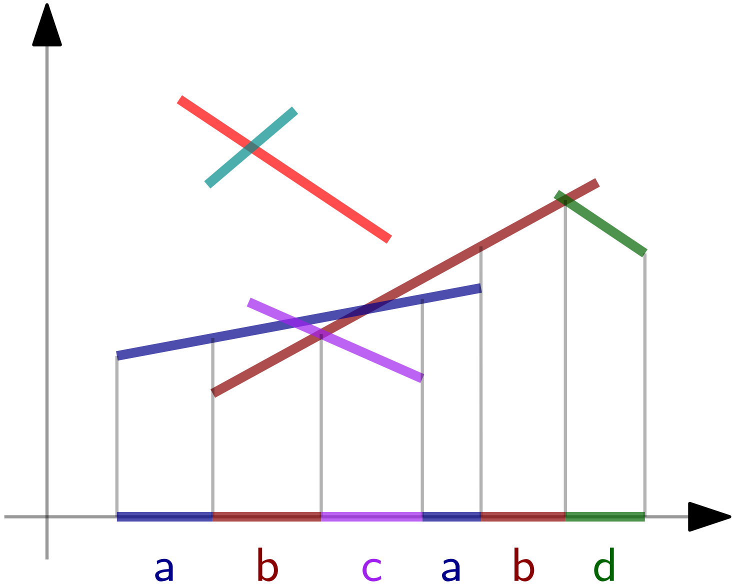

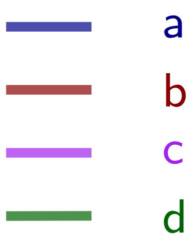

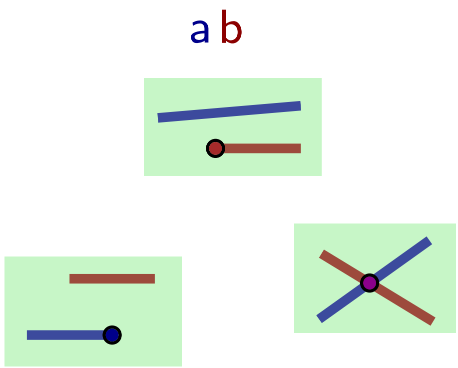

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Davenport-Schinzel sequence $\lambda_3(n) = \Theta(n\alpha(n))$.

Tight bound for this geometric problem!

Introduction

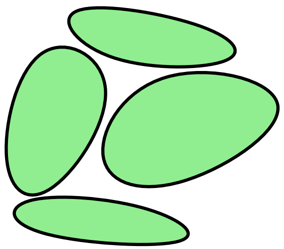
Line transversals and geometric permutations

More Davenport-Schinzel sequences

Excluded patterns

Extrapolation methods: VC dimension and shatter functions

Line transversals



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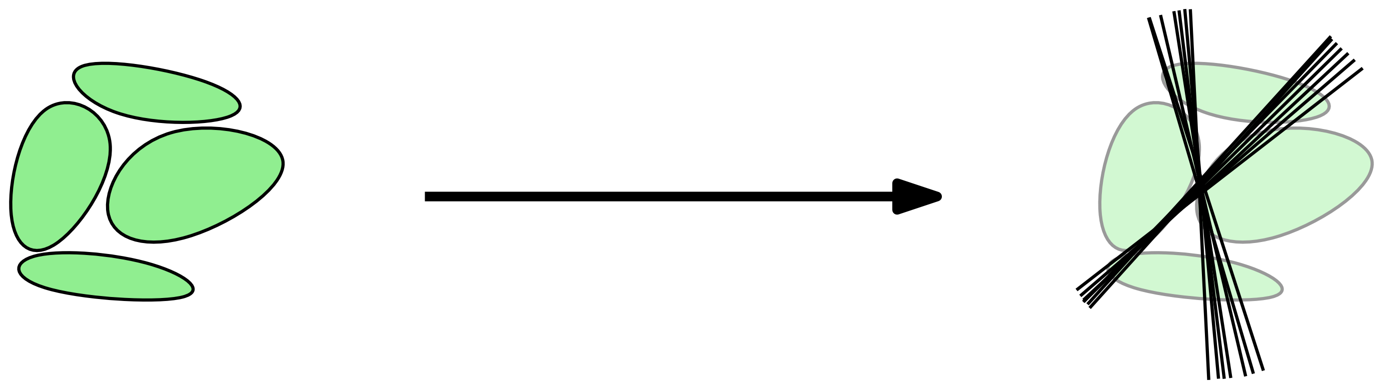

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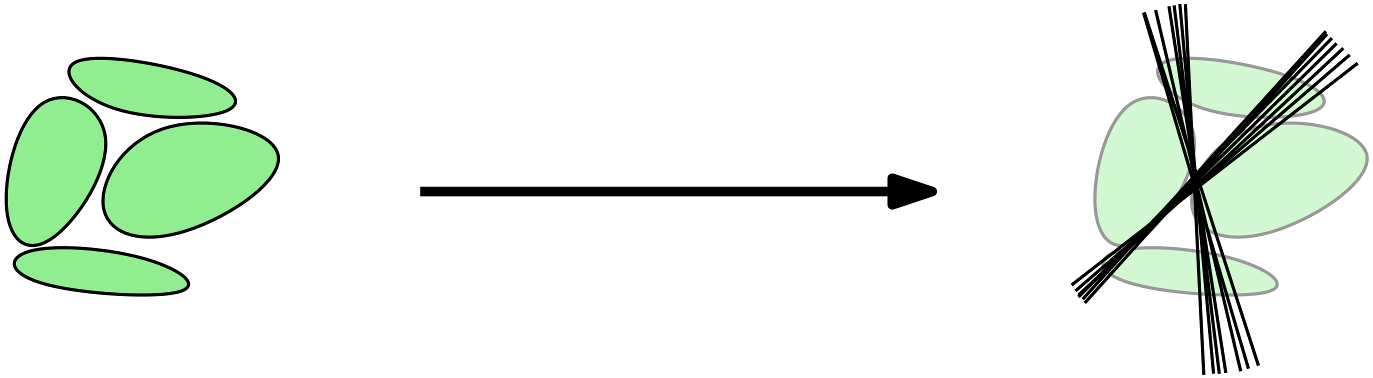

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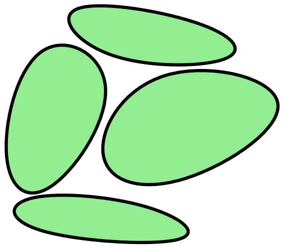

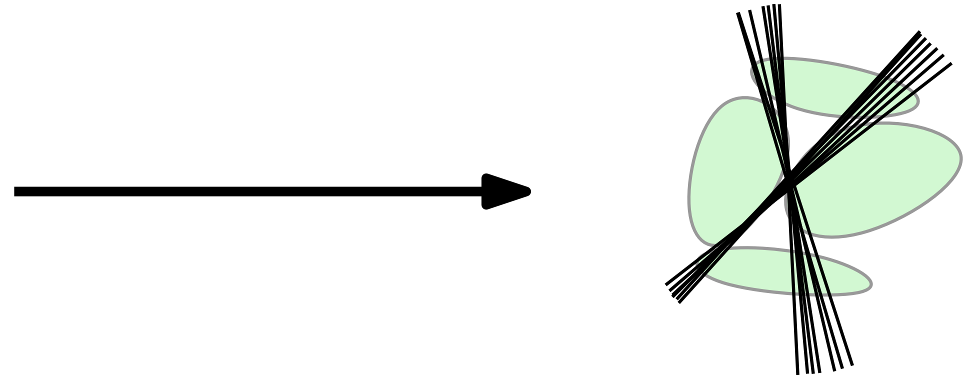

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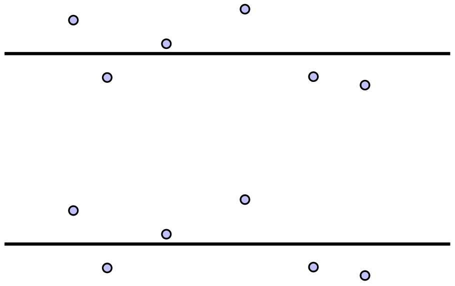

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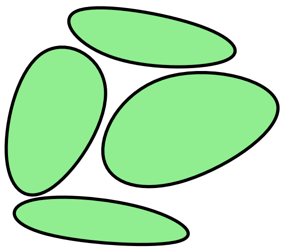

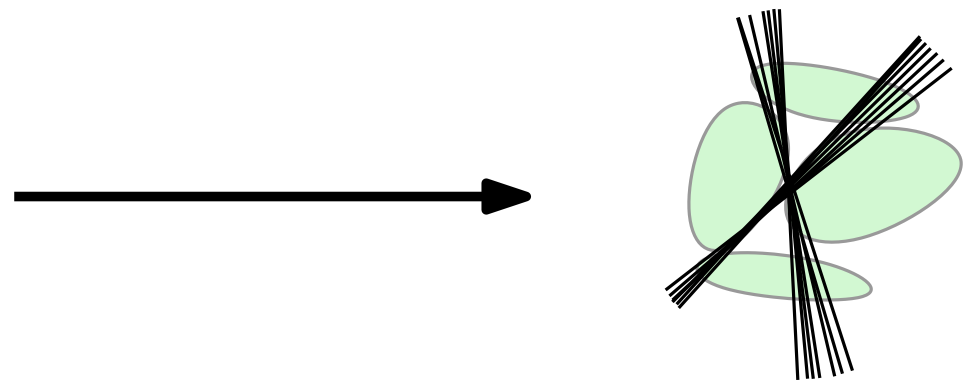

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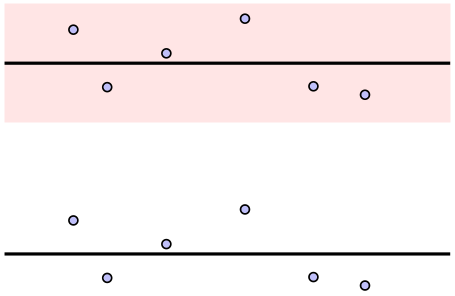

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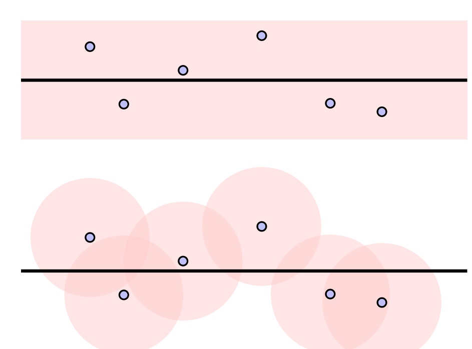

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Topologist says: *compute the Betti numbers of* $T(F)$.

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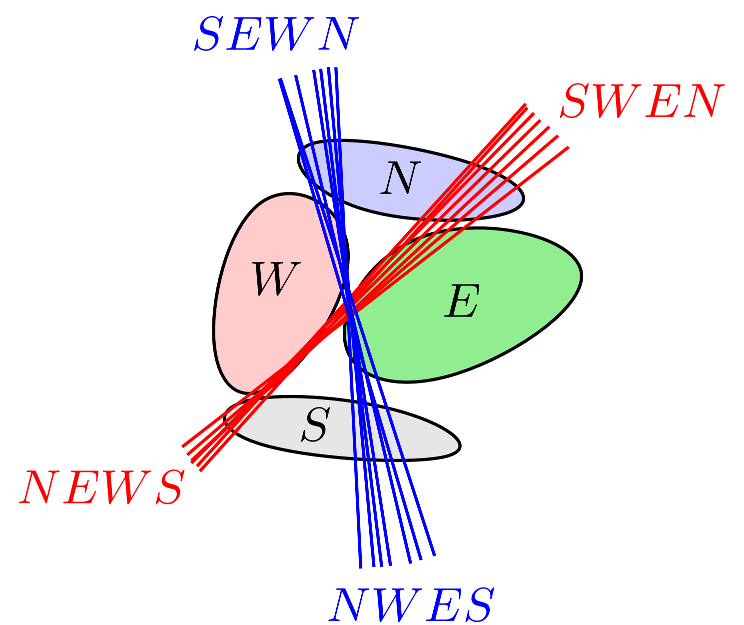
Oriented line transversal to disjoint convex sets

$\simeq$ permutation of these sets

Unoriented lines

$\simeq$ pair of (reverse) permutations

$=$ geometric permutation.



A hard nut

$g(d,n) = \max_F$ family of n disjoint convex sets in $\mathbb{R}^d$ #geom. perm. of F

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- $\star$ $g(2,n) = 2n-2$
- $\star$ at most 4 for disjoint translates of a planar convex set
- $\star$ at most 2 for $n \geq 9$ disjoint unit balls in $\mathbb{R}^d$

General case open for ∼20 years:

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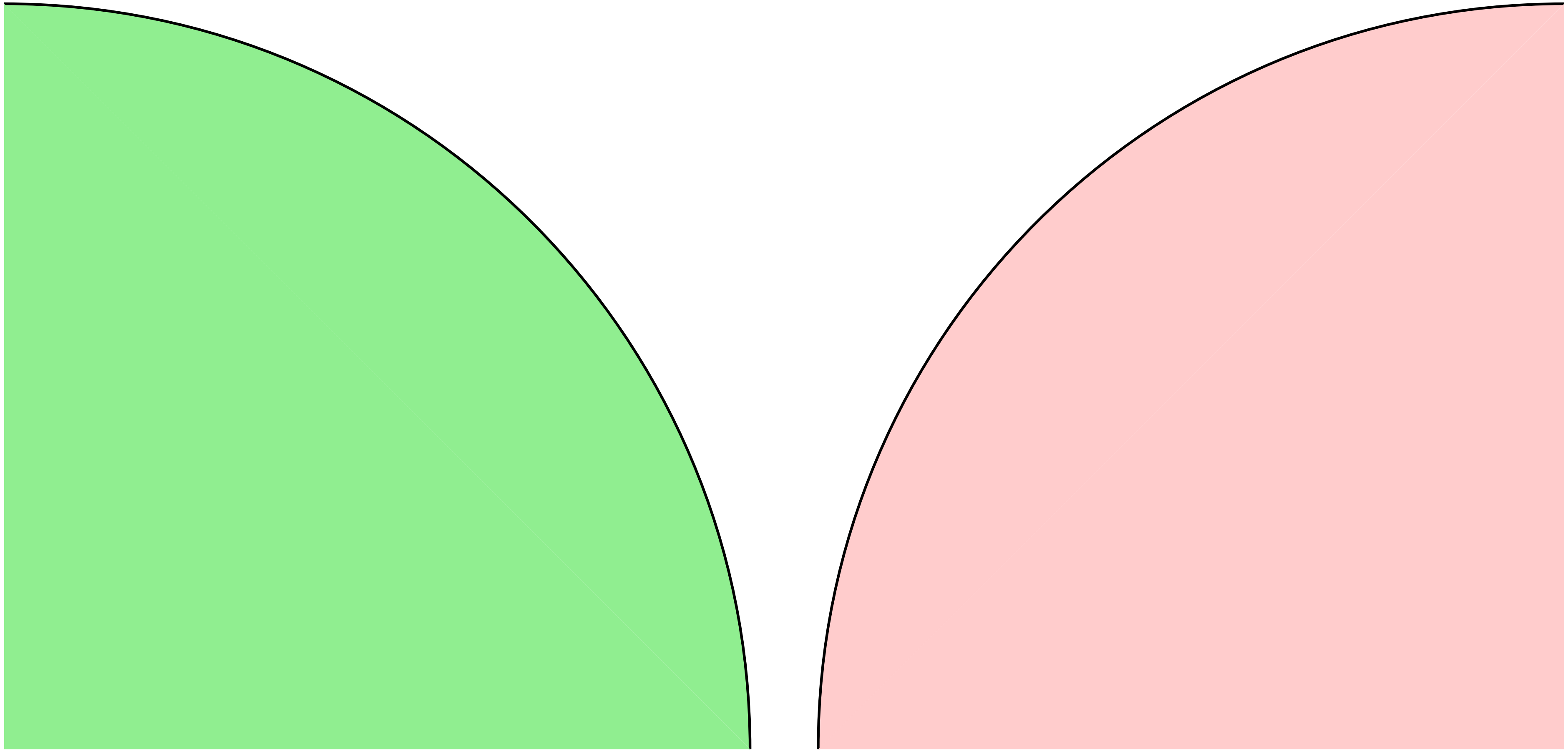
Construction showing that $g(2, n) \geq 2n - 2$.



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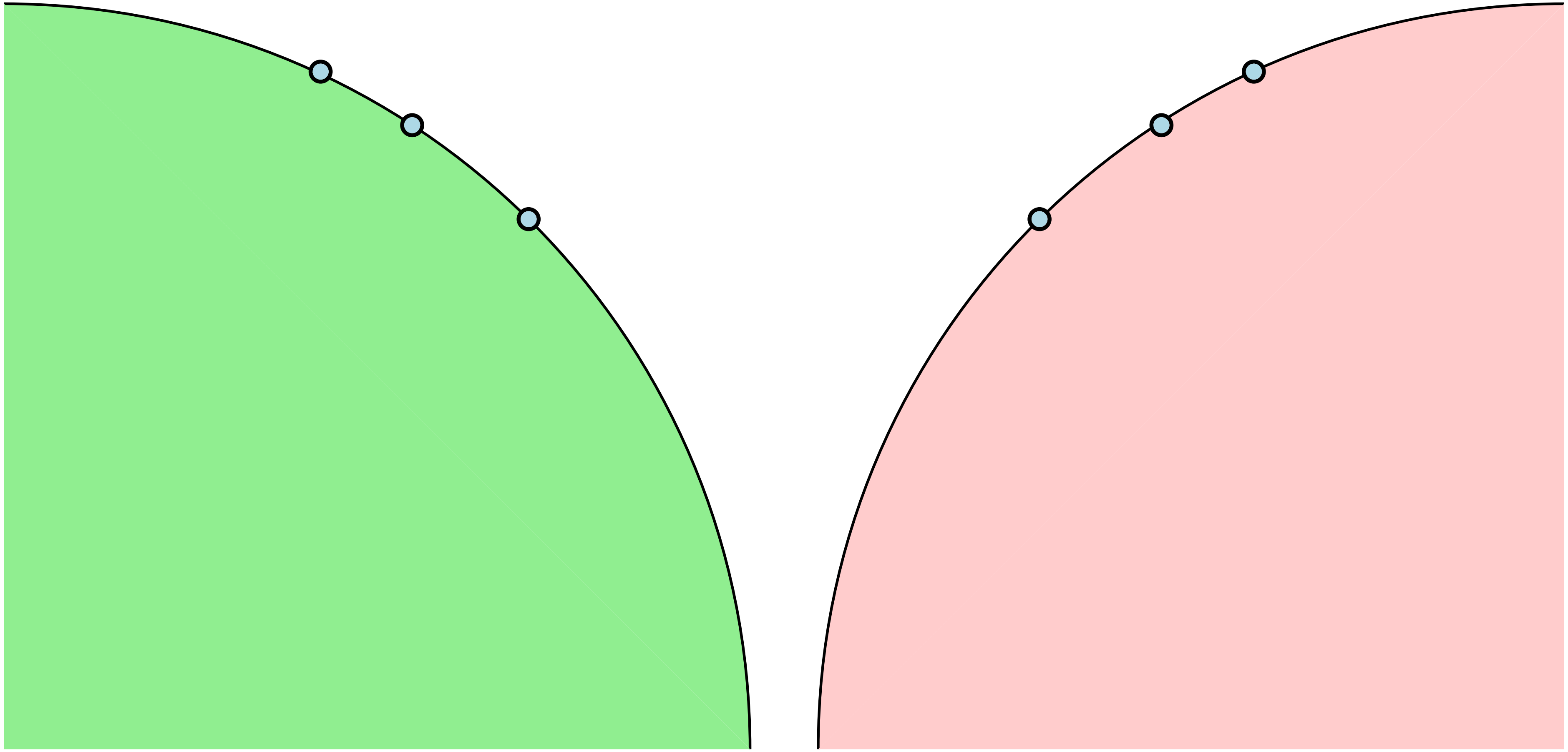

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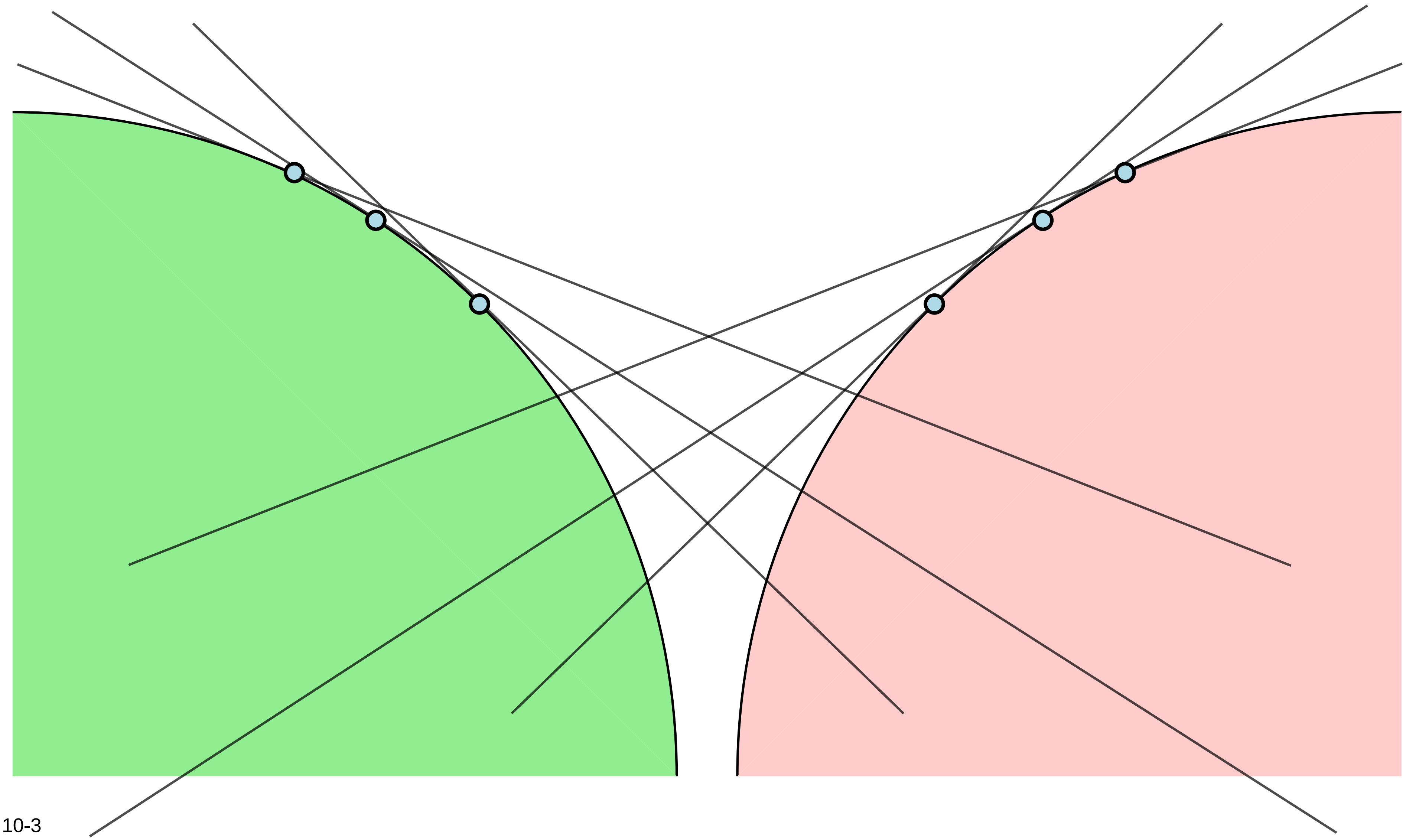

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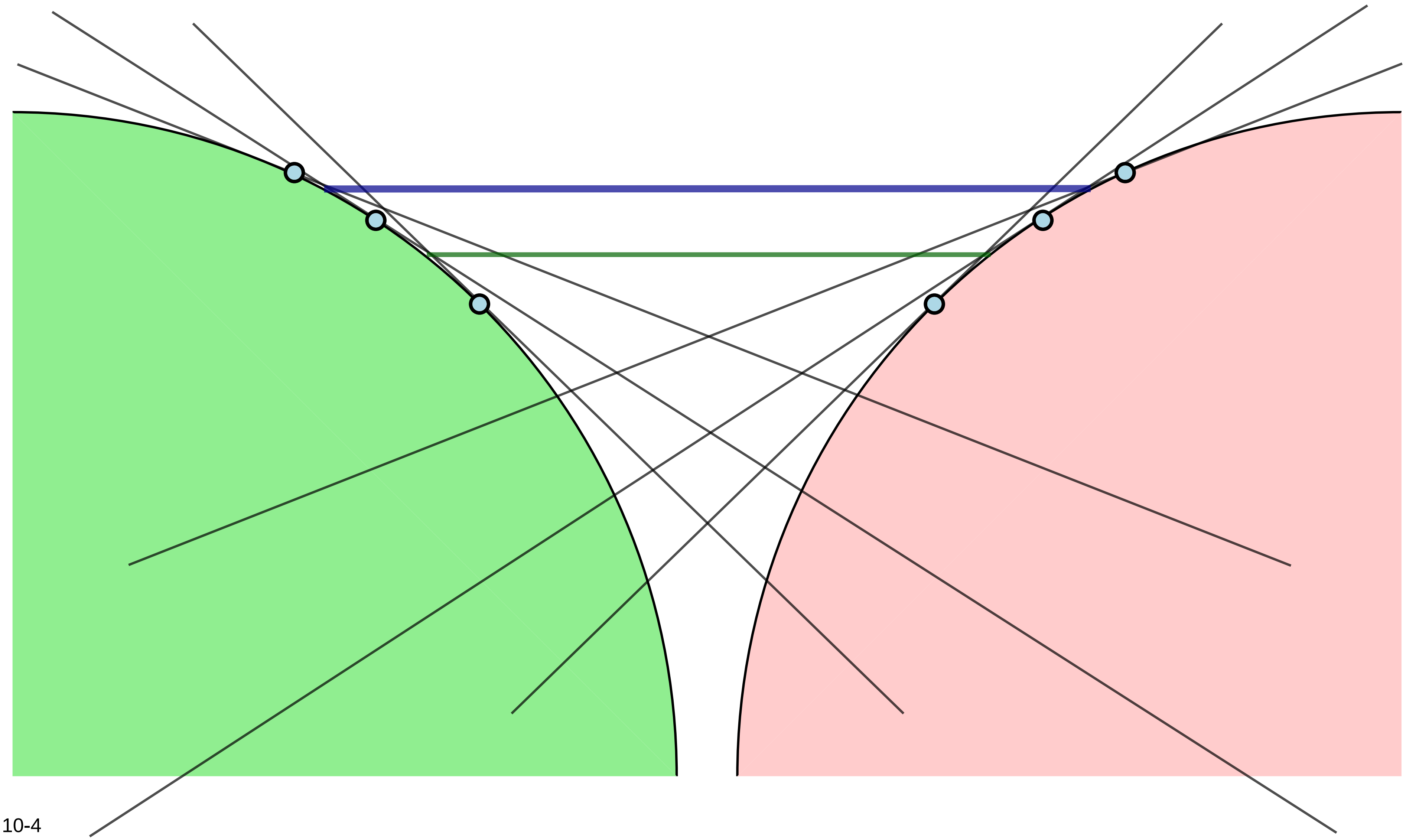

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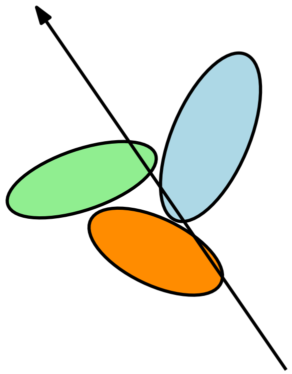

Charge every geometric permutation

to a LL bitangent line transversal.

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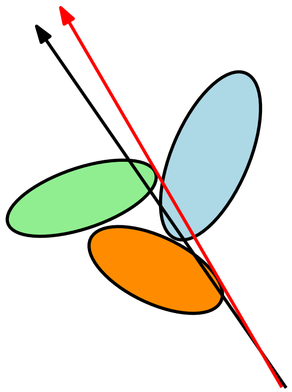

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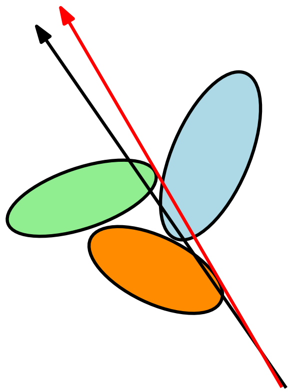

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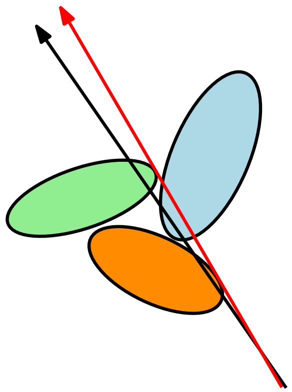

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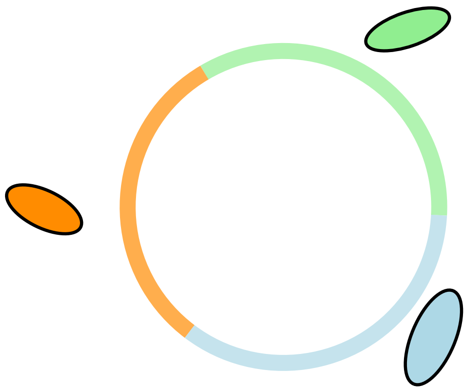

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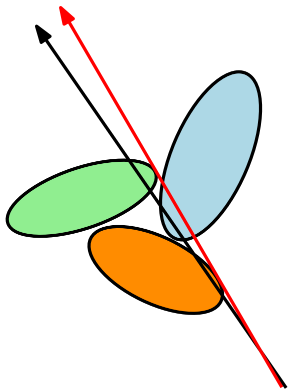

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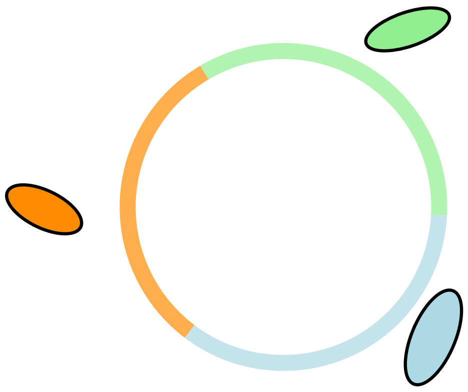
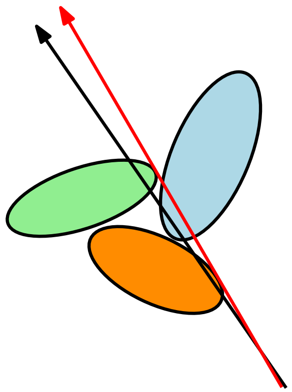
Get a circular word $w = L(\overrightarrow{u_1})L(\overrightarrow{u_2})\dots L(\overrightarrow{u_k})$



LL bitangent line transversals $\leq |w|$

Upper bound in the plane

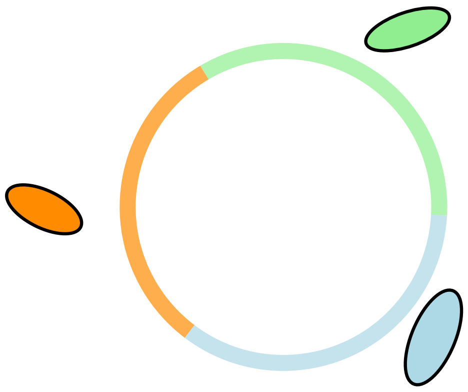


Charge every geometric permutation to a LL bitangent line transversal.

$L(\vec{u})$ = left-most object in direction $\vec{u}$.
(first object entirely on the left when sweeping from left to right)

Divide $\mathbb{S}^1$ in intervals with same $L(\cdot)$.

Get a circular word $w = L(\overrightarrow{u_1})L(\overrightarrow{u_2})\dots L(\overrightarrow{u_k})$



$\#$ LL bitangent line transversals $\leq |w|$

w has no $abab$ subword $\Rightarrow |w| \leq 2n - 2$.

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Another approach...

Let A, B, C and D be disjoint convex sets in the plane.

Observation. If $\{A, B, C, D\}$ has a line transversal in the order $ABCD$ then it cannot have a line transversal in the order $BADC$.

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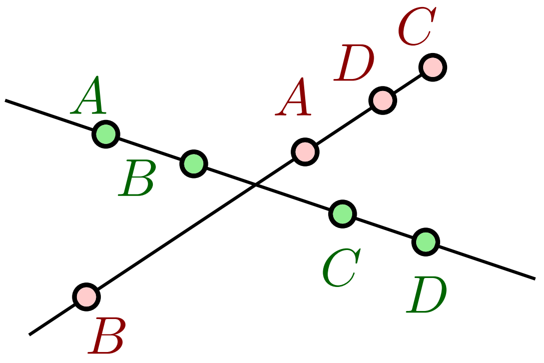

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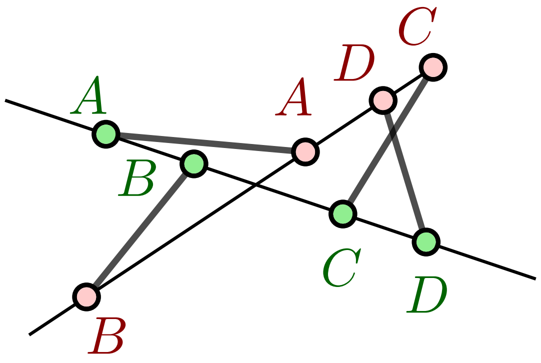

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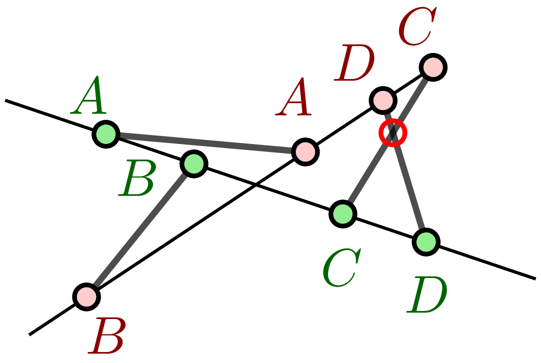

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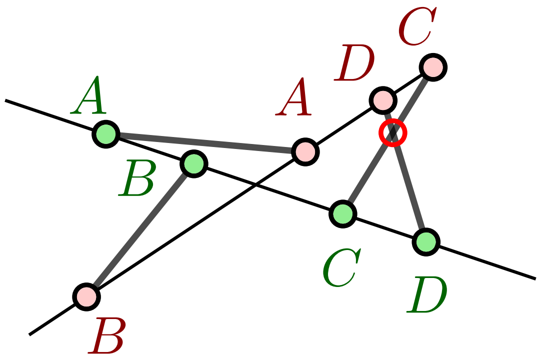

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Constraints on restrictions of geometric permutations.

1234567
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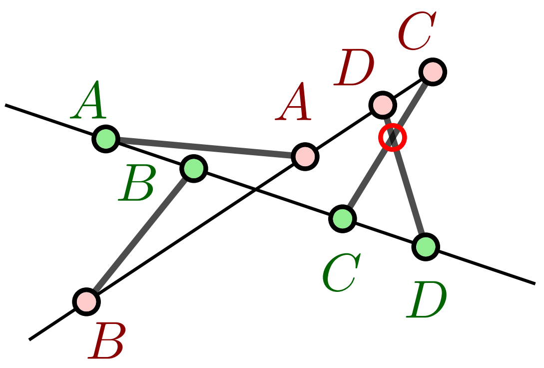

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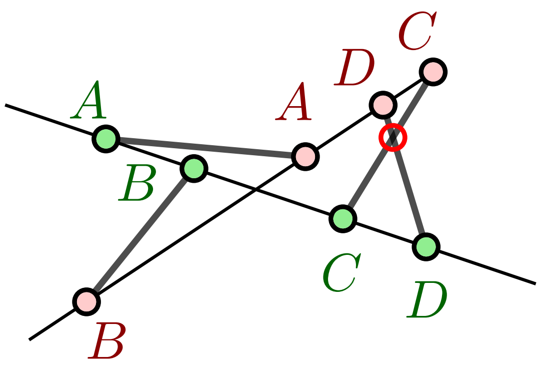

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Constraints on restrictions of geometric permutations.

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$(ABCD, BADC)$ is an excluded pattern for disjoint planar convex sets.

Excluded patterns: definition

Classical permutation patterns:

$\sigma \in S_n$ contains $\tau \in S_k$ if there exists $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

$\forall 1 \leq a, b \leq k, \quad \sigma^{-1}(i_a) < \sigma^{-1}(i_b) \quad \Leftrightarrow \quad \tau^{-1}(a) < \tau^{-1}(b)$

If σ does not contain τ then σ avoids τ.

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Patterns in geometric permutations:

$(\sigma_1, \sigma_2) \in (S_n)^2$ contains $(\tau_1, \tau_2) \in (S_k)^2$ if there exists $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that

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Previous example:

If F is a family of disjoint convex sets in $\mathbb{R}^2$,

Any pair of permutations of F induced by oriented line transversals avoids $(1234, 2143)$.

Excluded patterns in the plane

$(1234, 2143)$ is an excluded pattern for convex sets.

$(1234, 3214)$ is an excluded pattern for translates of a convex set.

$\Rightarrow$ disjoint translates of a convex set have at most 3 geometric permutations.

$(1234, 1342)$ and $(1234, 3142)$ are excluded pattern for unit disks.

$\Rightarrow n \geq 4$ disjoint unit disks have at most 2 geometric permutations.

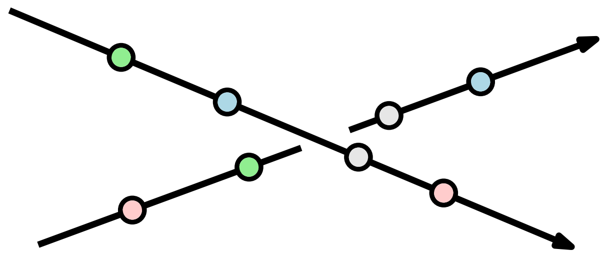
Application: Helly-type theorems for sets of line transversals.

Excluded patterns in higher dimension

All pairs of patterns are realizable!

Pick two non-coplanar lines.

Place points labelled from 1 to n in the desired orders.



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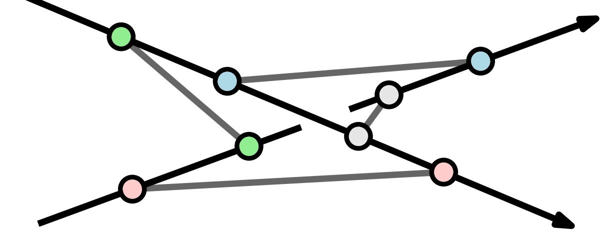

- Pick two non-coplanar lines.
- Place points labelled from 1 to n in the desired orders.
- Connect pairs of points with the same labels.
- Non-coplanarity of lines $\Rightarrow$ disjointedness of segments.

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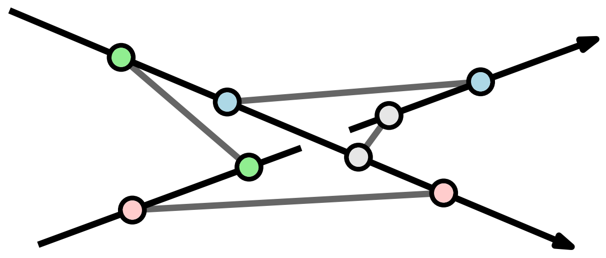

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There exist excluded triples...

$(123456, 456123, 246135)$ is excluded for convex sets in $\mathbb{R}^3$.

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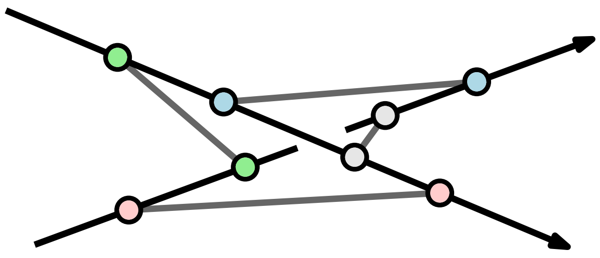

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There are excluded pairs in restricted settings...

$(1234, 4123)$, $(1234, 1432)$, $(1234, 3412)$ and $(1234, 3142)$ are excluded for unit balls in $\mathbb{R}^d$.

Contrary to the planar case, it is open whether $(1234, 1342)$ is excluded...

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A detour via hypergraphs

Consider a hypergraph $H \subseteq 2^V$ with vertex set V.

Associate to H the shatter function $f_H : \mathbb{N}^* \to \mathbb{N}$ defined by:

$$f_H(k) = \max_{X \in \binom{V}{k}} \#\{e \cap X \mid e \in H\}$$

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The largest k such that $f_H(k) = 2^k$ is the Vapnik-Chervonenkis (VC) dimension of H.

Applications in computational learning theory, approximation algorithms...

VC-dimension of families of permutations

Consider a family of permutations $F \subseteq S_n$.

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Generalizes excluded patterns and the Stanley-Wilf conjecture discussed in the next talk

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Recently disproved by Cibulka-Kyncl: the right bound is between $\alpha(n)^n$ and $(\log^* n)^n$.

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... a few open problems (come see me for more :))

Some bounds on Davenport-Schinzel sequences remain with gap.

$g(3, n)$ is only known to be $\Omega(n^2)$ and $O(n^3 \log n)$... The gap widens in higher dimension.

How to find excluded patterns in dimension 3 and higher ?

Incompatibility of $(1234, 1342)$ remains open (would close gaps and improve Helly numbers).

How hard is it to test if a d-tuple of permutations is excluded for convex sets in $\mathbb{R}^d$?

Can we refine the "bootstrapping" mechanism of the VC-dimension?

What does $f_H(k) = m$ guarantee in terms of asymptotic estimates when $m < 2^k$?

Same question for families of permutations...

Is there some reasonable shattering condition that would imply $g(3, n) = O(n^2)$?

A few pointers...

Davenport-Schinzel sequences and their geometric applications

Micha Sharir and Pankaj Agarwal, Cambridge Univ. Press

Improved bounds for geometric permutations

Nathan Rubin, Haim Kaplan and Micha Sharir, to appear in SICOMP (FOCS 2010)

Geometric permutations in the plane and in Euclidean spaces of higher dimension

Andrei Asinowski, PhD thesis (2005)

Geometric permutations of disjoint unit spheres

Otfried Cheong, X. G. and Hyeon-Suk Na

Comp. Geom. Theor. and Appl. 30: 253–270 (2005).

ϵ-nets and simplex range queries

David Haussler and Emo Welzl, Discrete & Computational Geometry 2:127-151 (1987)

VC-Dimension of Sets of Permutations

Ran Raz, Combinatorica 20: 1-15 (2000)

Tight bounds on the maximum size of a set of permutations with bounded VC-dimension

Jan Kyncl and Josef Cibulka, arXiv:1104.5007v2 (SODA 2012)