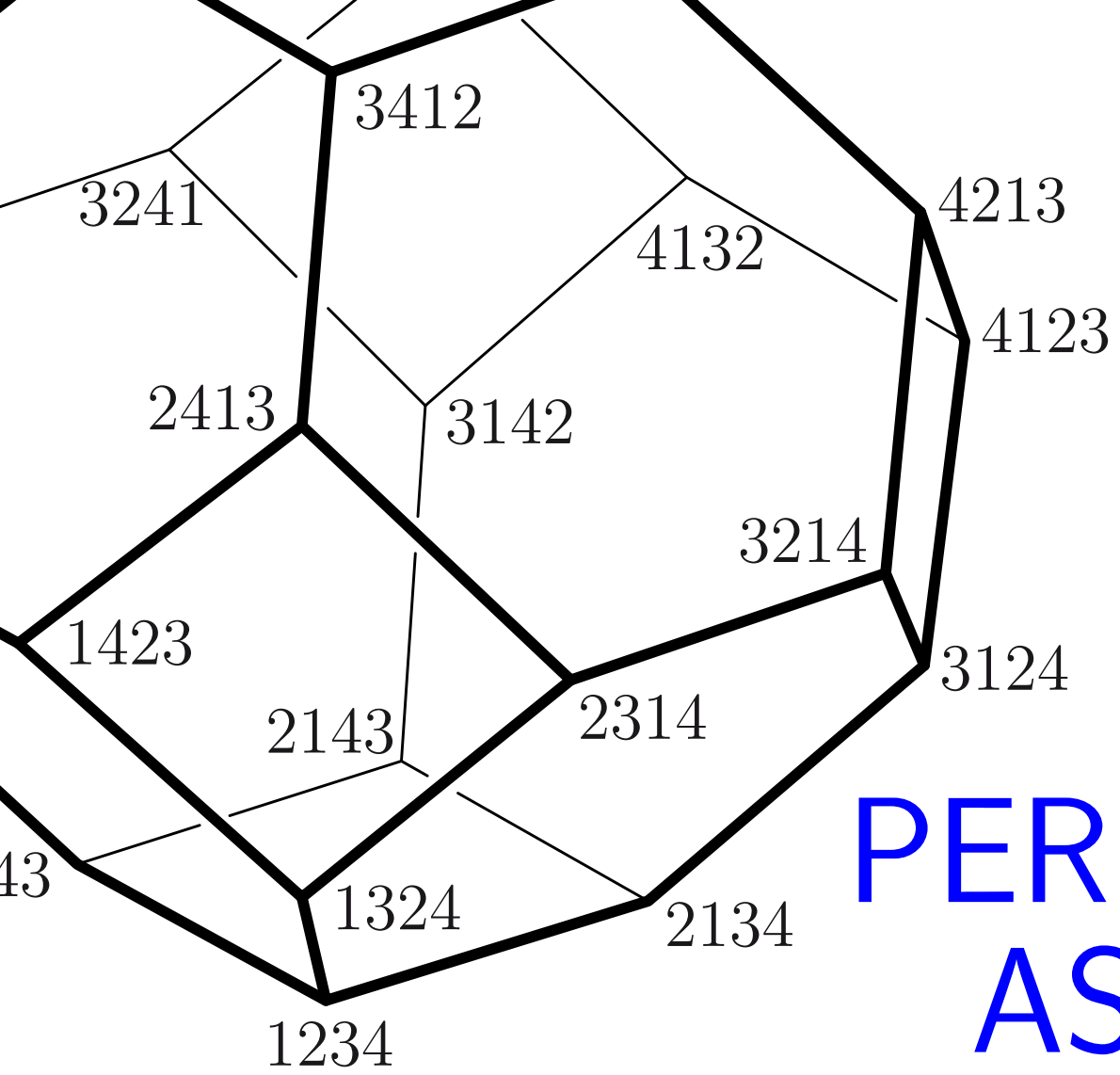


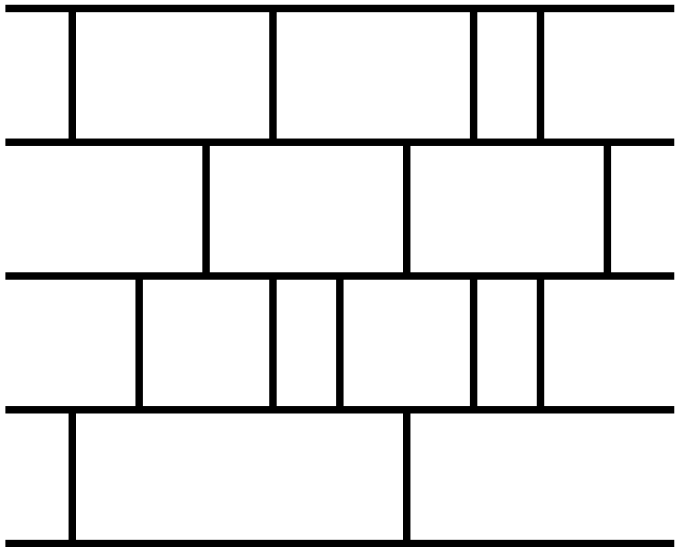
Algorithms & Permutations 2012

PERMUTAHEDRA, ASSOCIAHEDRA & SORTING NETWORKS

Vincent PILAUD

PRIMITIVE SORTING NETWORKS
— & —
PSEUDOLINE ARRANGEMENTS

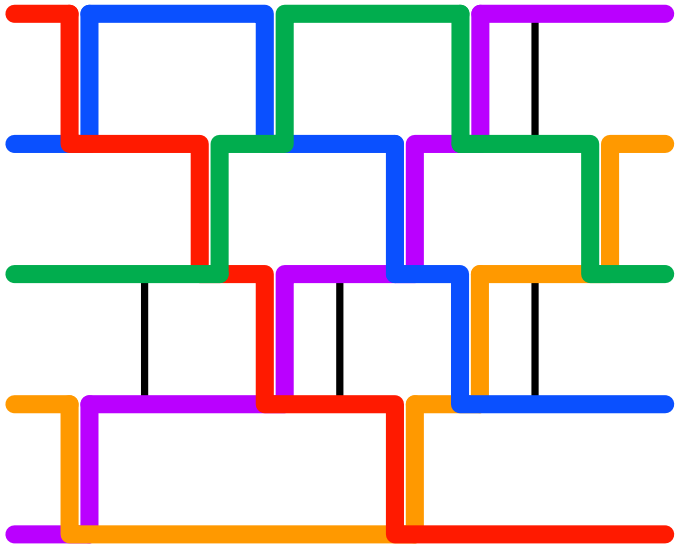
PRIMITIVE SORTING NETWORKS



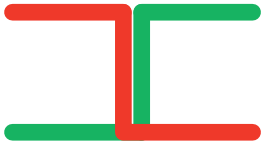
network $\mathcal{N} = n$ horizontal levels and m vertical commutators

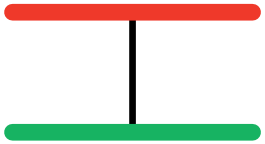
bricks of $\mathcal{N} =$ bounded cells

PSEUDOLINE ARRANGEMENTS ON A NETWORK



pseudoline = abscissa-monotone path

crossing = 

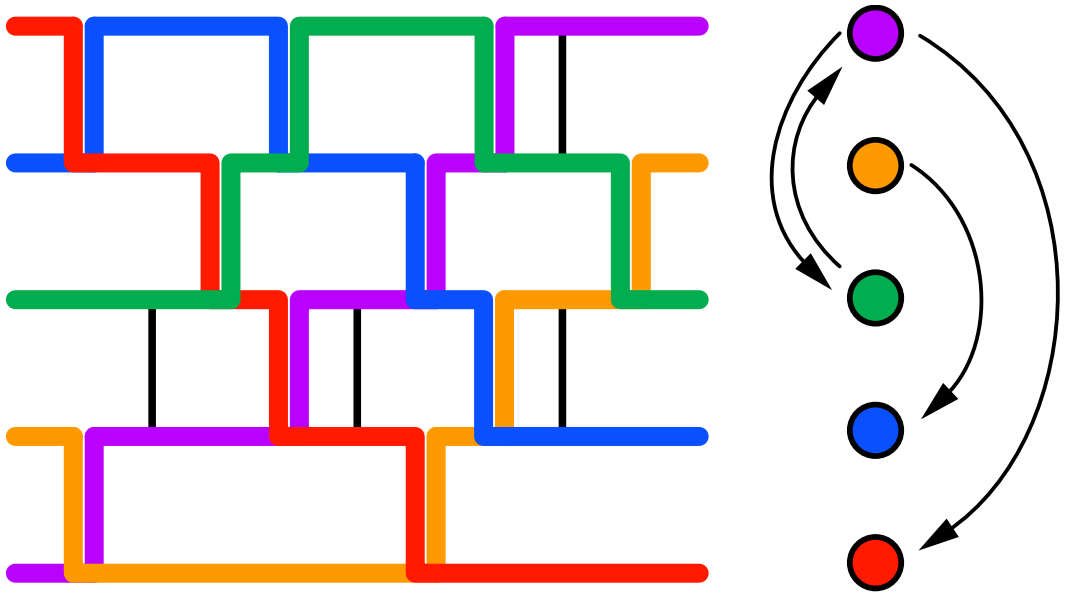
contact = 

pseudoline arrangement (with contacts) = n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, possibly some contacts, and no other intersection

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

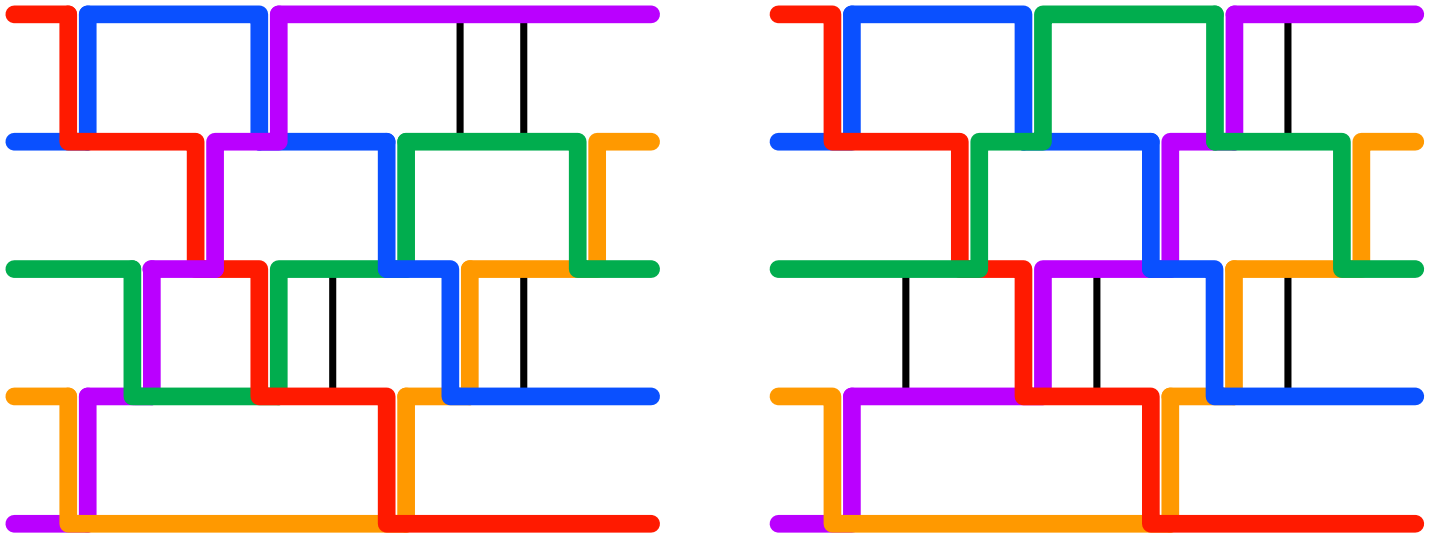
contact graph $\Lambda^{\#}$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact of Λ oriented from top to bottom



FLIPS

flip = exchange an arbitrary contact with the corresponding crossing



Combinatorial and geometric properties of the graph of flips $G(\mathcal{N})$?

VP & M. Pocchiola, *Multitriangulations, pseudotriangulations and sorting networks*, 2012$^+$

VP & F. Santos, *The brick polytope of a sorting network*, 2012

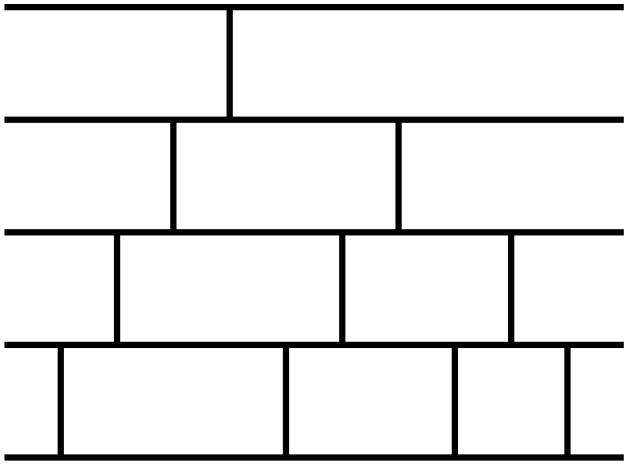
A. Knutson & E. Miller, *Subword complexes in Coxeter groups*, 2004

C. Ceballos, J.-P. Labbé & C. Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, 2012$^+$

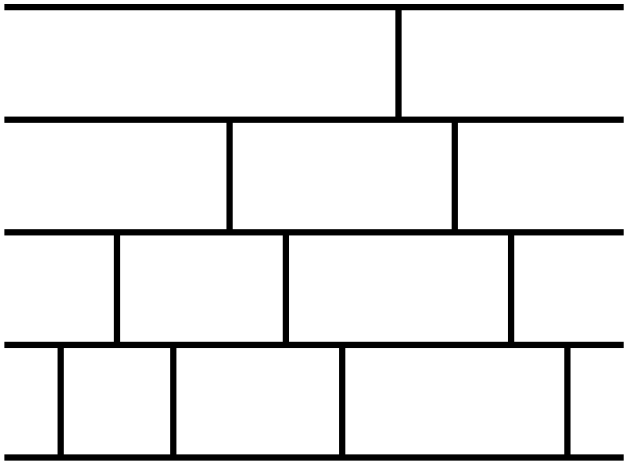
VP & C. Stump, *Brick polytopes of spherical subword complexes [. . .]*, 2012$^+$

POINT SETS
— & —
MINIMAL SORTING NETWORKS

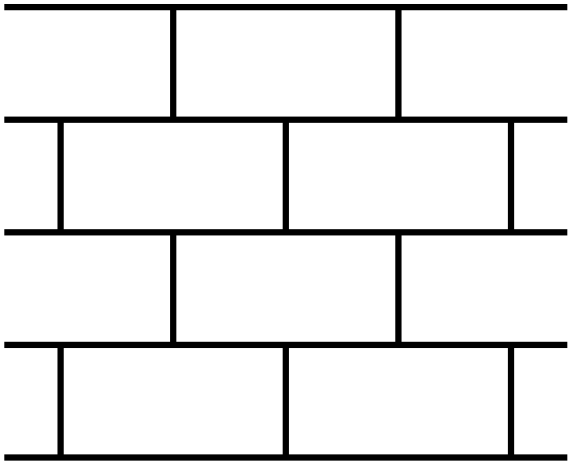
MINIMAL SORTING NETWORKS



bubble sort



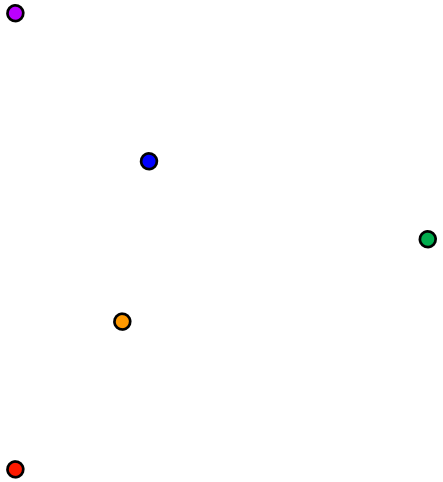
insertion sort



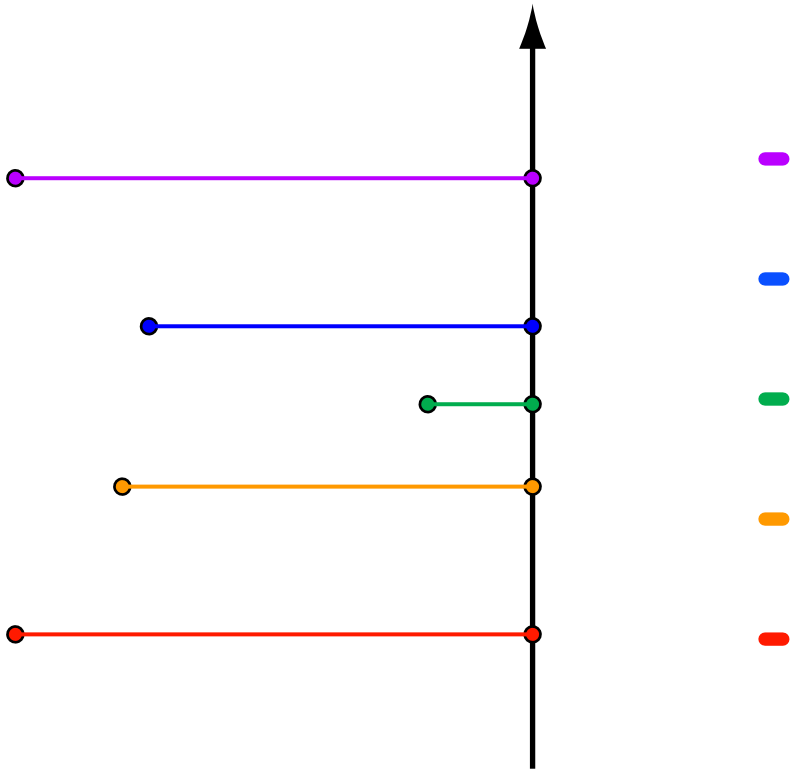
even-odd sort

D. Knuth, *The art of Computer Programming* (Vol. 3, Sorting and Searching), 1997

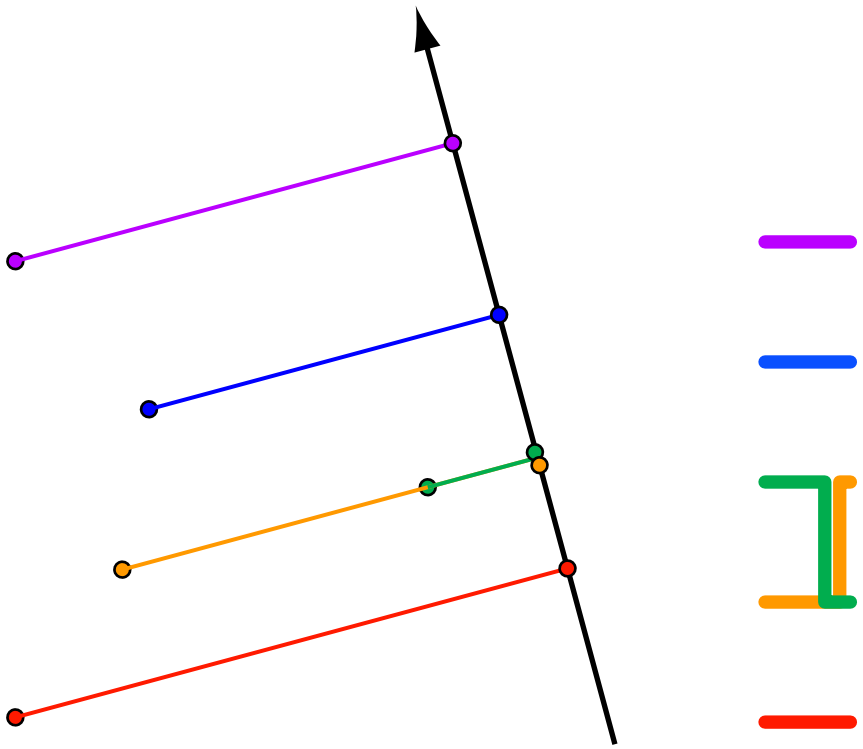
POINT SETS & MINIMAL SORTING NETWORKS



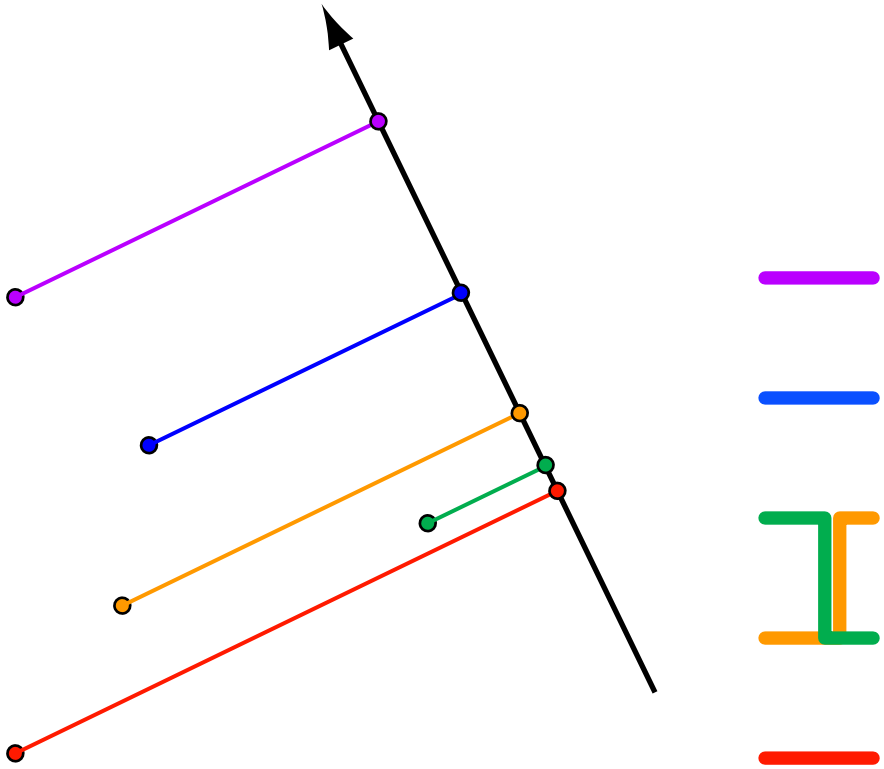
POINT SETS & MINIMAL SORTING NETWORKS



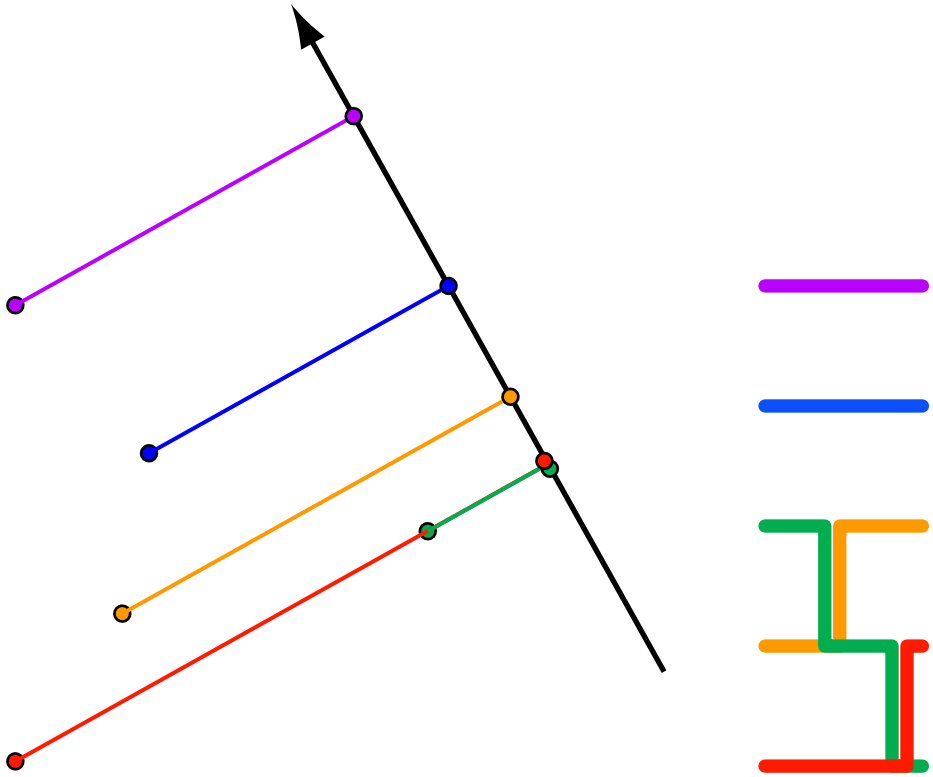
POINT SETS & MINIMAL SORTING NETWORKS



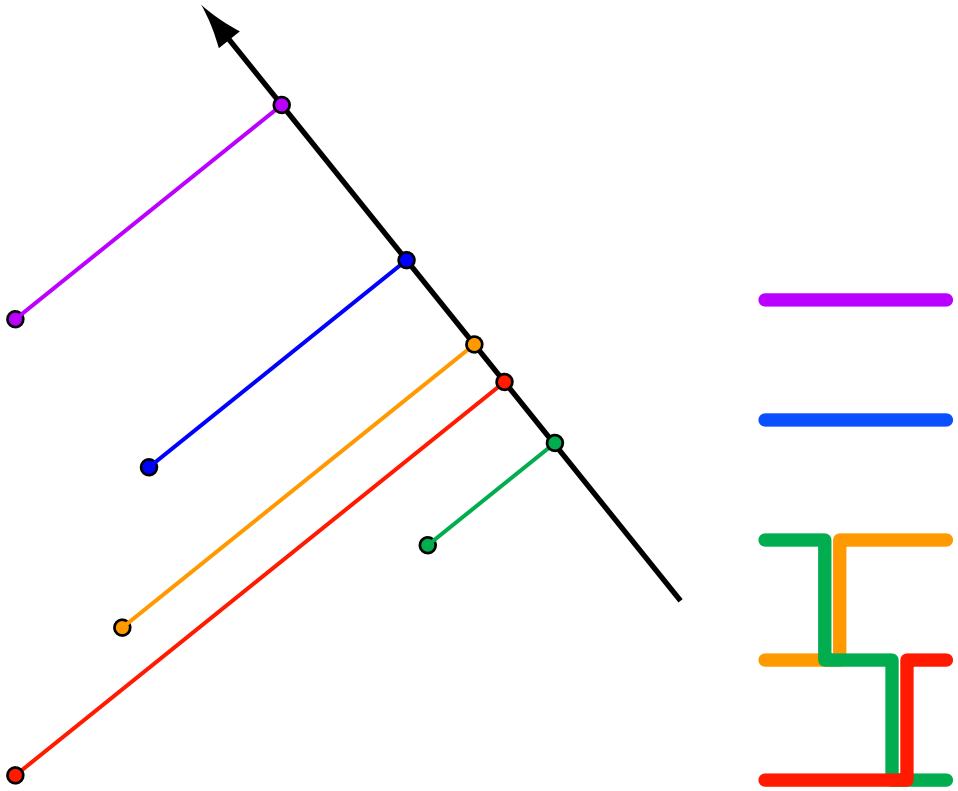
POINT SETS & MINIMAL SORTING NETWORKS



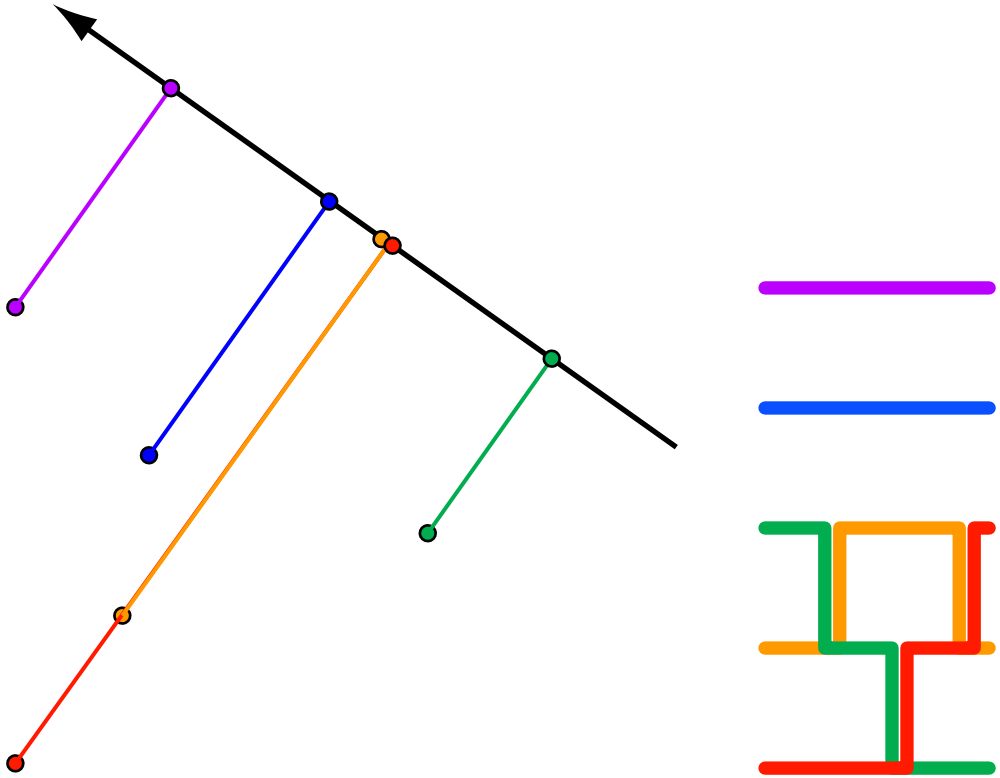
POINT SETS & MINIMAL SORTING NETWORKS



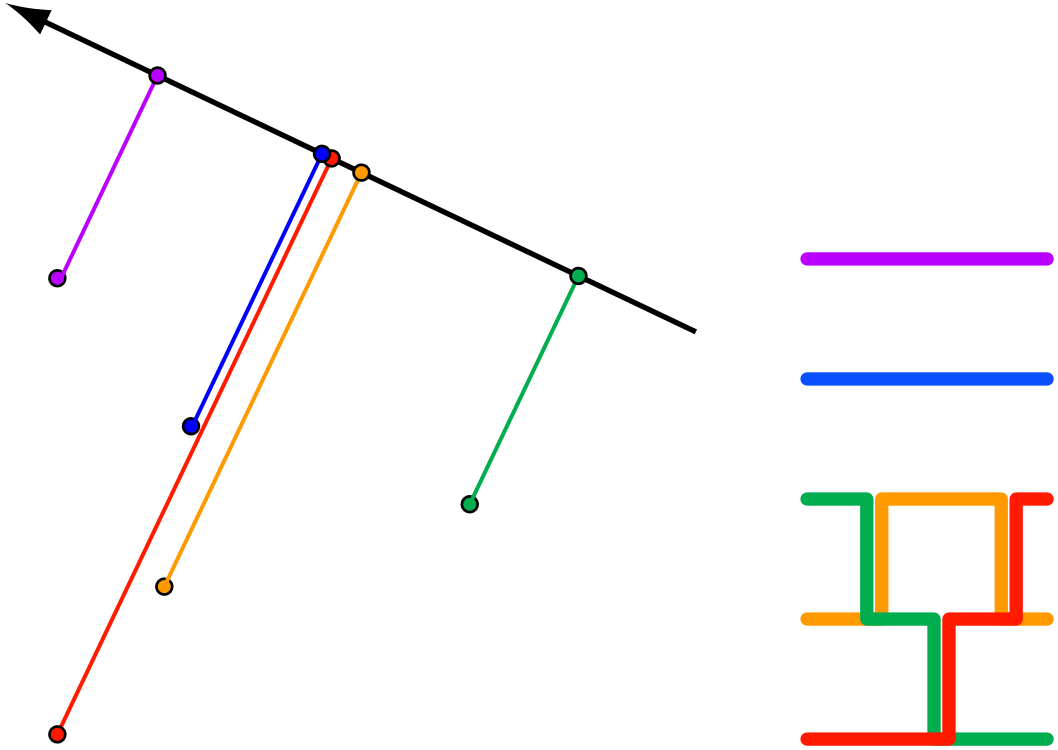
POINT SETS & MINIMAL SORTING NETWORKS



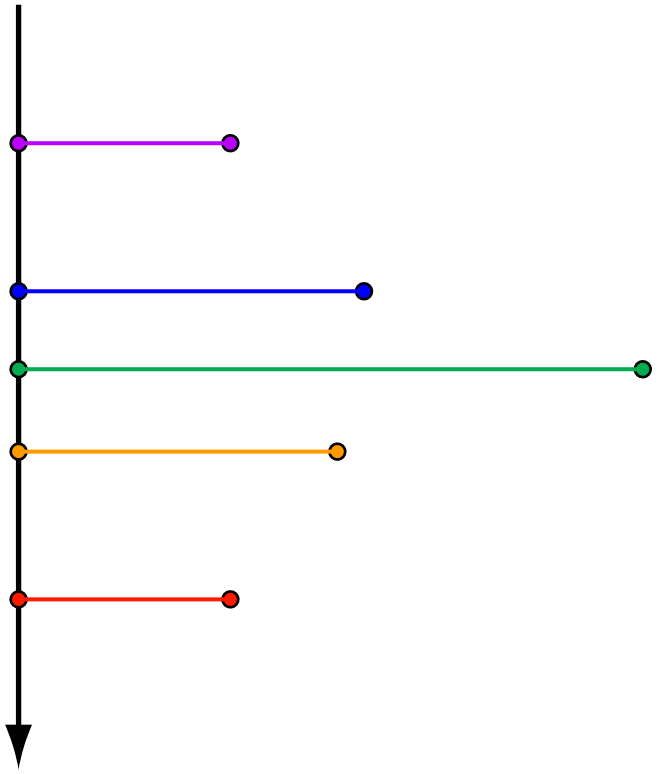
POINT SETS & MINIMAL SORTING NETWORKS

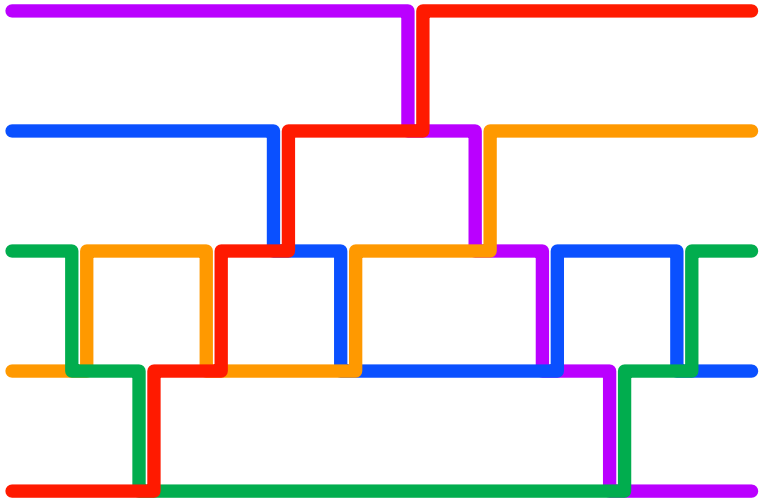


POINT SETS & MINIMAL SORTING NETWORKS

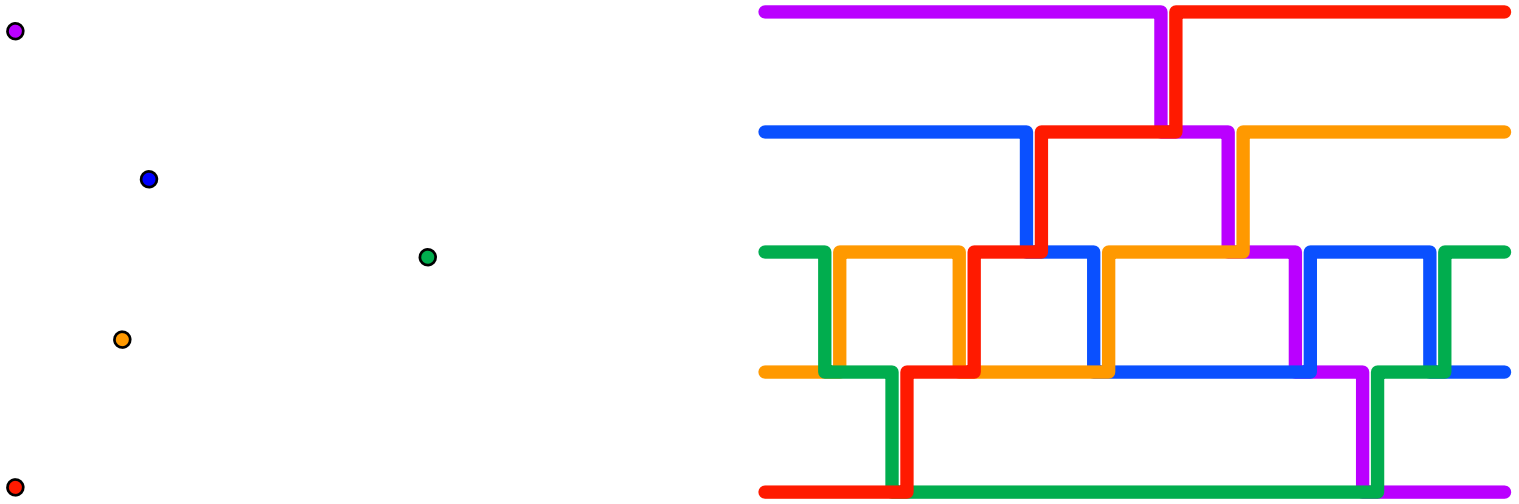


POINT SETS & MINIMAL SORTING NETWORKS





POINT SETS & MINIMAL SORTING NETWORKS



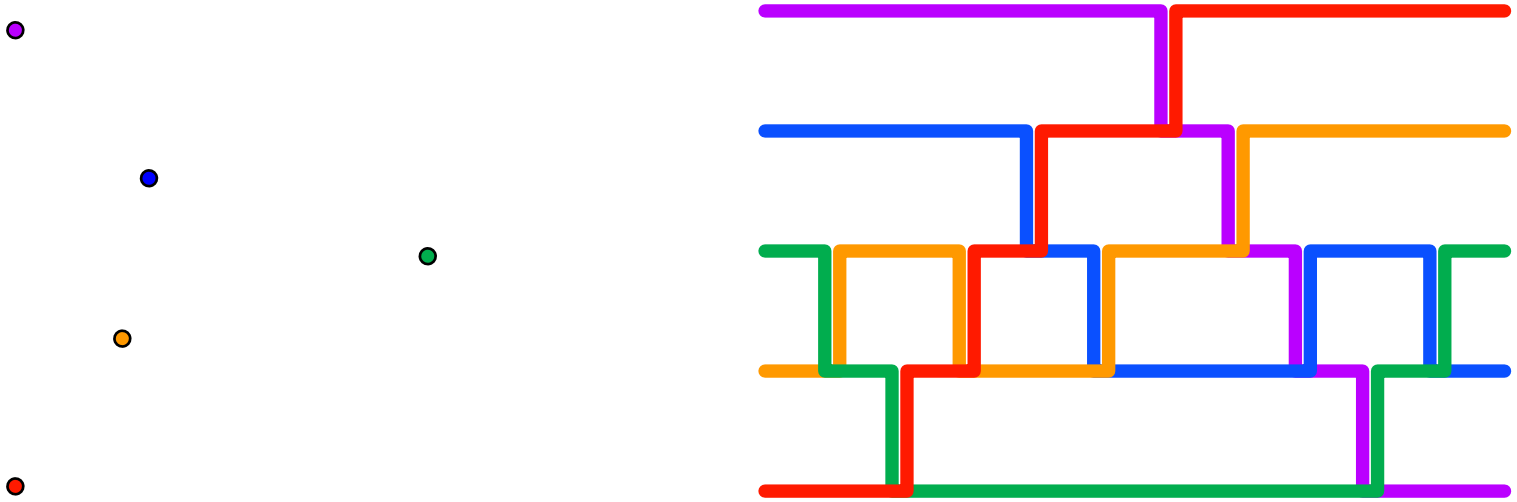
n points in $\mathbb{R}^2$ $\Longrightarrow$ minimal primitive sorting network with n levels

point $\longleftrightarrow$ pseudoline

edge $\longleftrightarrow$ crossing

boundary edge $\longleftrightarrow$ external crossing

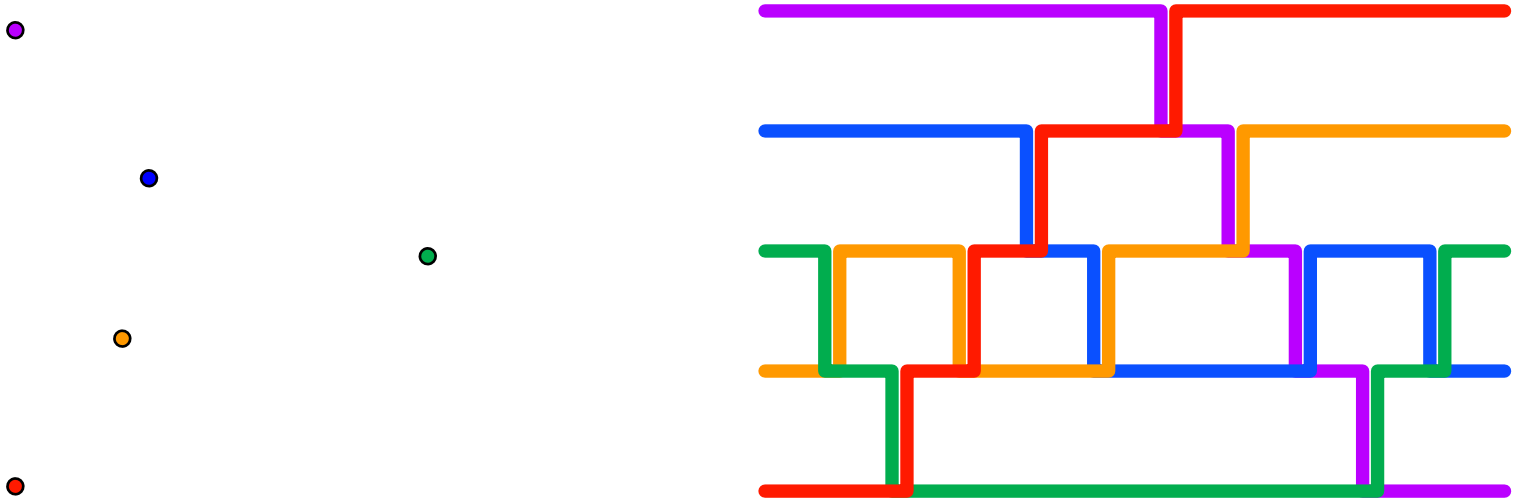
POINT SETS & MINIMAL SORTING NETWORKS



n points in $\mathbb{R}^2$ $\Longrightarrow$ minimal primitive sorting network with n levels

not all minimal primitive sorting networks correspond to points sets of $\mathbb{R}^2$
$\Longrightarrow$ realizability problems

POINT SETS & MINIMAL SORTING NETWORKS



J. Goodmann & R. Pollack, *On the combinatorial classification of nondegenerate configurations in the plane*, 1980

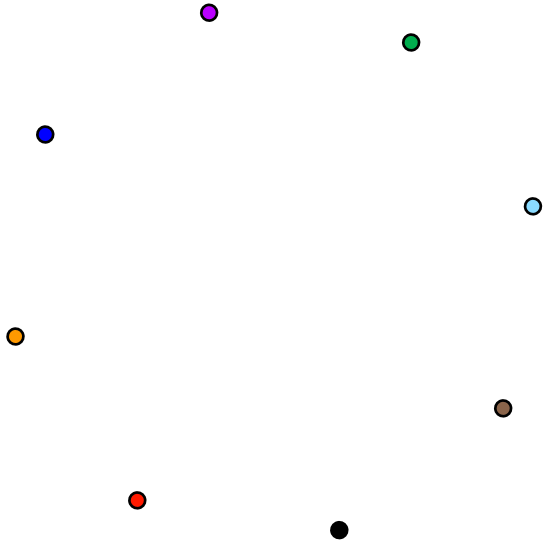
D. Knuth, *Axioms and Hulls*, 1992

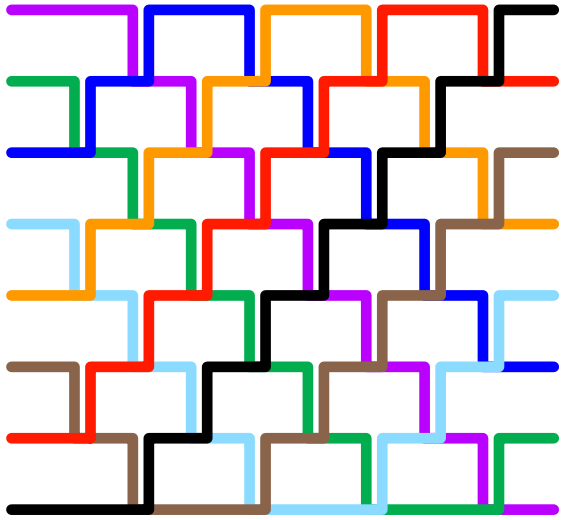
A. Björner, M. Las Vergnas, B. Sturmfels, N. White, & G. Ziegler, *Oriented Matroids*, 1999

J. Bokowski, *Computational oriented matroids*, 2006

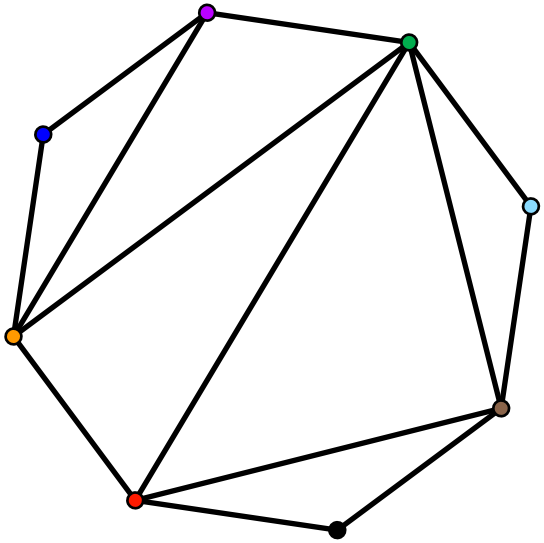
TRIANGULATIONS
— & —
ALTERNATING SORTING NETWORKS

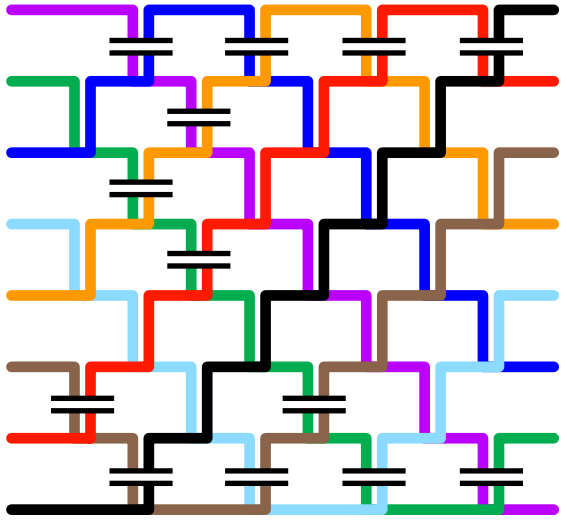
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



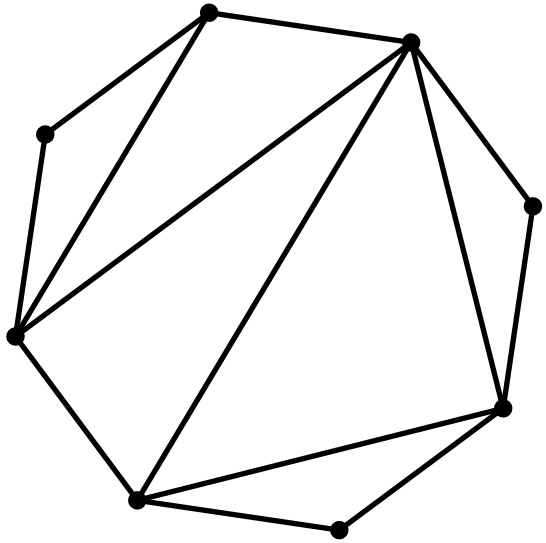


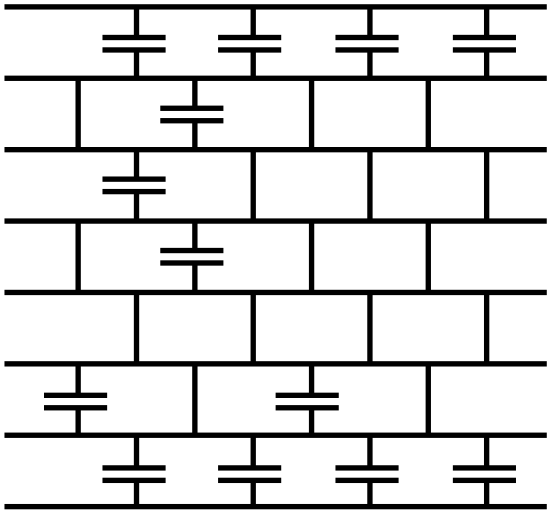
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



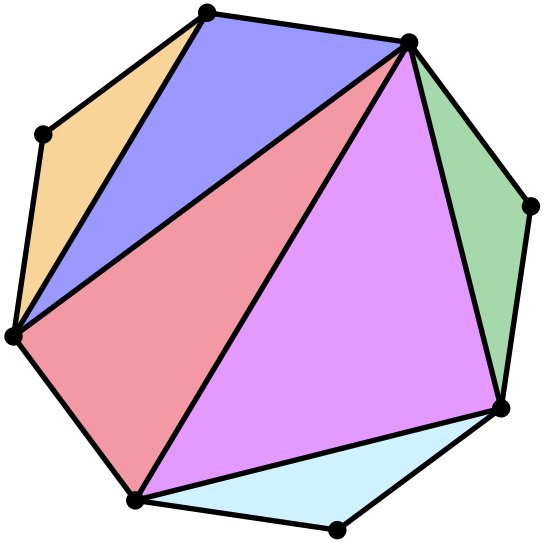


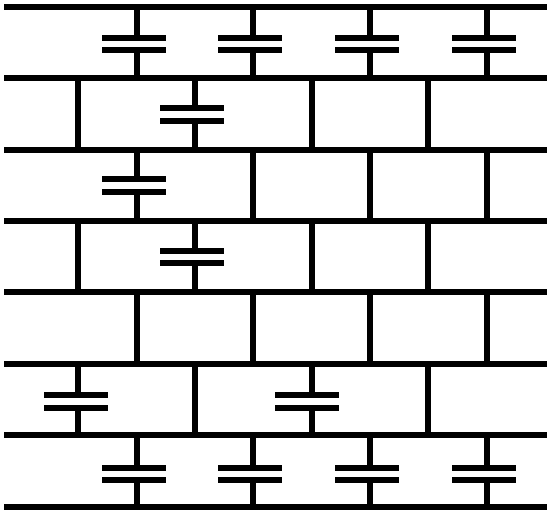
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



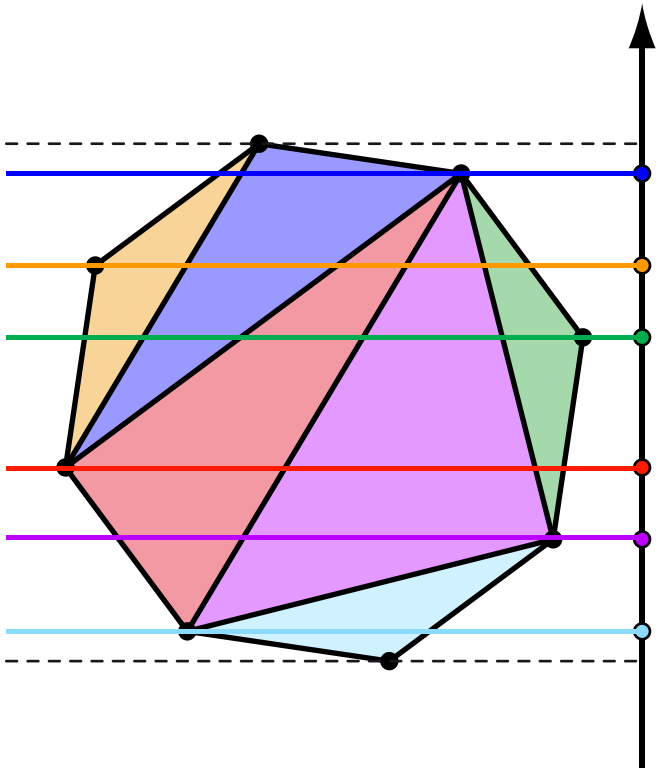


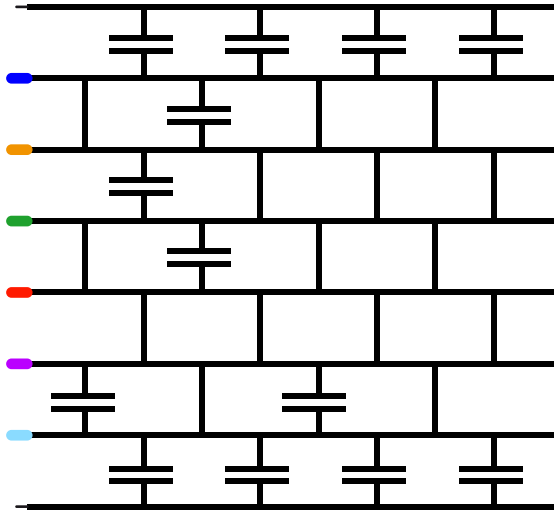
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



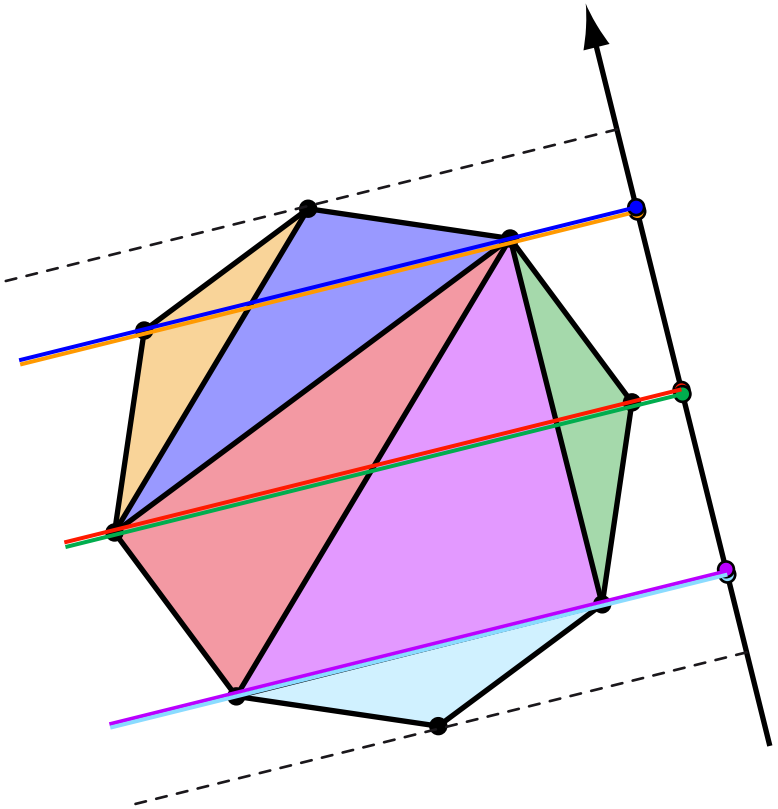


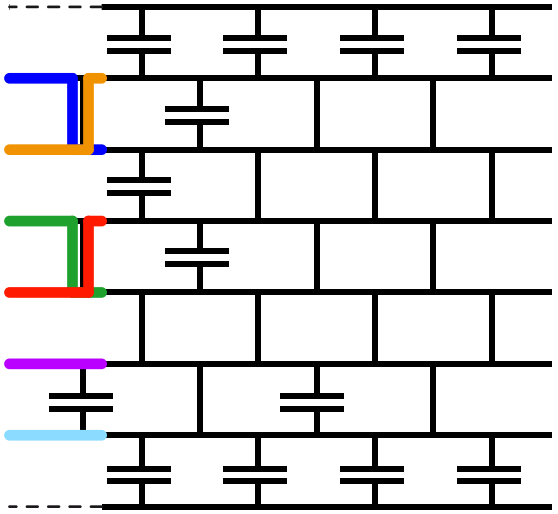
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



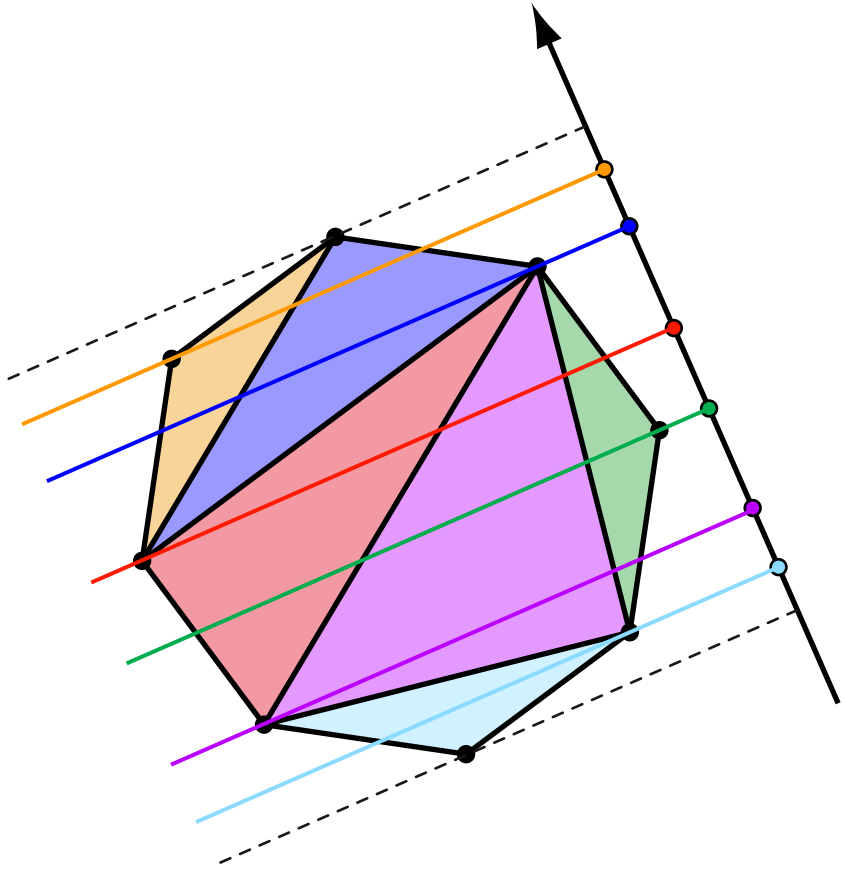


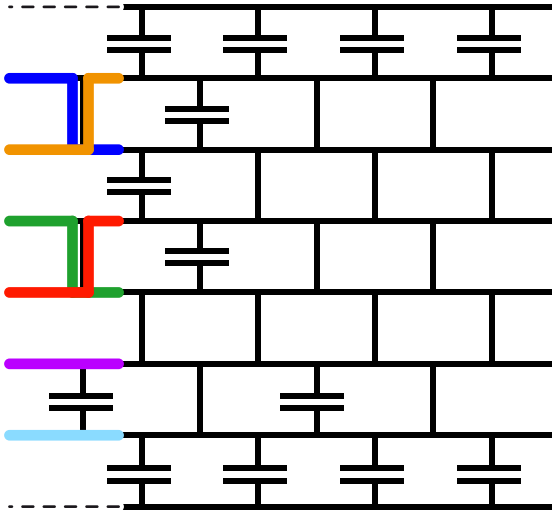
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



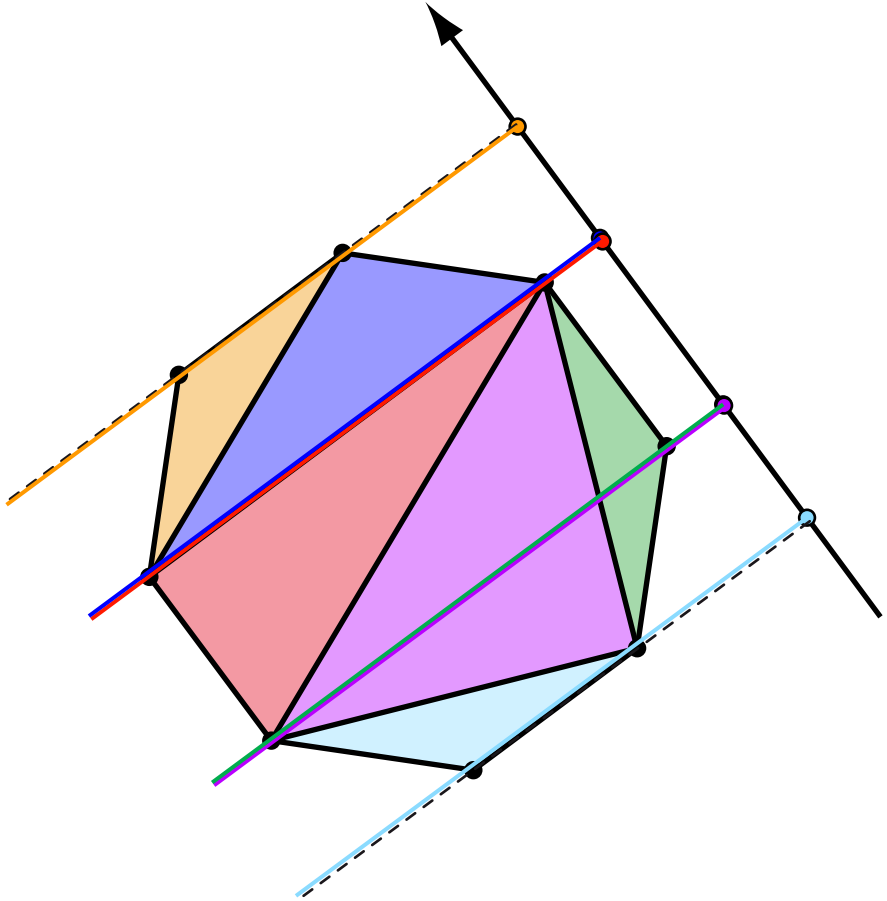


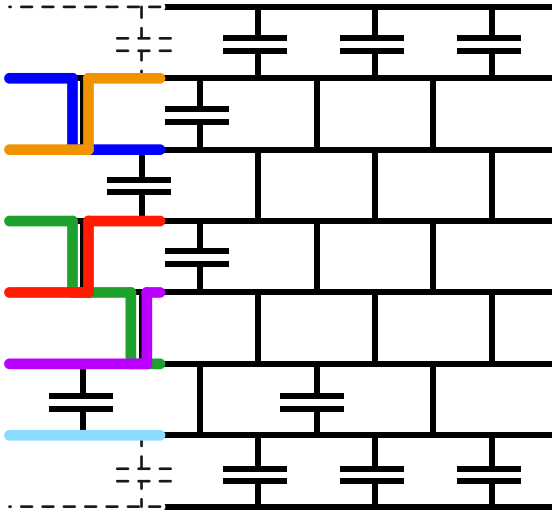
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



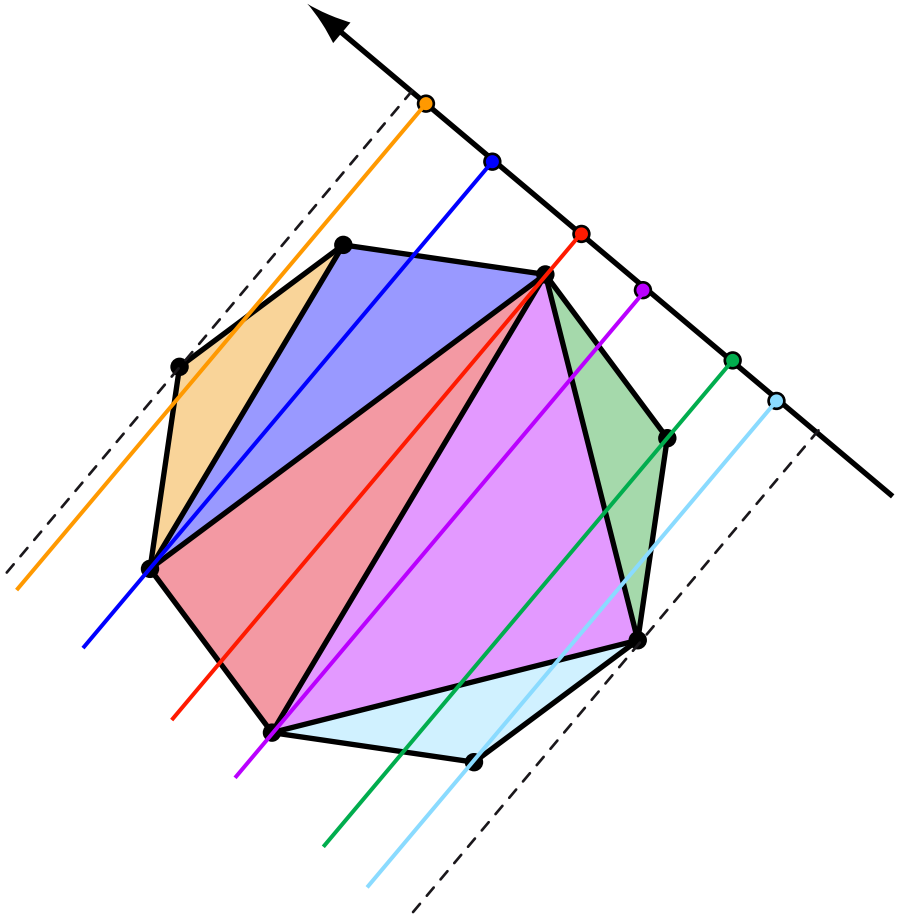


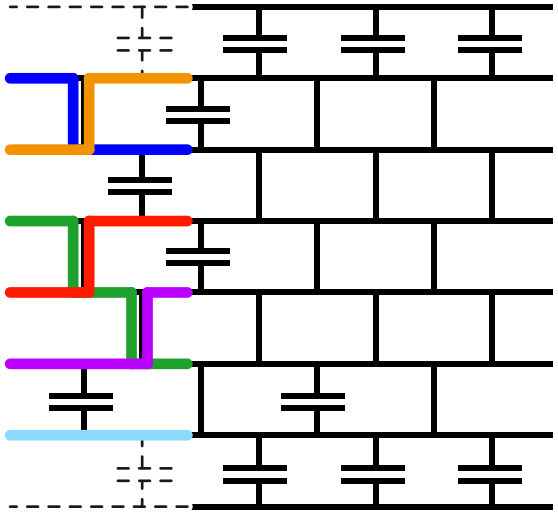
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



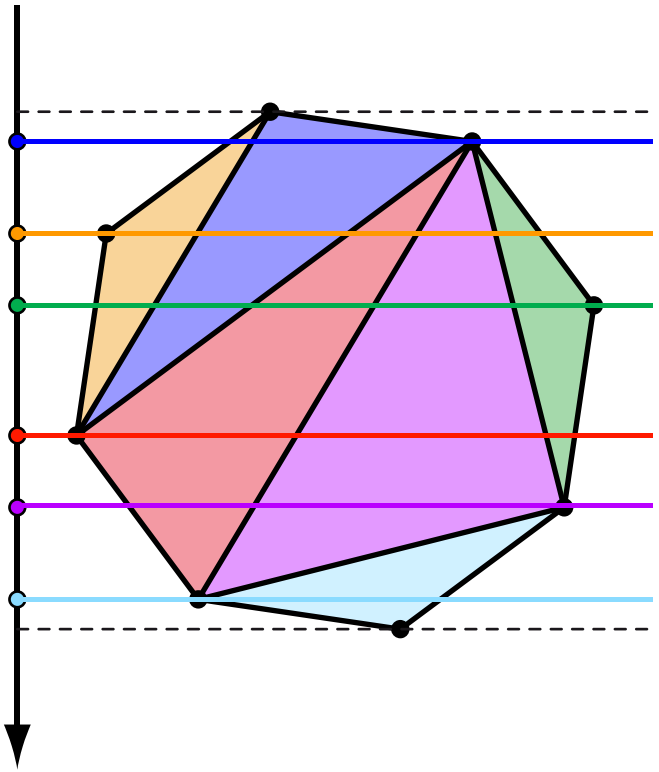


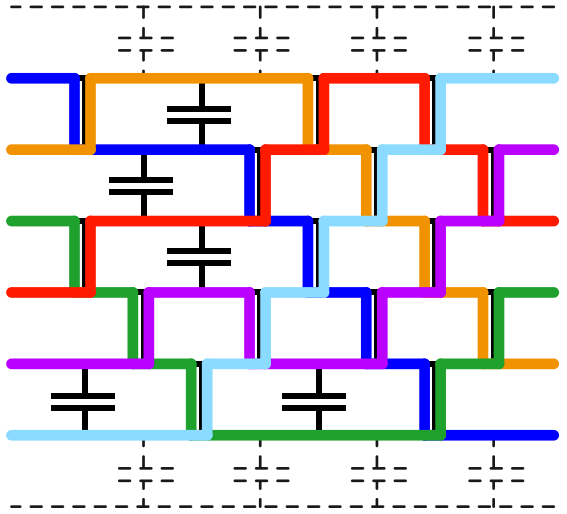
TRIANGULATIONS & ALTERNATING SORTING NETWORKS



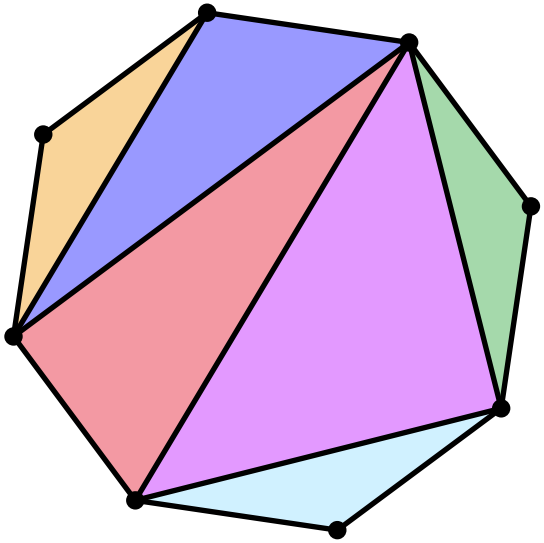


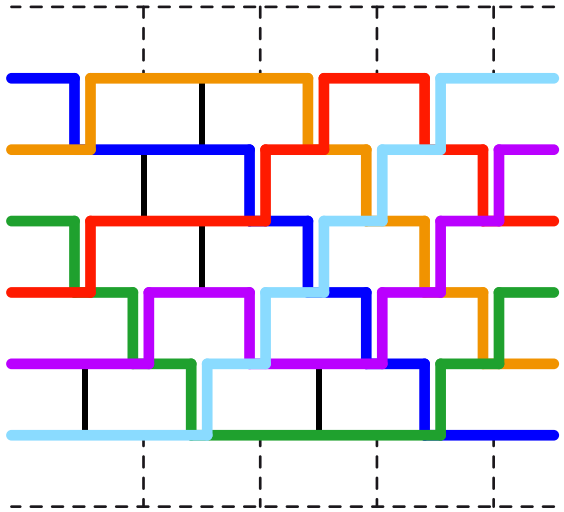
TRIANGULATIONS & ALTERNATING SORTING NETWORKS





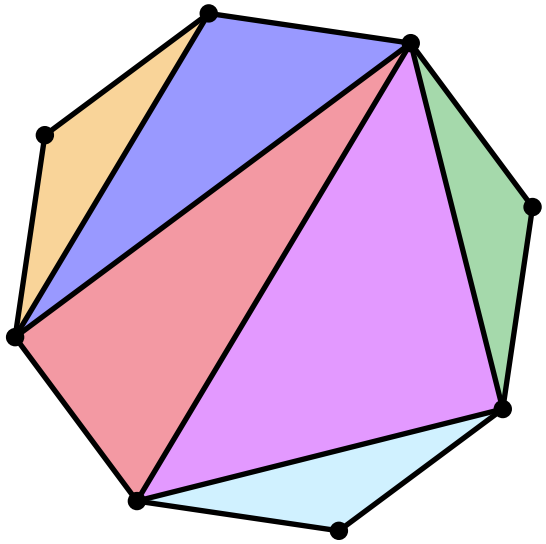
TRIANGULATIONS & ALTERNATING SORTING NETWORKS

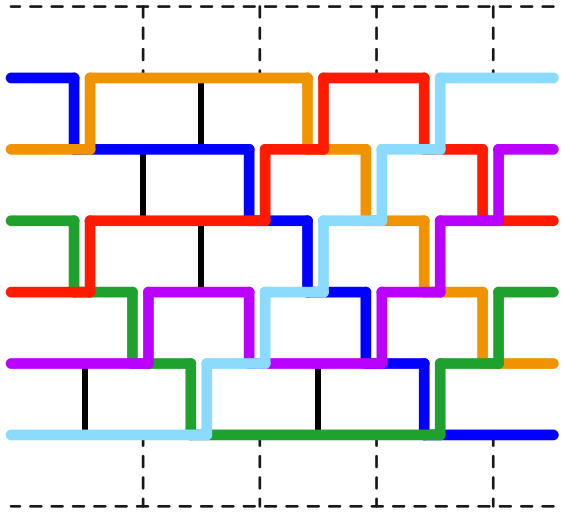


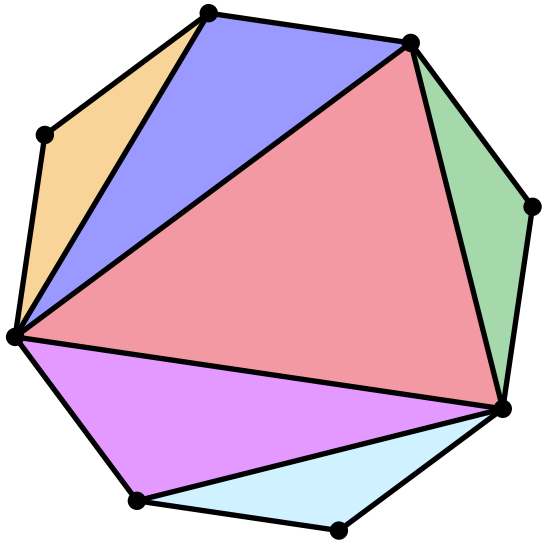


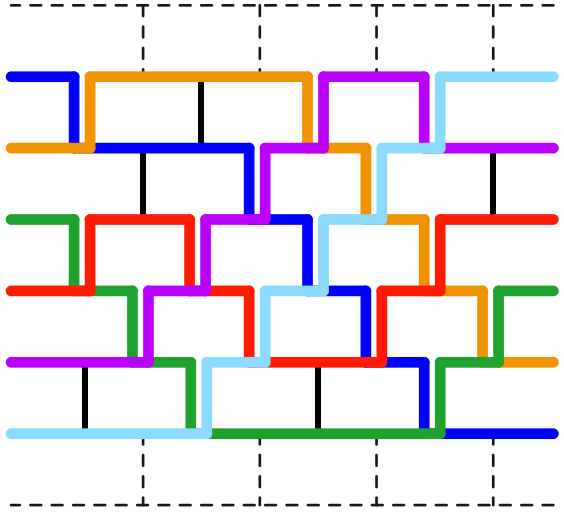
triangulation of the n-gon	$\longleftrightarrow$	pseudoline arrangement
triangle	$\longleftrightarrow$	pseudoline
edge	$\longleftrightarrow$	contact point
common bisector	$\longleftrightarrow$	crossing point
dual binary tree	$\longleftrightarrow$	contact graph

FLIPS









PROPERTIES OF THE FLIP GRAPH

The diameter of the graph of flips on triangulations of the n-gon is precisely $2n - 10$ when n is large enough.

D. Sleator, R. Tarjan, & W. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, 1988

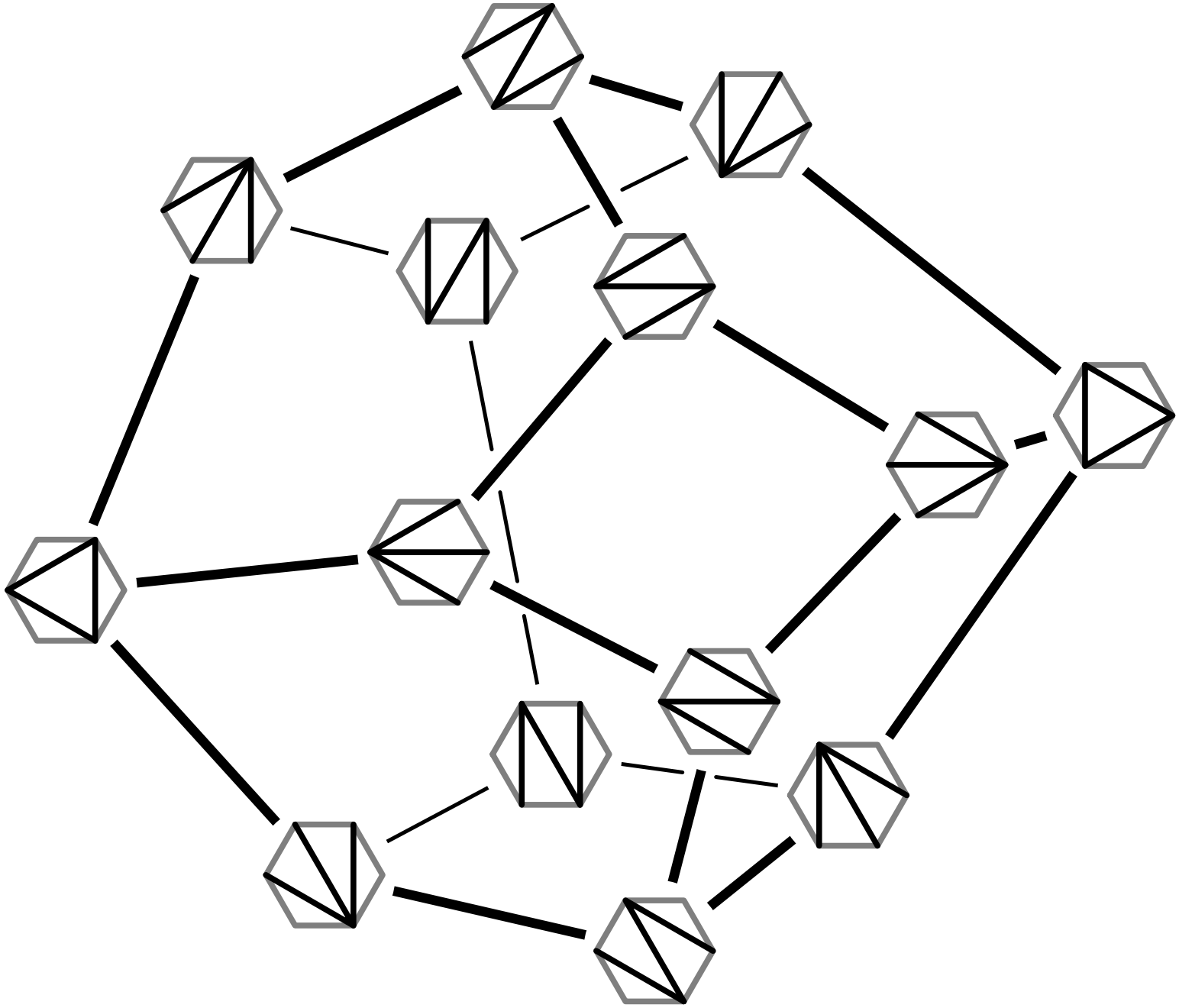
The graph of flips on triangulations of the n-gon is Hamiltonian.

L. Lucas, *The rotation graph of binary trees is Hamiltonian*, 1988
F. Hurado & M. Noy, *Graph of triangulations of a convex polygon and tree of triangulations*, 1999

The graph of flips on triangulations of the n-gon is polytopal.

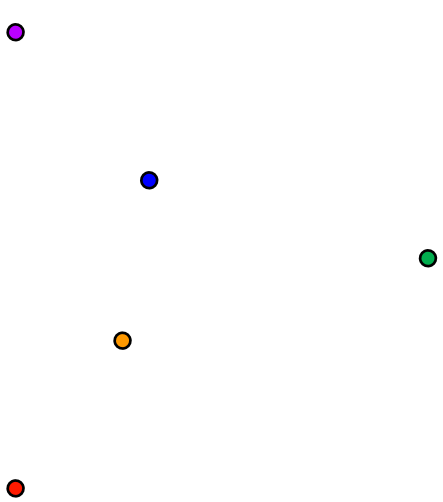
C. Lee, *The associahedron and triangulations of the n-gon*, 1989
L. Billera, P. Filliman, & B. Strumfels, *Construction and complexity of secondary polytopes*, 1990
J.-L. Loday, *Realization of the Stasheff polytope*, 2004
C. Holhweg & C. Lange, *Realizations of the associahedron and cyclohedron*, 2007
A. Postnikov, *Permutahedra, associahedra, and beyond*, 2009
VP & F. Santos, *The brick polytope of a sorting network*, 2012
C. Ceballos, F. Santos, & G. Ziegler, *Many non-equivalent realizations of the associahedron*, 2012$^+$

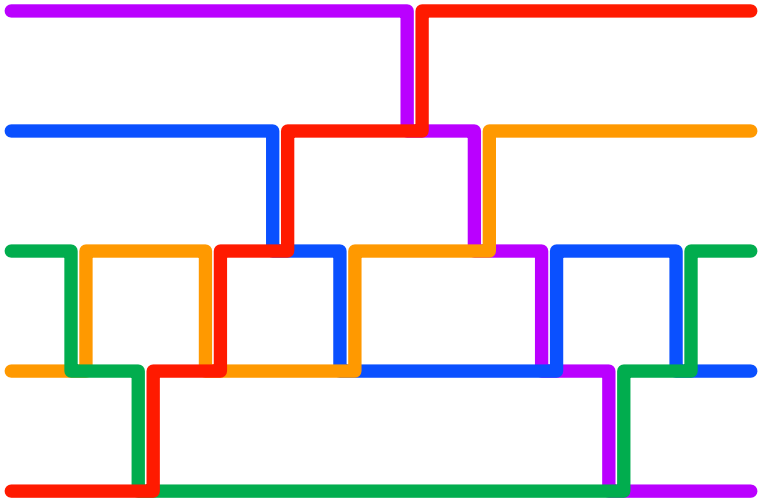
ASSOCIAHEDRA



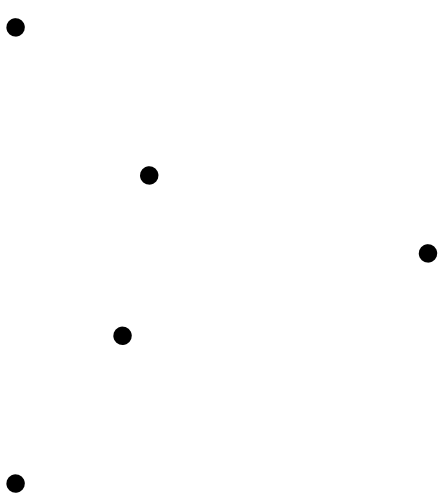
PSEUDOTRIANGULATIONS
— & —
MULTITRIANGULATIONS

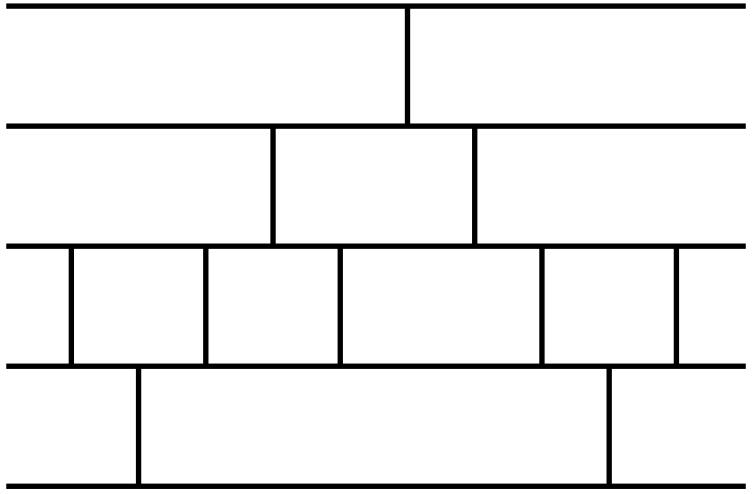
PSEUDOTRIANGULATIONS



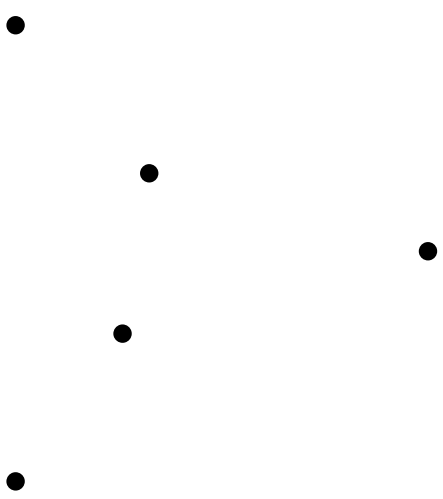


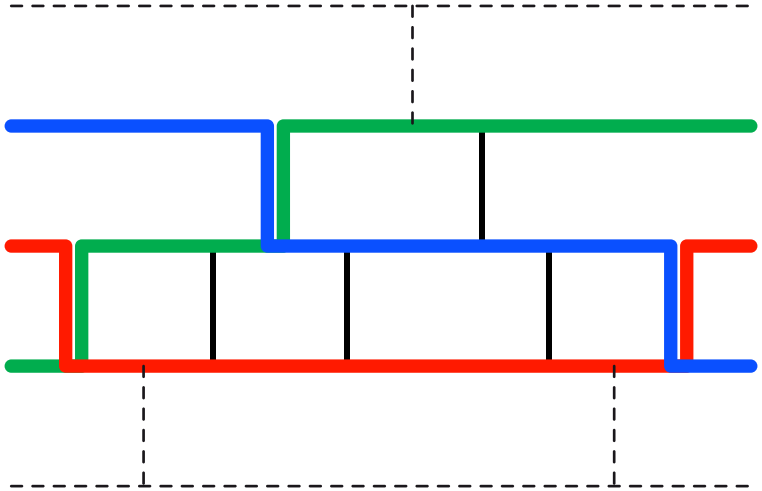
PSEUDOTRIANGULATIONS



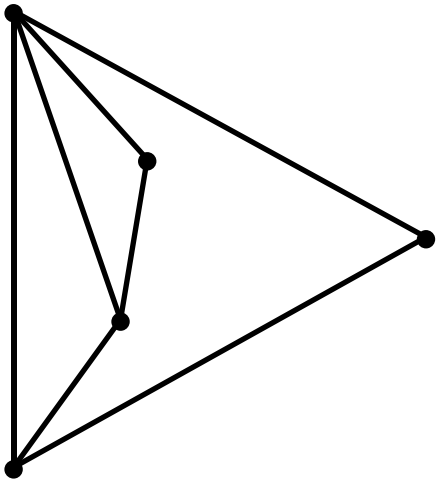


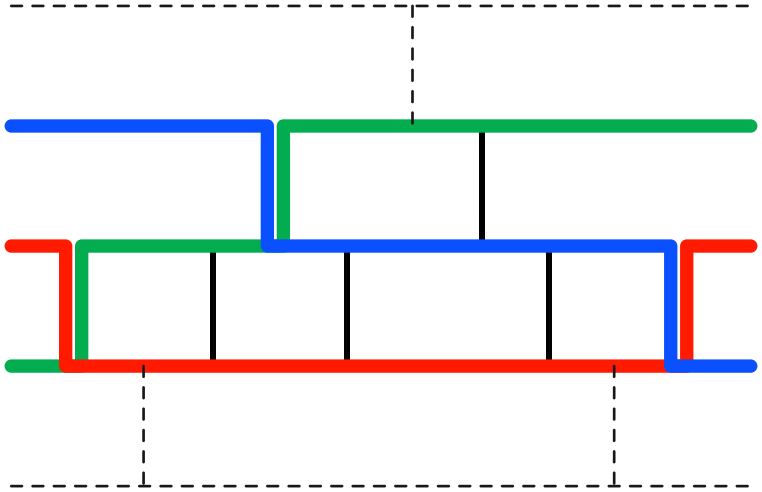
PSEUDOTRIANGULATIONS





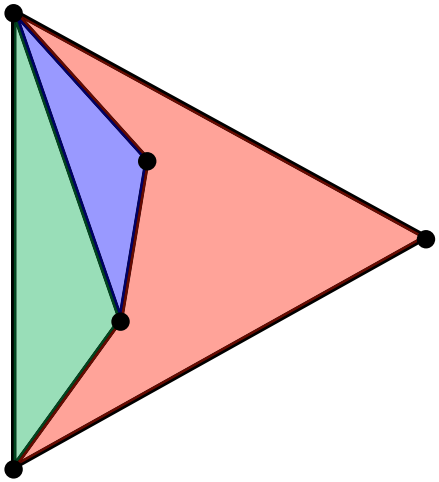
PSEUDOTRIANGULATIONS

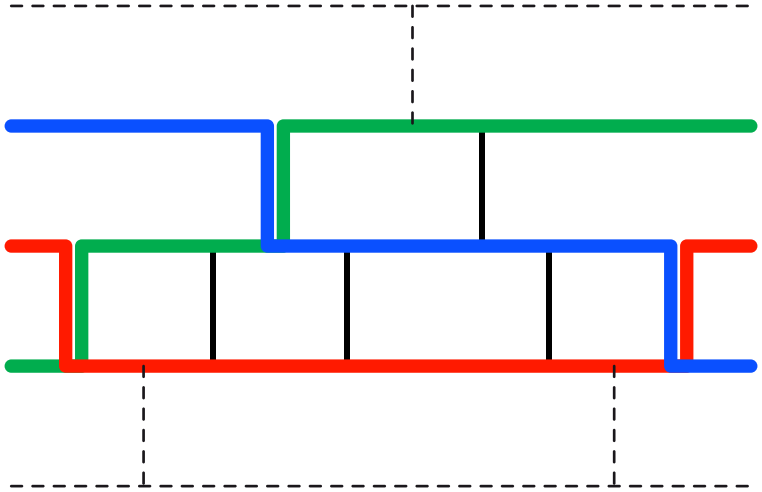




pseudotriangulation of P = maximal crossing-free and pointed set of edges on P

PSEUDOTRIANGULATIONS

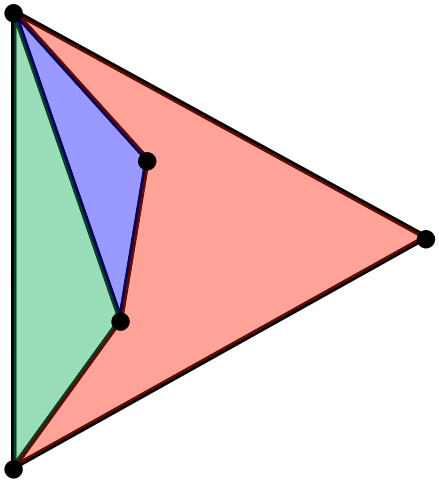


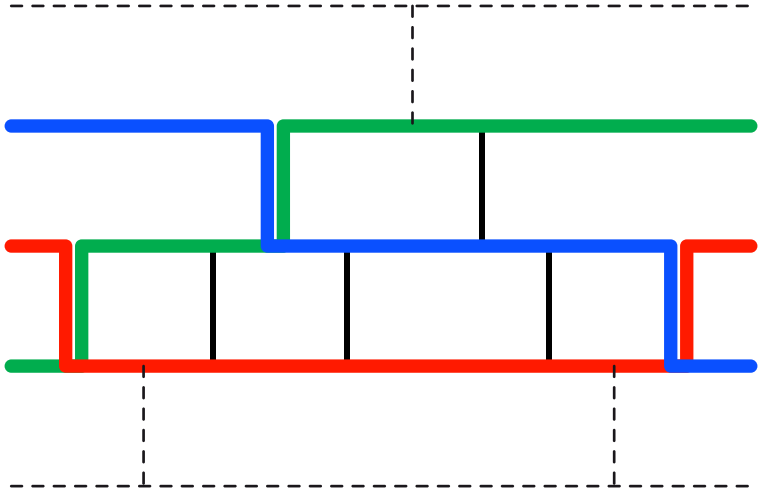


pseudotriangulation of P $=$ maximal crossing-free and pointed set of edges on P

$=$ complex of pseudotriangles

PSEUDOTRIANGULATIONS





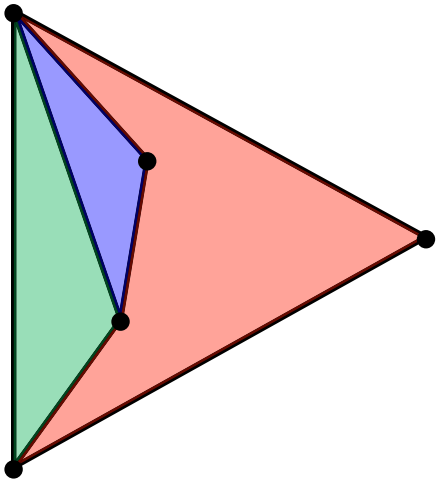
pseudotriangulation of P $=$ maximal crossing-free and pointed set of edges on P

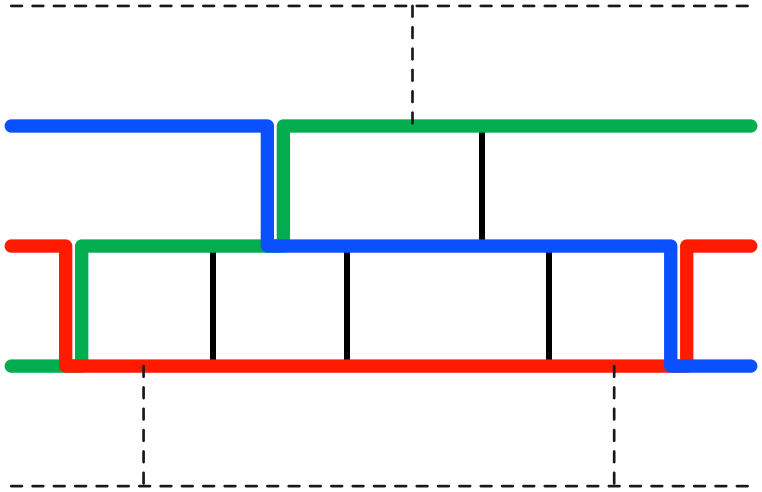
$=$ complex of pseudotriangles

object from computational geometry

applications to visibility, rigidity, motion planning, ...

PSEUDOTRIANGULATIONS



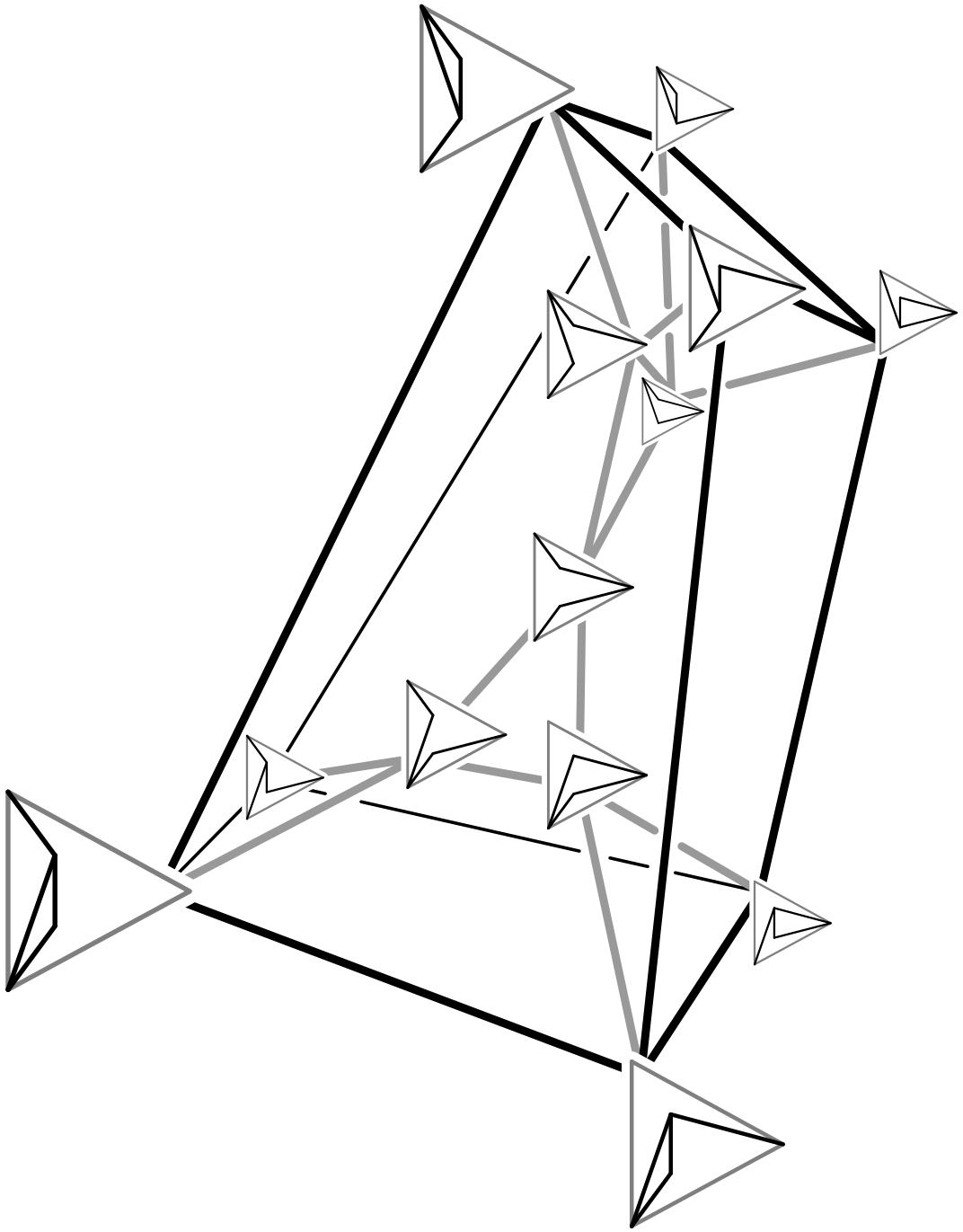


pseudotriangulation of P = maximal crossing-free and pointed set of edges on P
= complex of pseudotriangles

object from computational geometry
applications to visibility, rigidity, motion planning, ...

properties of the flip graph: $\Omega(n) \leq$ diameter $\leq O(n \ln n)$
graph of the pseudotriangulation polytope

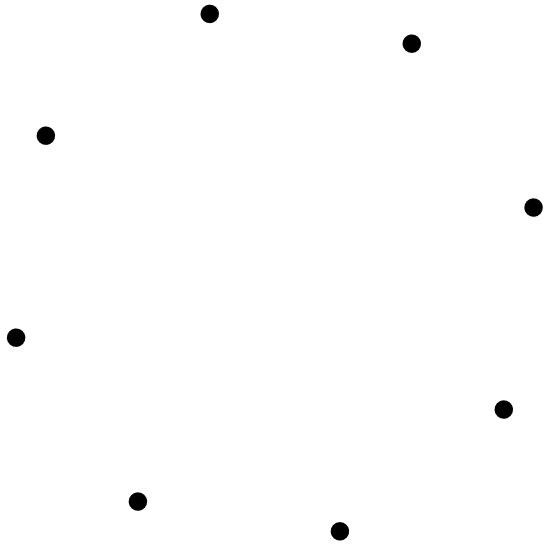
PSEUDOTRIANGULATIONS

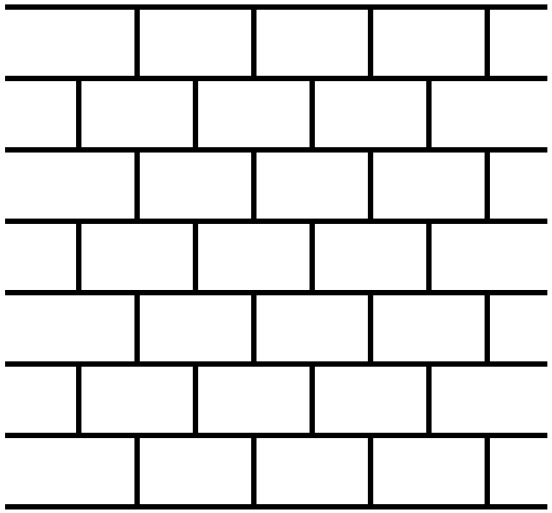


The flip graph on pseudotriangulations of a planar point set P is polytopal

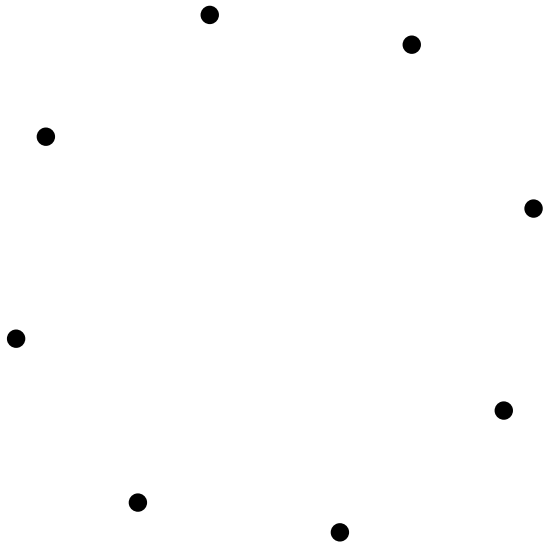
G. Rote, F. Santos, I. Streinu,
Expansive motions and the polytope
of pointed pseudotriangulations, 2008

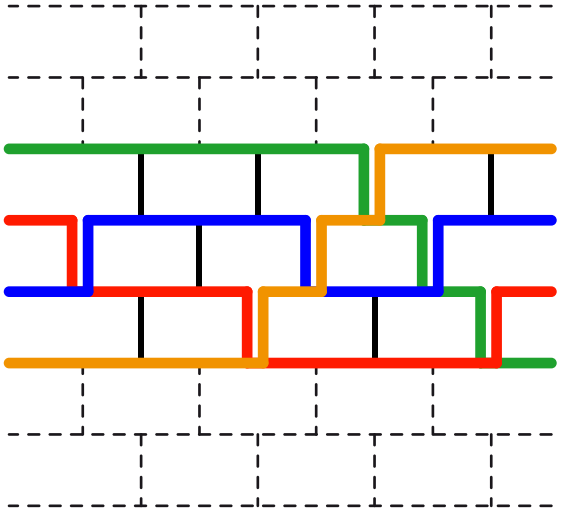
MULTITRIANGULATIONS



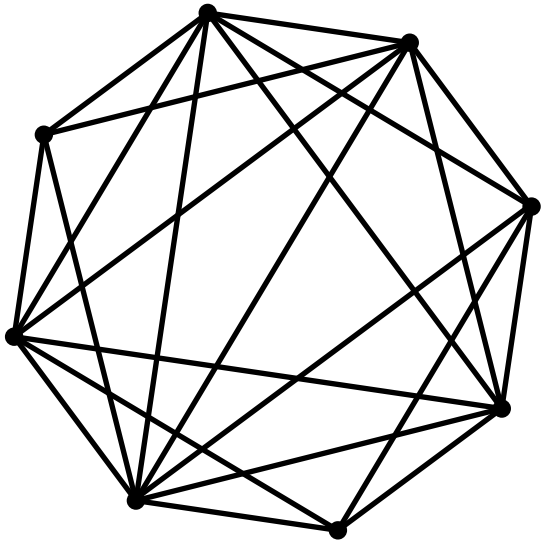


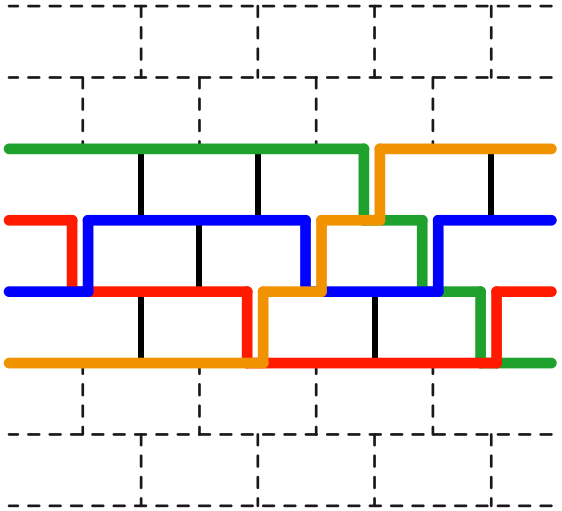
MULTITRIANGULATIONS





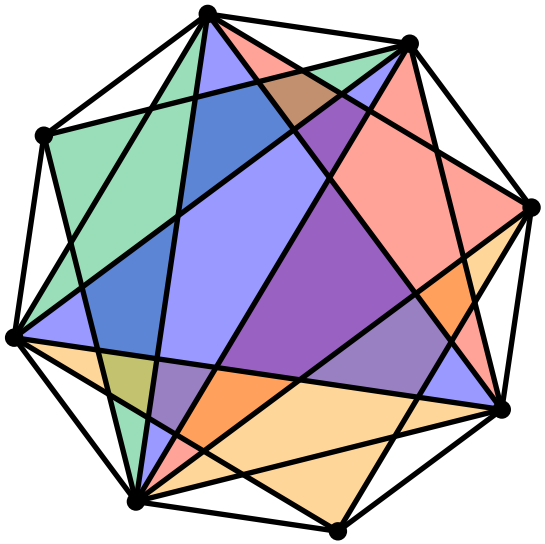
MULTITRIANGULATIONS

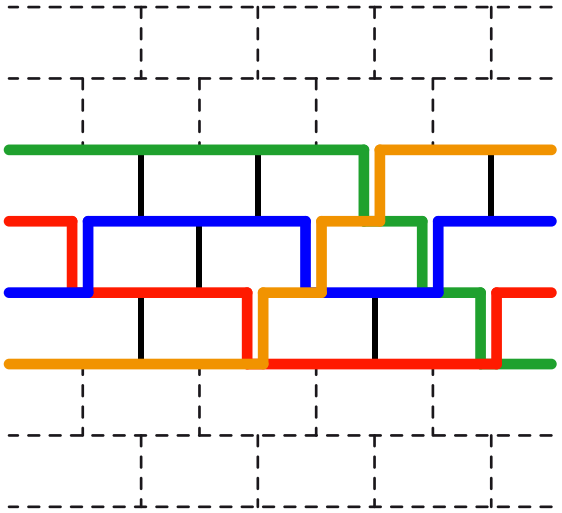




k-triangulation of the n-gon $=$ maximal $(k+1)$-crossing-free set of edges

MULTITRIANGULATIONS

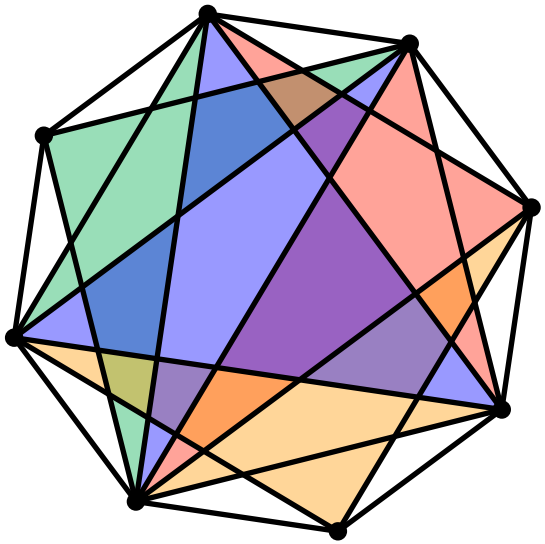


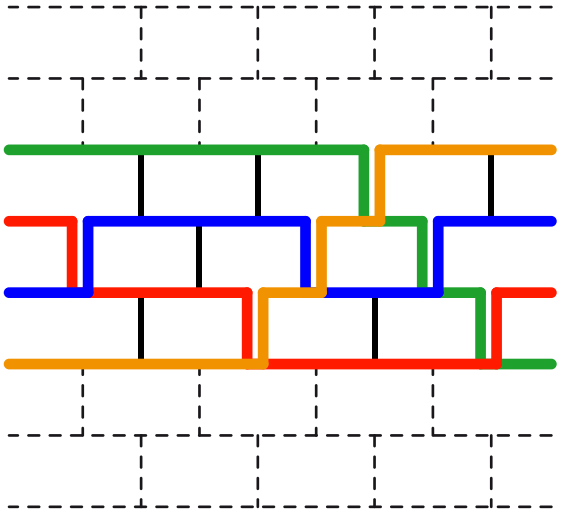


k-triangulation of the n-gon $=$ maximal $(k+1)$-crossing-free set of edges

$=$ complex of k-stars

MULTITRIANGULATIONS



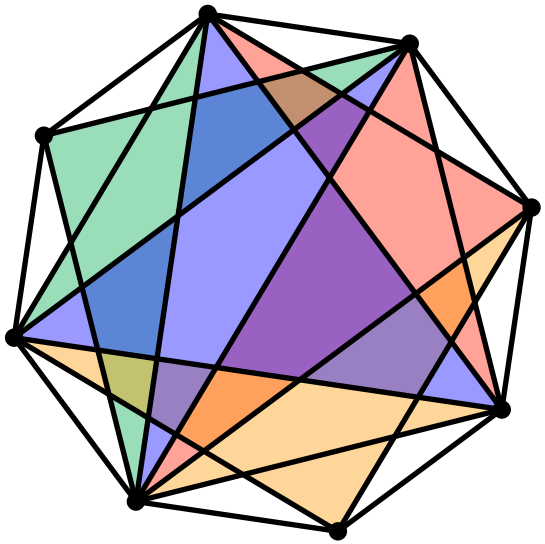


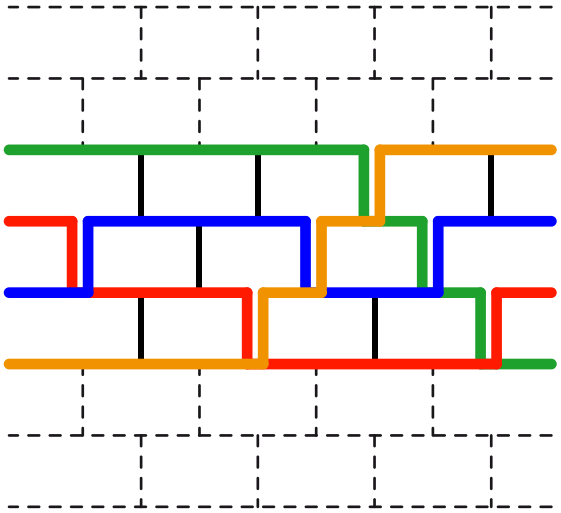
k-triangulation of the n-gon $=$ maximal $(k+1)$-crossing-free set of edges
$=$ complex of k-stars

object from combinatorics

counted by the Hankel determinant $\det([C_{n-i-j}]_{1\leq i,j\leq n})$ of Catalan numbers, ...

MULTITRIANGULATIONS





k-triangulation of the n-gon $=$ maximal $(k+1)$-crossing-free set of edges
$=$ complex of k-stars

object from combinatorics
counted by the Hankel determinant $\det([C_{n-i-j}]_{1\leq i,j\leq n})$ of Catalan numbers, ...

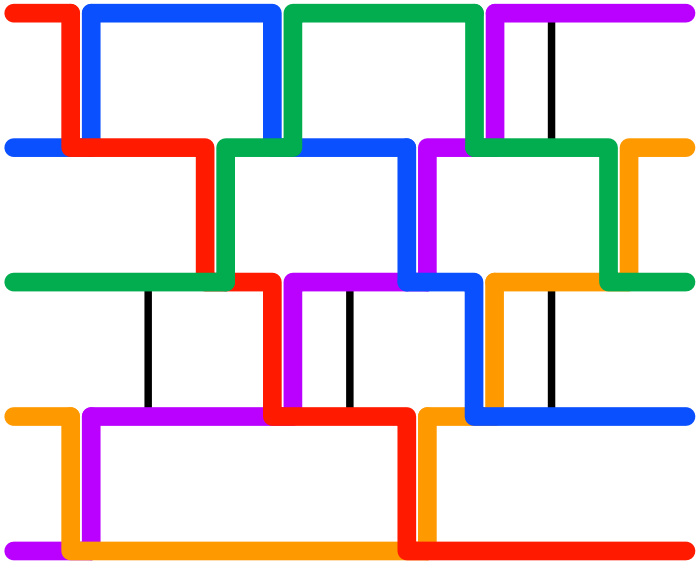
properties of the flip graph: $(k+1/2)n \leq$ diameter $\leq 2kn$
graph of a combinatorial sphere

BRICK POLYTOPE

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

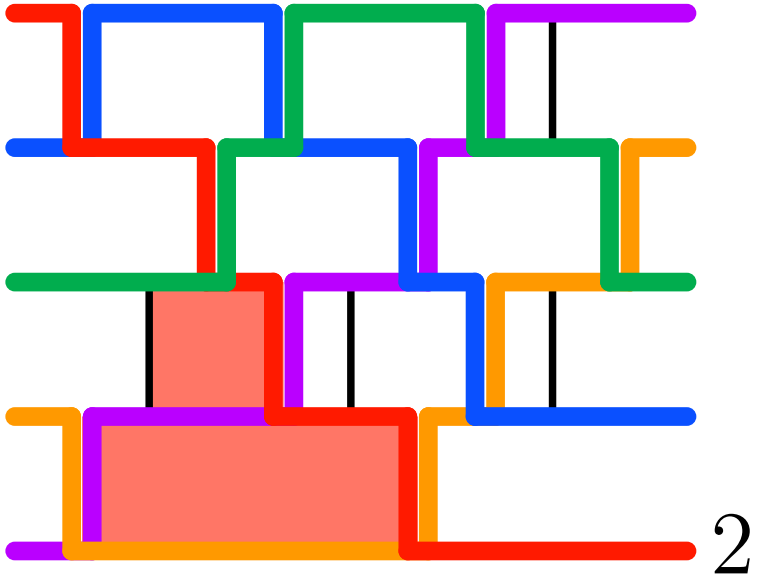


Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

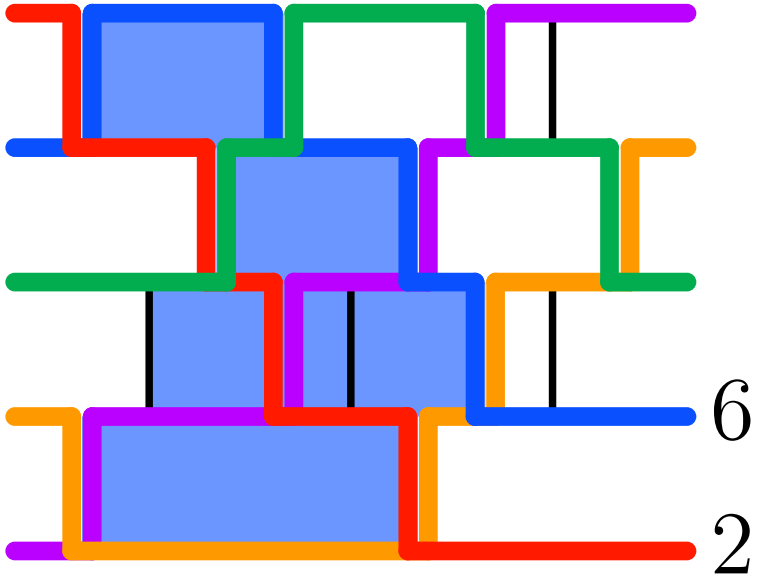


Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

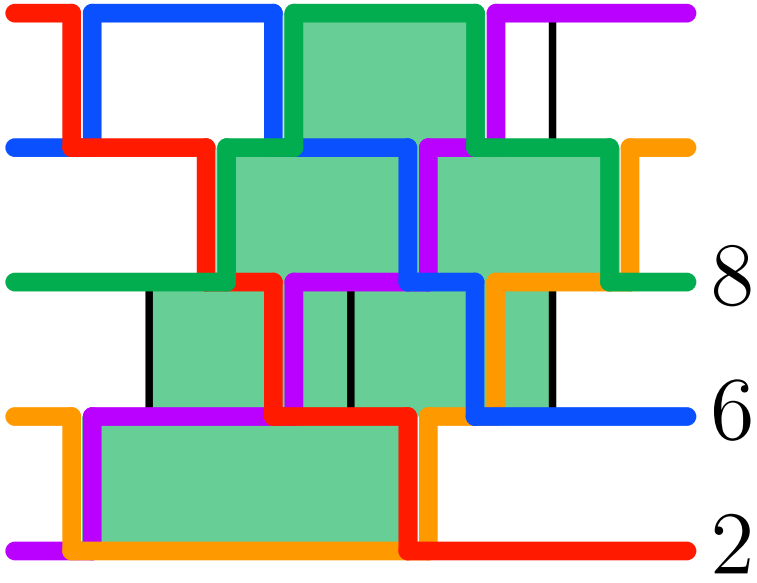


Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

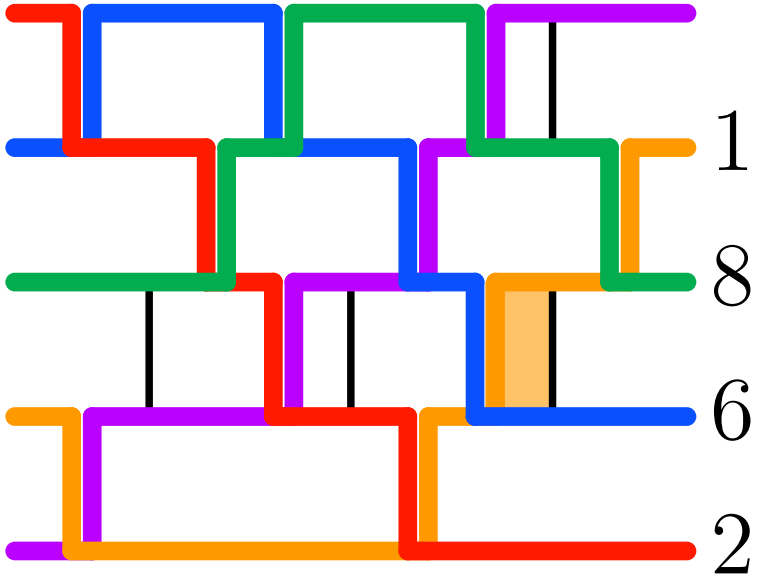


Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ

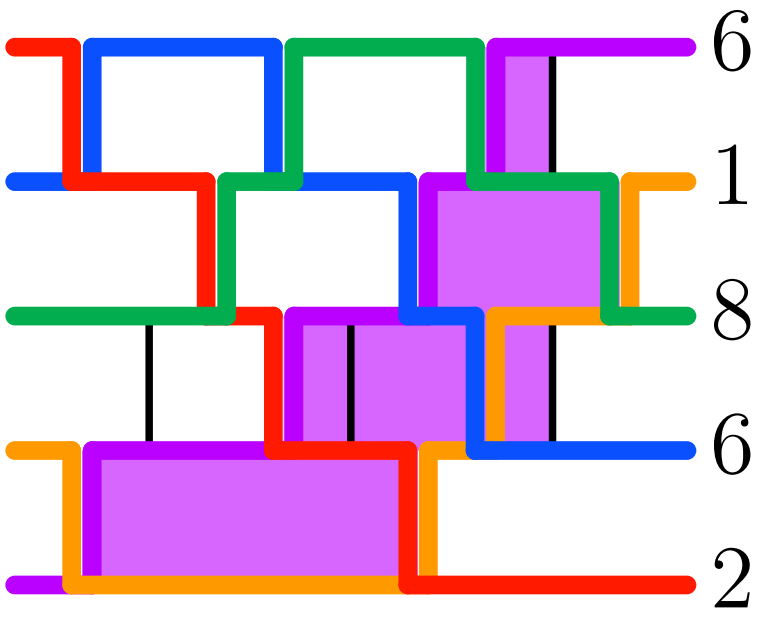


Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ



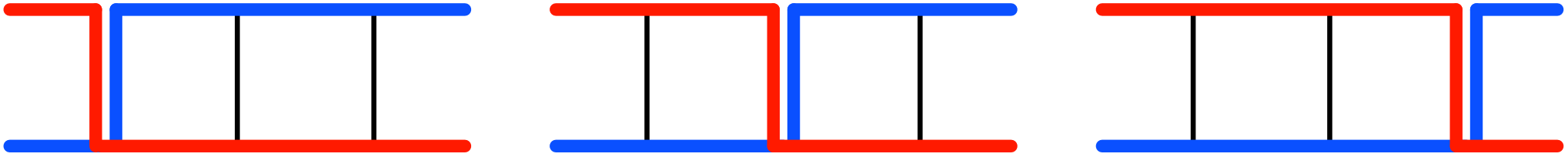
Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$

BRICK POLYTOPE

$\mathcal{X}_m =$ network with two levels and m commutators

graph of flips $G(\mathcal{X}_m) =$ complete graph K_m

brick polytope $\Omega(\mathcal{X}_m) = \text{conv}\left\{ \binom{m-i}{i-1} \;\middle|\; i \in [m] \right\} = \left[\binom{m-1}{0}, \binom{0}{m-1} \right]$

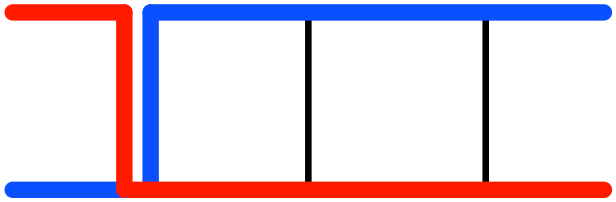
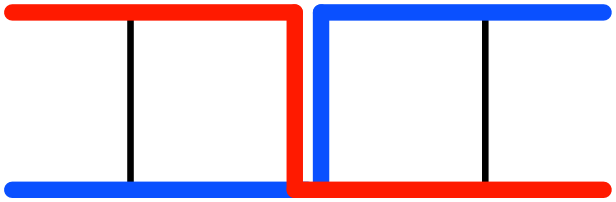
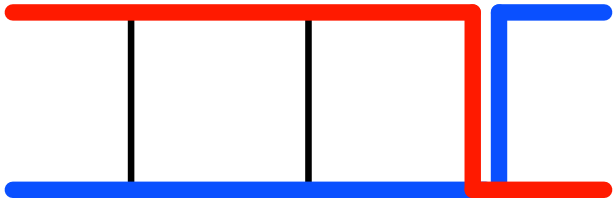


BRICK POLYTOPE

$\mathcal{X}_m$ = network with two levels and m commutators

graph of flips $G(\mathcal{X}_m)$ = complete graph K_m

brick polytope $\Omega(\mathcal{X}_m) = \text{conv}\left\{ \binom{m-i}{i-1} \;\middle|\; i \in [m] \right\} = \left[\binom{m-1}{0}, \binom{0}{m-1} \right]$

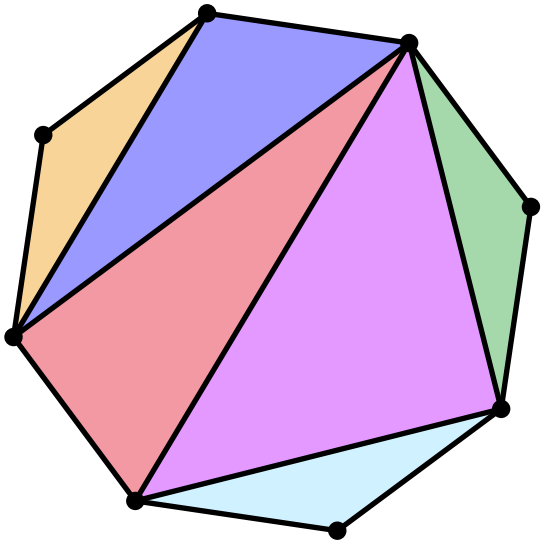




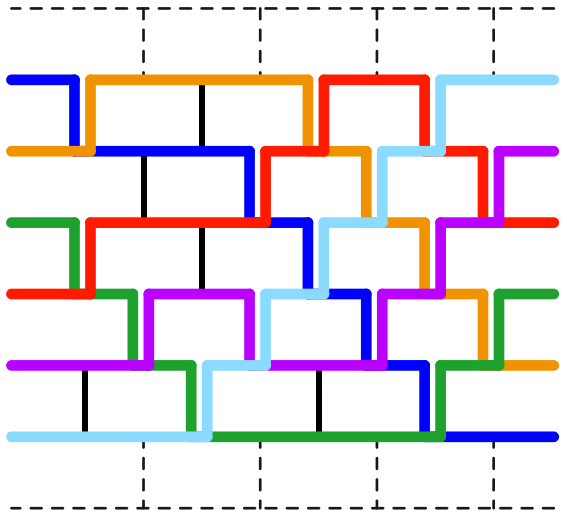
The brick vector $\omega(\Lambda)$ is a vertex of $\Omega(\mathcal{N})$ $\iff$ the contact graph $\Lambda^\#$ is acyclic
The graph of the brick polytope $\Omega(\mathcal{N})$ is a subgraph of the flip graph $G(\mathcal{N})$

The graph of the brick polytope $\Omega(\mathcal{N})$ coincides with the graph of flips $G(\mathcal{N})$
$\iff$ the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$ are forests

ASSOCIAHEDRA
— & —
PERMUTAHEDRA

ALTERNATING NETWORKS & ASSOCIAHEDRA



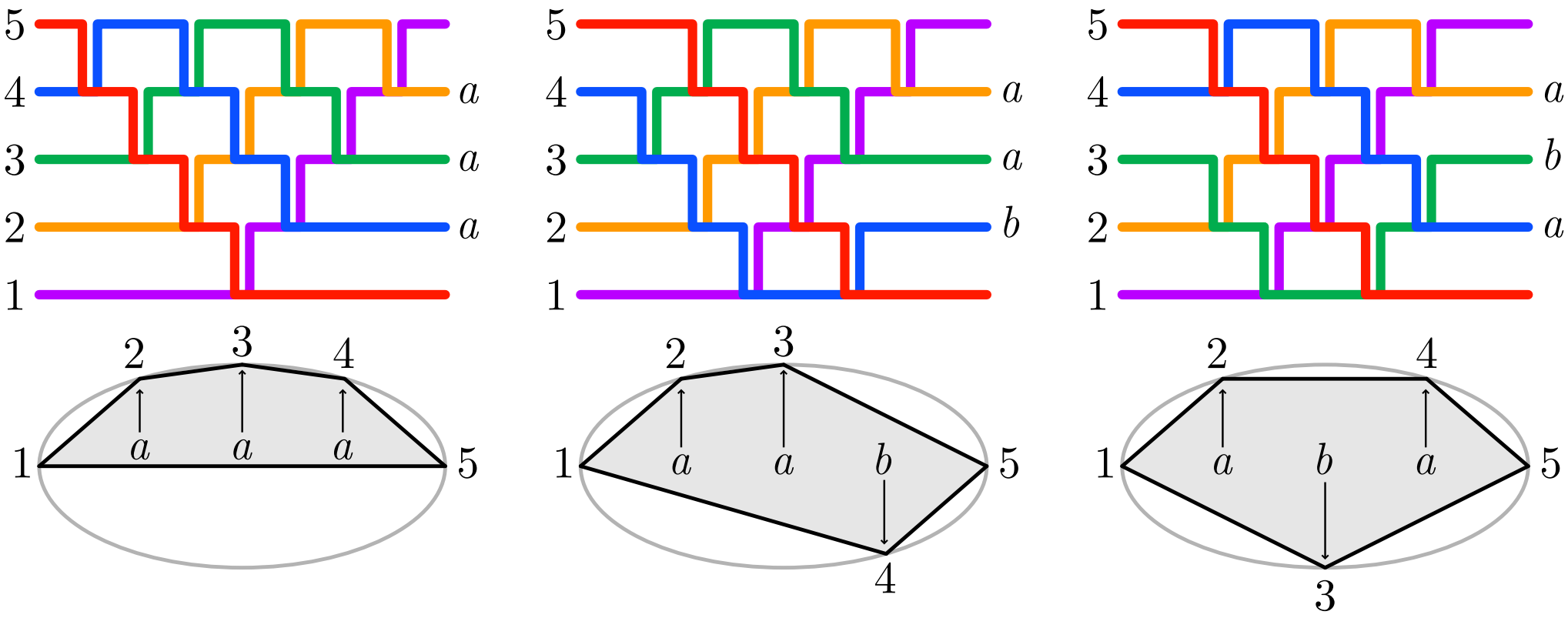


triangulation of the n-gon	$\longleftrightarrow$	pseudoline arrangement
triangle	$\longleftrightarrow$	pseudoline
edge	$\longleftrightarrow$	contact point
common bisector	$\longleftrightarrow$	crossing point
dual binary tree	$\longleftrightarrow$	contact graph

The brick polytope is an associahedron.

ALTERNATING NETWORKS & ASSOCIAHEDRA

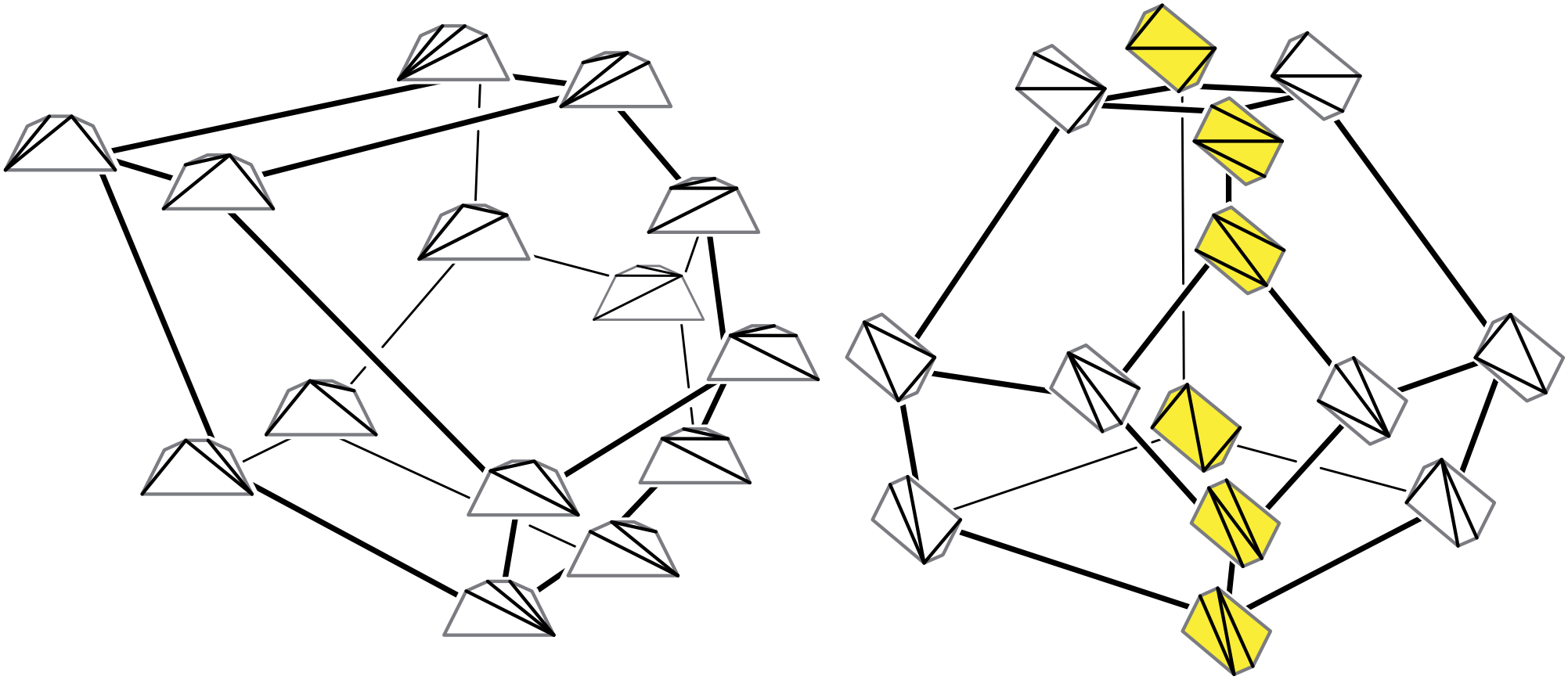
for $x \in \{a, b\}^{n-2}$, define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$



Pseudoline arrangements on $\mathcal{N}_x^1 \longleftrightarrow$ triangulations of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORKS & ASSOCIAHEDRA

For any word $x \in \{a, b\}^{n-2}$, the brick polytope $\Omega(\mathcal{N}_x^1)$ is an associahedron



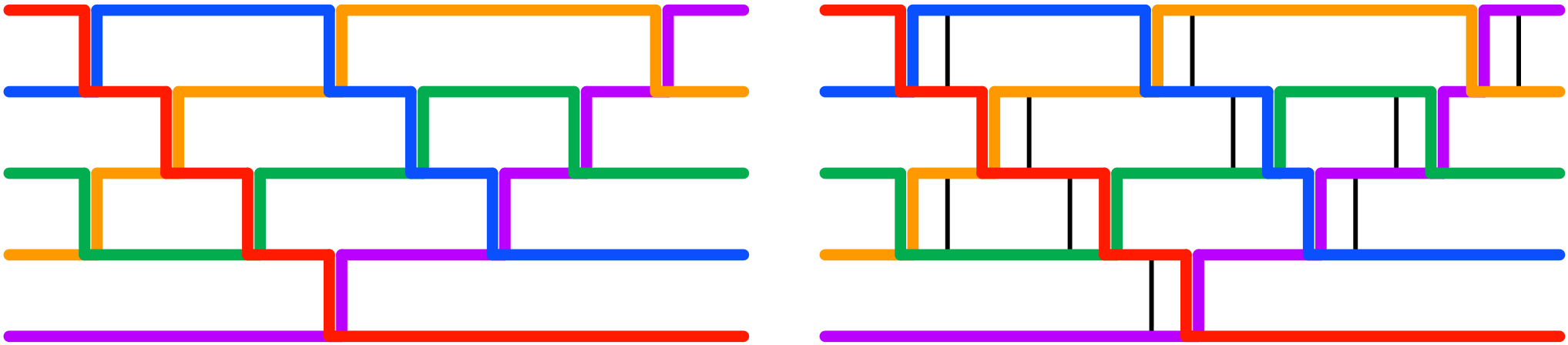
C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007

VP & F. Santos, The brick polytope of a sorting network, 2012

DUPLICATED NETWORKS & PERMUTAHEDRA

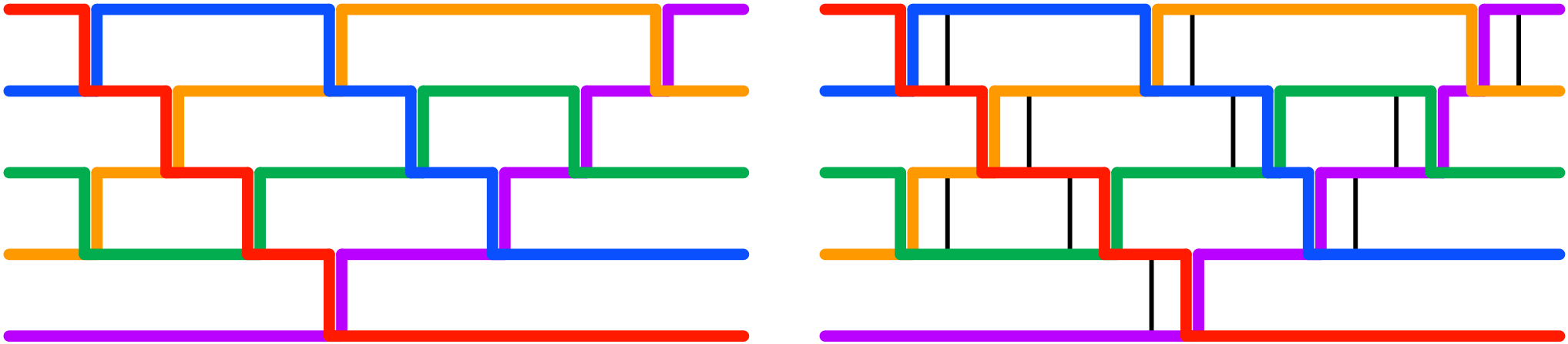
reduced network $=$ network with n levels and $\binom{n}{2}$ commutators
it supports only one pseudoline arrangement

duplicated network $\Pi =$ network with n levels and $2\binom{n}{2}$ commutators obtained by duplicating each commutator of a reduced network



Any pseudoline arrangement supported by Π has one contact
and one crossing among each pair of duplicated commutators.

DUPLICATED NETWORKS & PERMUTAHEDRA

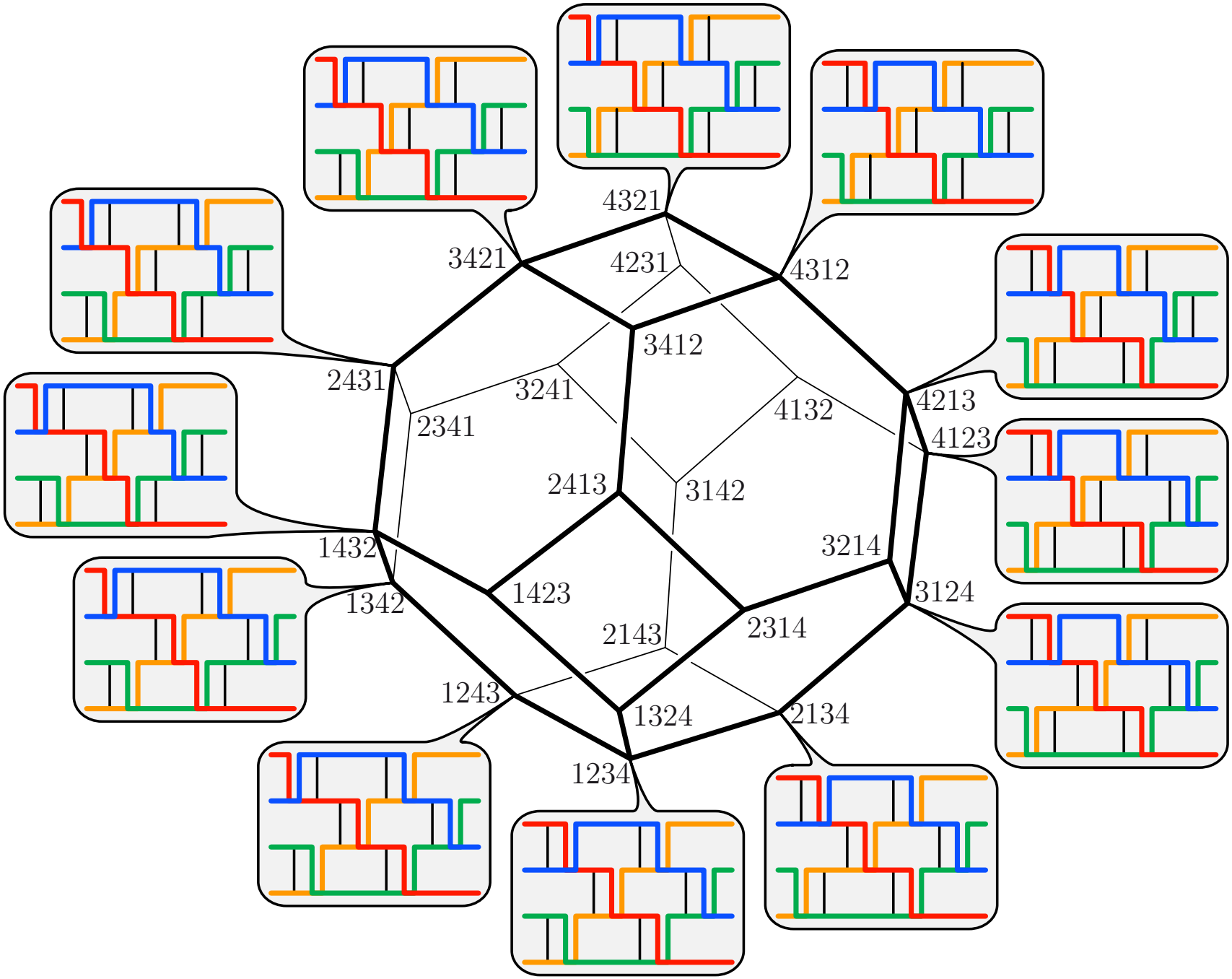


Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators $\implies$ The contact graph $\Lambda^{\#}$ is a tournament.

Vertices of $\Omega(\Pi)$ $\iff$ acyclic tournaments $\iff$ permutations of $[n]$

Brick polytope $\Omega(\Pi) =$ permutahedron

DUPLICATED NETWORKS & PERMUTAHEDRA



THANK YOU