The expected number of inversions after $n$ adjacent transpositions

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A Markov chain

- Consider the symmetric group $\mathcal{S}_{d+1}$ on the elements $\{0, 1, \ldots, d\}$.

- Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

- Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

- Repeat.

Example: $d = 2$

```
\begin{center}
\begin{tikzpicture}
  \node (012) at (0,0) {012};
  \node (120) at (-1,-1) {120};
  \node (201) at (1,-1) {201};
  \node (102) at (-1,1) {102};
  \node (021) at (1,1) {021};
  \node (210) at (0,2) {210};
  \draw (012) -- (120);
  \draw (012) -- (201);
  \draw (012) -- (102);
  \draw (012) -- (021);
  \draw (120) -- (102);
  \draw (201) -- (021);
  \draw (210) -- (120);
  \draw (210) -- (201);
  \draw (210) -- (102);
  \draw (210) -- (021);
\end{tikzpicture}
\end{center}
```

$\pi^{(0)} = 012$
A Markov chain

- Consider the symmetric group $\mathcal{S}_{d+1}$ on the elements $\{0, 1, \ldots, d\}$.

- Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

- Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

- Repeat.

**Example:** $d = 2$

\[
\begin{array}{ccc}
120 & 210 & 201 \\
102 & & 021 \\
012 & & \\
\end{array}
\]

$\pi^{(1)} = 102$
A Markov chain

- Consider the symmetric group $\mathcal{G}_{d+1}$ on the elements $\{0, 1, \ldots, d\}$.

- Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

- Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

- Repeat.

Example: $d = 2$

```
π(2) = 120
```

```
\begin{tikzpicture}
    \node (120) at (0,0) {120};
    \node (210) at (1,1) {210};
    \node (201) at (2,0) {201};
    \node (021) at (1,-1) {021};
    \node (012) at (0,-2) {012};
    \draw (120) -- (210);
    \draw (210) -- (201);
    \draw (201) -- (021);
    \draw (021) -- (012);
    \draw (012) -- (120);
\end{tikzpicture}
```
A Markov chain

- Consider the symmetric group $\mathcal{S}_{d+1}$ on the elements \( \{0, 1, \ldots, d\} \).

- Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

- Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

- Repeat.

**Example:** $d = 2$

\[
\begin{align*}
210 & \quad 012 \\
120 & \quad 102 & \quad 201 \\
102 & \quad 021 \\
012
\end{align*}
\]

$\pi^{(3)} = 102$
A Markov chain

- Consider the symmetric group $\mathcal{G}_{d+1}$ on the elements $\{0, 1, \ldots, d\}$.

- Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

- Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

- Repeat.

Example: $d = 2$

```
\begin{tikzpicture}
  \node (210) at (0,2) {210};
  \node (120) at (-1,-1) {120};
  \node (201) at (1,-1) {201};
  \node (102) at (-1,-2) {102};
  \node (021) at (1,-2) {021};
  \node (012) at (0,-3) {012};

  \path[->] (210) edge (120);
  \path[->] (210) edge (201);
  \path[->] (120) edge (102);
  \path[->] (120) edge (021);
  \path[->] (201) edge (021);
  \path[->] (021) edge (012);

  \node at (0,1) {$\pi^{(4)} = 012$};
\end{tikzpicture}
```
A Markov chain

• Consider the symmetric group $\mathcal{G}_{d+1}$ on the elements $\{0, 1, \ldots, d\}$.

• Start from the identity permutation $\pi^{(0)} = 012 \cdots d$.

• Apply an adjacent transposition, taken uniformly at random (probability $1/d$ for each).

• Repeat.

Example: $d = 2$

$\begin{array}{c}
\pi^{(5)} = 012 \\
\end{array}$
Periodicity

This chain, \( \pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots \) is periodic of period 2: it takes an even number of steps to return to a point.

```
      210
     /   \
    120   201
   /     \
 102     021
 /       \
012
```
An aperiodic variant

This chain, $\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots$ is periodic of period 2: it takes an even number of steps to return to a point.

- Do nothing with probability $1/(d+1)$, and otherwise apply an adjacent transposition chosen uniformly.

This chain is aperiodic, irreducible and symmetric, and thus converges to the uniform distribution on $\mathcal{S}_{d+1}$. 
Motivations

• 1980 → 2012: Random walks in finite groups (Aldous, Diaconis, Letac, Saloff-Coste, Wilson...)

Tools: coupling techniques, representation theory...
Motivations


Tools: coupling techniques, representation theory...

- More recently: Computational biology (N. Beresticky, Durrett, Eriksen, Hultman, H. Eriksson, K. Eriksson, Sjöstrand...)

A transposition: a gene mutation
What do we ask? What do we expect?

- **Mixing time:** How much time does it take to “reach” the uniform distribution?

The total variation distance between the distribution at time $n$ and the uniform distribution on $\mathcal{G}_{d+1}$:

- $d = 3$
- $d = 4$
- $d$ large

![Graphs showing the total variation distance for different values of $d$]
What do we ask? What do we expect?

- **Mixing time**: How much time does it take to “reach” the uniform distribution?

The **total variation distance** between the distribution at time $n$ and the uniform distribution on $\mathcal{G}_{d+1}$:

$d = 3$

$$d = 4$$

$d$ large

$$\frac{1}{\pi^2 d^3 \log d}$$

[Aldous 83, Diaconis & Saloff-Coste 93, Wilson 04]
What do we ask? What do we expect?

- **Focus on observables**, for instance the inversion number $I_{d,n} = \text{inv}(\pi^{(n)})$.

The expected value of the inversion number, $l_{d,n} := \mathbb{E}(I_{d,n})$:

$$d = 3$$

$$d = 4$$

$\Rightarrow$ Estimate the number of transpositions (mutations) that have occurred, and hence the evolutionary distance between species.

Of particular interest: what happens before mixing.
A formula for the expected inversion number

**Theorem:** The expected number of inversions after $n$ adjacent transpositions in $\mathfrak{S}_{d+1}$ is

$$l_{d,n} = \frac{d(d+1)}{4} - \frac{1}{8(d+1)^2} \sum_{k,j=0}^{d} \frac{(c_j + c_k)^2}{s_j^2 s_k^2} x_{jk} n,$$

where

$$c_k = \cos \alpha_k, \quad s_k = \sin \alpha_k, \quad \alpha_k = \frac{(2k+1)\pi}{2d+2},$$

and

$$x_{jk} = 1 - \frac{4}{d} (1 - c_j c_k).$$

**Remark:** For $d$ large enough ($d \geq 8$), $l_{d,n}$ increases, as $n$ grows, to $\frac{d(d+1)}{4}$, which is the average inversion number of a random permutation in $\mathfrak{S}_{d+1}$. 
Another formula for the expected inversion number [Eriksen 05]

\[ |d, n| = \sum_{r=1}^{n} \frac{1}{d^r} \binom{n}{r} \sum_{s=1}^{r-1} \frac{1}{s-1} (-4)^{r-s} g_{s,d} h_{s,d}, \]

with

\[ g_{s,d} = \sum_{\ell=0}^{d} \sum_{k \geq 0} (-1)^k (p - 2\ell) \left( 2\left\lfloor s/2 \right\rfloor - 1 \right) \left( \left\lfloor s/2 \right\rfloor + \ell + k(d+1) \right) \]

and

\[ h_{s,d} = \sum_{j \in \mathbb{Z}} (-1)^j \left( 2\left\lfloor s/2 \right\rfloor \right) \left( \left\lfloor s/2 \right\rfloor + j(d+1) \right). \]

Based on [Eriksson & Eriksson & Sjöstrand 00]

Beresticky & Durrett 08: “it is far from obvious how to extract useful asymptotic from this formula”.

Combinatorialists could not throw in the sponge!
A formula for the expected inversion number

Theorem: The expected number of inversions after $n$ adjacent transpositions in $S_{d+1}$ is

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The expected inversion number: asymptotics

Three regimes, as $d$ and $n$ tend to $\infty$

- When $n$ is “small”, each step of the chain increases the inversion number with high probability. For example,

$$\text{inv}(\pi^{(1)}) = 1, \quad \mathbb{P}\left( \text{inv}(\pi^{(2)}) = 2 \right) = 1 - \frac{1}{d}, \quad \mathbb{P}\left( \text{inv}(\pi^{(n)}) = n \right) = 1 - O\left( \frac{1}{d} \right).$$
The expected inversion number: asymptotics

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$$\text{inv}(\pi^{(1)}) = 1, \quad P\left(\text{inv}(\pi^{(2)}) = 2\right) = 1 - \frac{1}{d}, \quad P\left(\text{inv}(\pi^{(n)}) = n\right) = 1 - O\left(\frac{1}{d}\right).$$

- When $n$ is “large”, the expected inversion number must approach its limit value $d(d + 1)/4 \sim d^2/4$. 

The expected inversion number: asymptotics

Three regimes, as \( d \) and \( n \) tend to \( \infty \)

- When \( n \) is “small”, each step of the chain increases the inversion number with high probability. For example,

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\text{inv}(\pi^{(1)}) = 1, \quad P\left(\text{inv}(\pi^{(2)}) = 2\right) = 1 - \frac{1}{d}, \quad P\left(\text{inv}(\pi^{(n)}) = n\right) = 1 - O\left(\frac{1}{d}\right).
\]

- When \( n \) is “large”, the expected inversion number must approach its limit value \( d(d + 1)/4 \sim d^2/4 \).

- An intermediate regime?
Small times: linear and before

- Sub-linear regime. If $n = o(d)$,

$$\frac{1_{d,n}}{n} = 1 + O(n/d).$$
Small times: linear and before

- **Sub-linear regime.** If \( n = o(d) \),
  \[
  \frac{1_{d,n}}{n} = 1 + O(n/d).
  \]

- **Linear regime.** If \( n \sim \kappa d \),
  \[
  \frac{1_{d,n}}{n} = f(\kappa) + O(1/d)
  \]
  where
  \[
  f(\kappa) = \frac{1}{2\pi\kappa} \int_{0}^{\infty} \frac{1 - \exp(-8\kappa t^2/(1 + t^2))}{t^2(1 + t^2)} dt
  = \sum_{j \geq 0} (-1)^j \frac{(2j)!}{j!(j + 1)!^2 (2\kappa)^j}.
  \]
  The function \( f(\kappa) \) decreases from \( f(0) = 1 \) to \( f(\infty) = 0 \).
Large times: cubic and beyond

• Super-cubic regime. If $n \gg d^3$,

$$\frac{l_{d,n}}{d^2} \to \frac{1}{4}.$$
Large times: cubic and beyond

- Super-cubic regime. If \( n \gg d^3 \),

\[
\frac{l_{d,n}}{d^2} \to \frac{1}{4}.
\]

- Cubic regime. If \( n \sim \kappa d^3 \),

\[
\frac{l_{d,n}}{d^2} \sim g(\kappa)
\]

where

\[
g(\kappa) = \frac{1}{4} - \frac{16}{\pi^4} \left( \sum_{j \geq 0} e^{-\kappa \pi^2 (2j+1)^2/2} \frac{(2j + 1)^2}{(2j + 1)^2} \right)^2.
\]

The function \( g(\kappa) \) increases from \( g(0) = 0 \) to \( g(\infty) = 1/4 \).
The intermediate regime

• If $d \ll n \ll d^3$,

$$\frac{I_{d,n}}{\sqrt{dn}} \to \sqrt{\frac{2}{\pi}}.$$ 

**Remark.** For a related continuous time chain, the normalized inversion number $I_{d,n}/\sqrt{dn}$ converges in probability to $\sqrt{2/\pi}$ [Beresticky & Durrett 08]
A formula for the expected inversion number

**Theorem:** The expected number of inversions after \( n \) adjacent transpositions in \( \mathfrak{S}_{d+1} \) is

\[
I_{d,n} = \frac{d(d+1)}{4} - \frac{1}{8(d+1)^2} \sum_{k,j=0}^{d} \frac{(c_j + c_k)^2}{s_j^2 s_k^2} x_{jk}^n,
\]

where

\[
c_k = \cos \alpha_k, \quad s_k = \sin \alpha_k, \quad \alpha_k = \frac{(2k + 1)\pi}{2d + 2},
\]

and

\[
x_{jk} = 1 - \frac{4}{d} (1 - c_j c_k).
\]
Where are the inversions? [Eriksson et al. 00]

For $i \leq j$, let $p_{i,j}^{(n)}$ be the probability that there is an inversion at time $n$ in the positions $i$ and $j + 1$:

$$p_{i,j}^{(n)} = \mathbb{P}(\pi_i^{(n)} > \pi_{j+1}^{(n)}).$$

- The expected number of inversions at time $n$ is

$$I_{d,n} = \sum_{0 \leq i \leq j < d} p_{i,j}^{(n)}.$$
Where are the inversions? [Eriksson et al. 00]

For $i \leq j$, let $p^{(n)}_{i,j}$ be the probability that there is an inversion at time $n$ in the positions $i$ and $j + 1$:

$$p^{(n)}_{i,j} = \mathbb{P}(\pi_i^{(n)} > \pi_{j+1}^{(n)}).$$

- The numbers $p^{(n)}_{i,j}$ can be described recursively by examining where were the values $\pi_i^{(n)}$ and $\pi_{j+1}^{(n)}$ at time $n - 1$. 

Where are the inversions? [Eriksson et al. 00]

For $i \leq j$, let $p_{i,j}^{(n)}$ be the probability that there is an inversion at time $n$ in the positions $i$ and $j + 1$:

$$p_{i,j}^{(n)} = \mathbb{P}(\pi_i^{(n)} > \pi_{j+1}^{(n)}).$$

- The numbers $p_{i,j}^{(n)}$ can be described recursively by examining where were the values $\pi_i^{(n)}$ and $\pi_{j+1}^{(n)}$ at time $n - 1$. For instance:
  - If $i = j$ and the $n$th transposition has switched the $i$th and $i + 1$st values:
    $$
    \left(1 - p_{i,j}^{(n-1)}\right) \frac{1}{d}
    $$
  - etc.
A recursion for the inversion probabilities

**Lemma.** The inversion probabilities $p_{i,j}^{(n)}$ are characterized by:

$$p_{i,j}^{(0)} = 0 \quad \text{for} \quad 0 \leq i \leq j < d,$$

and for $n \geq 0$,

$$p_{i,j}^{(n+1)} = p_{i,j}^{(n)} + \frac{1}{d} \sum_{(k,\ell) \leftrightarrow (i,j)} (p_{k,\ell}^{(n)} - p_{i,j}^{(n)}) + \frac{\delta_{i,j}}{d} \left(1 - 2p_{i,j}^{(n)}\right),$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, and the neighbour relations $\leftrightarrow$ are those of the following graph:

\[\begin{array}{c}
\text{(d - 1)} \\
\text{0} \quad \text{1} \quad \text{(d - 1)} \\
\end{array}\]

\[\text{A (weighted) walk in a triangle.}\]
A functional equation for the GF of the inversion probabilities

Let $P(t; u, v)$ be the generating function of the numbers $p^{(n)}_{i,j}$:

$$P(t; u, v) \equiv P(u, v) := \sum_{n \geq 0} t^n \sum_{0 \leq i \leq j < d} p^{(n)}_{i,j} u^i v^j.$$
A functional equation for the GF of the inversion probabilities

Let $P(t; u, v)$ be the generating function of the numbers $p_{i,j}^{(n)}$:

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The above recursion translates as

$$\left(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\right) P(u, v) =$$

$$t \left(\frac{1 - u^d v^d}{(1 - u v)(1 - t)} - (\bar{u} - 1)P_{\ell}(v) - (v - 1) v^{d-1}P_t(u) - (u + \bar{v})P_{\delta}(u v)\right),$$

where $\bar{u} = 1/u$, $\bar{v} = 1/v$, and the series $P_{\ell}$, $P_t$ and $P_{\delta}$ describe the numbers $p_{i,j}^{(n)}$ on the boundaries of the graph:
Back to the inversion number

We are interested in

\[ l_m(t) = \sum_{n \geq 0} l_{d,n} t^n = P(t; 1, 1), \]

which, according to the functional equation, may be rewritten

\[ l_m(t) = \frac{t}{(1 - t)^2} - \frac{2tP_{\delta}(1)}{d(1 - t)}. \]

\[ P(t; u, v) \equiv P(u, v) := \sum_{n \geq 0} t^n \sum_{0 \leq i \leq j < d} p_{i,j}^{(n)} u^i v^j. \]

\[ \left( 1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v}) \right) P(u, v) = \]

\[ \frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - uv)(1 - t)} - (\bar{u} - 1) P_{\ell}(v) - (v - 1)v^{d-1} P_t(u) - (u + \bar{v}) P_{\delta}(uv) \right), \]
\[
\left( 1 - t + \frac{t}{d} (4 - u - \bar{u} - v - \bar{v}) \right) P(u, v) = \\
\frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - uv)(1 - t)} - (\bar{u} - 1) P_\ell(v) - (v - 1) v^{d-1} P_t(u) - (u + \bar{v}) P_\delta(uv) \right)
\]
What a beautiful equation!

\[
(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})) P(u, v) = \\
\frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - uv)(1 - t)} - (\bar{u} - 1)P_\ell(v) - (v - 1)v^{d-1}P_t(u) - (u + \bar{v})P_\delta(uv) \right)
\]

Analogies with:

- Walks with steps ±1 in a strip of height \( d \):
  \[
  (1 - t(u + \bar{u}))P(u) = 1 - t\bar{u}P_0 - tu^{d+1}P_d
  \]

- Walks in the quarter plane
  \[
  (1 - t(u + \bar{u} + v + \bar{v}))P(u, v) = \\
  1 - t\bar{u}P(0, v) - t\bar{v}P(u, 0)
  \]

- and others...
The ingredients of the solution

\[
\left(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\right) P(u, v) = \\
\frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - uv)(1 - t)} - (\bar{u} - 1) P_\ell(v) - (v - 1) v^{d-1} P_t(u) - (u + \bar{v}) P_\delta(uv) \right)
\]

- Cancel the kernel \( \left(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\right) \) by coupling \( u \) and \( v \)
The ingredients of the solution

\[ P(u, v) = \left( 1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v}) \right) \]

\[ = \frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - uv)(1 - t)} \right) - \left( \bar{u} - 1 \right) P_\ell(v) - \left( v - 1 \right) v^{d-1} P_t(u) - \left( u + \bar{v} \right) P_\delta(uv) \]

- Cancel the kernel \( \left( 1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v}) \right) \) by coupling \( u \) and \( v \)
- Exploit the symmetries of this kernel, which is invariant by \( (u, v) \mapsto (\bar{u}, v) \)
  \( (u, v) \mapsto (u, \bar{v}), (u, v) \mapsto (\bar{u}, \bar{v}) \) (the reflection principle)
The ingredients of the solution

\[
\left(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\right) P(u, v) = \\
\frac{t}{d} \left( \frac{1 - u^d v^d}{(1 - u v)(1 - t)} - (\bar{u} - 1) P_\ell(v) - (v - 1) v^{d-1} P_t(u) - (u + \bar{v}) P_\delta(u v) \right)
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- Plus one more coupling between \( u \) and \( v \).
The ingredients of the solution

\[
\left(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\right) P(u, v) =
\]
\[
\frac{t}{d} \left(\frac{1-u^d v^d}{(1-uv)(1-t)} - (\bar{u} - 1)P_\ell(v) - (v - 1)v^{d-1}P_t(u) - (u + \bar{v})P_\delta(uv)\right)
\]

- Cancel the kernel \(1 - t + \frac{t}{d}(4 - u - \bar{u} - v - \bar{v})\) by coupling \(u\) and \(v\)
- Exploit the symmetries of this kernel, which is invariant by \((u, v) \mapsto (\bar{u}, v)\) \((u, v) \mapsto (u, \bar{v})\), \((u, v) \mapsto (\bar{u}, \bar{v})\) (the reflection principle)
- Plus one more coupling between \(u\) and \(v\).

One obtains an explicit expression of \(P_\delta(q)\) at every \(q \neq -1\) such that \(q^{d+1} = -1\), and this is enough to reconstruct the whole polynomial \(P_\delta(u)\) (and in particular, \(P_\delta(1)\)) by interpolation.
The final result

The generating function $I_d(t) = \sum_{n \geq 0} I_{d,n} t^n$ is

$$I_d(t) = \frac{d(d + 1)}{4(1 - t)} - \frac{1}{8(d + 1)^2} \sum_{k,j=0}^{d} \frac{(c_j + c_k)^2}{s_j^2 s_k^2} \frac{1}{1 - tx_{jk}}$$

with

$$c_k = \cos \alpha_k, \quad s_k = \sin \alpha_k, \quad \alpha_k = \frac{(2k + 1)\pi}{2d + 2},$$

and

$$x_{jk} = 1 - \frac{4}{d} (1 - c_j c_k).$$
Perspectives

- **Other generators** (ex: all transpositions [Sjöstrand 10], transpositions \((0, i)\), block transpositions...)
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- **Other statistics**: inversion number \(\mapsto\) measure of the “distance” between the identity and a permutation (ex: [Eriksen & Hultman 04], expected transposition distance after \(n\) transpositions)
• **Other generators** (ex: all transpositions [Sjöstrand 10], transpositions \((0, i)\), block transpositions...)

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• **Other groups:** mostly, finite irreducible Coxeter groups, with the length as the distance statistics ([Troili 02]: the case of \(I_2(d)\)). When the generators are all reflections, see [Sjöstrand 10].
Perspectives

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Thank you!
Around the mixing time (super-cubic regime)

Assume \( n \sim \kappa d^3 \log d \).

- If \( \kappa < 1/\pi^2 \), there exists \( \gamma > 0 \) such that
  \[
  |d, n| \leq \frac{d(d+1)}{4} - \Theta(d^{1+\gamma}),
  \]

- If \( \kappa > 1/\pi^2 \), there exists \( \gamma > 0 \) such that
  \[
  |d, n| = \frac{d(d+1)}{4} - O(d^{1-\gamma}).
  \]

- For the critical value \( \kappa = 1/\pi^2 \), the following refined estimate holds: if
  \( n \sim 1/\pi^2 d^3 \log d + \alpha d^3 + o(d^3) \), then
  \[
  |d, n| = \frac{d(d+1)}{4} - \frac{16d}{\pi^4} e^{-\alpha\pi^2} (1 + o(1)).
  \]
What do we ask? What do we expect?

Let \( Q = (Q_{\sigma, \tau}) \) be the transition matrix of the chain:

\[
Q = \begin{pmatrix}
0 & 1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 1/2 \\
0 & 0 & 0 & 1/2 & 1/2 & 0
\end{pmatrix}
\]

- Then for all \( \sigma \in \mathcal{S}_{d+1} \),
  \[
  \sum_{n \geq 0} \mathbb{P}(\pi^{(n)} = \sigma) \ t^n = \sum_{n \geq 0} Q_{\text{id}, \sigma}^n t^n = (1 - tQ)^{-1}_{\text{id}, \sigma}
  \]
  is a rational series in \( t \).

- The GF of the expected inversion number
  \[
  \sum_{n \geq 0} \mathbb{E}(I_{\text{id}, n}) t^n = \sum_{n \geq 0} \left( \sum_{\sigma \in \mathcal{S}_{d+1}} \text{inv}(\sigma) \mathbb{P}(\pi^{(n)} = \sigma) \right) t^n = \sum_{\sigma \in \mathcal{S}_{d+1}} \text{inv}(\sigma) G_{\sigma}(t)
  \]
  is rational as well.