A branch and bound method to compute a median permutation

Irène Charon, Olivier Hudry

Télécom ParisTech

olivier.hudry@telecom-paristech.fr

Algorithms & Permutations 2012

TELECOM ParisTech

A permutation problem in voting theory

- Given a profile $\Pi = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ of m permutations (i.e. linear orders) σ_i $(1 \leq i \leq m)$ on a set X of $n = |X|$ elements, how to aggregate them into a unique permutation which summarizes Π as accurately as possible?
- In voting theory (Condorcet, 1784): we want to rank n candidates from the rankings provided by m voters.

Example

- $X = \{a, b, c, d, e, f\}, m = 5$

  voter 1: $\sigma_1 = a > b > c > f > d > e$

  voter 2: $\sigma_2 = a > c > f > b > d > e$

  voter 3: $\sigma_3 = e > d > a > f > b > c$

  voter 4: $\sigma_4 = b > c > d > e > f > a$

  voter 5: $\sigma_5 = c > f > b > e > a > d.$

A combinatorial optimization problem

- Symmetric difference distance d between R and R':
  $d(R, R') = |\{(x, y) \in X^2$ with [xRy and not $xR'y$] or [not xRy and $xR'y$]$\}|$.

- Let Σ be the set of all the permutations defined on X.
  Then, for $\Pi = (\sigma_1, \sigma_2, \ldots, \sigma_m)$:

$$\text{Minimize } \rho_\Pi(\sigma) = \sum_{i=1}^{m} d(\sigma, \sigma_i) \text{ for } \sigma \in \Sigma$$

(cf. J.-P. Barthélemy, B. Monjardet, 1981)

- $d(R, R')$ measures the number of disagreements between R and R'.
- $\rho_\Pi(\sigma)$ (= *remoteness of* σ *from* Π) measures the total number of disagreements between σ and Π.
- σ^* minimizing ρ_Π over Σ is called a *median permutation* (or a *median linear order*) of Π.
- **Theorem** (J.J. Bartholdi III *et alii*, 1989; O. Hudry, 1989; C. Dwork *et alii*, 2001): The computation of σ^* is NP-hard.

A 0-1 linear programming problem

- $\sigma = (\sigma_{xy})_{(x,\, y)\in X^2}$ with $\sigma_{xy} = 1$ if σ ranks x better than y ($x >_\sigma y$) and $\sigma_{xy} = 0$ otherwise.
- $m_{xy} = m - 2|\{i: 1 \le i \le m \text{ and } x >_{\sigma i} y\}| = -m_{yx}$
- Then: $\rho_\Pi(\sigma) = C + \sum_{(x,y)\in X^2} m_{xy}\,\sigma_{xy}$

with :

$\forall\, x \in X,\ \sigma_{xx} = 1$ (reflexivity)

$\forall\, (x, y) \in X^2,\ x \neq y,\ \sigma_{xy} + \sigma_{yx} = 1$ (antisymmetry)

$\forall\, (x, y, z) \in X^3,\ \sigma_{xy} + \sigma_{yz} - \sigma_{xz} \le 1$ (transitivity)

$\forall\, (x, y) \in X^2,\ \sigma_{xy} \in \{0, 1\}$ (binarity)

Lagrangean relaxation

- Relaxation of the transitivity constraints:

$$\forall\ (x, y, z) \in X^3,\ \sigma_{xy} + \sigma_{yz} - \sigma_{xz} \leq 1$$

- Lagrangean function L for $\sigma = (\sigma_{xy})_{(x,\,y)\in X^2}$ with $\sigma_{xy} \in \{0, 1\}$, $\sigma_{xx} = 1$, $\sigma_{xy} + \sigma_{yx} = 1$, and $\Lambda = (\lambda_{xyz})_{(x,\,y,\,z)\in X^3}$ with $\lambda_{xyz} \geq 0$:

$$L(\sigma, \Lambda) = \rho_\Pi(\sigma) + \sum_{(x,y,z)\in X^3} \lambda_{xyz}\left(\sigma_{xy} + \sigma_{yz} - \sigma_{xz} - 1\right)$$

$$= C + \sum_{(x,y)\in X^2} a_{xy}(\Lambda)\sigma_{xy} - \sum_{(x,y,z)\in X^3} \lambda_{xyz}$$

with $a_{xy}(\Lambda) = m_{xy} + \sum_{z\in X} (\lambda_{xyz} + \lambda_{zxy} - \lambda_{xzy})$

Lagrangean relaxation (end)

- Dual function for $\Lambda = (\lambda_{xyz})_{(x,\, y,\, z) \in X^3}$ with $\lambda_{xyz} \geq 0$:

$$D(\Lambda) = \min\{L(\sigma, \Lambda) \text{ with } \sigma \in \mathcal{A}\}$$

with $\mathcal{A}$ = {reflexive and antisymmetric relations defined on X}.

- Dual problem: maximize $D(\Lambda)$ for $\Lambda \geq 0$.
- The maximum of D gives a lower bound of the minimum of ρ_Π.
- Computation of $D(\Lambda)$ for a given Λ:

$$\text{if } a_{xy} \geq 0, \text{ set } \sigma_{xy} = 0, \text{ and } \sigma_{xy} = 1 \text{ otherwise.}$$

- Resolution of the dual problem by subgradient methods.

The components of the BB algorithm

- Initial bound: found by a metaheuristic (a self-tuned noising method; I. Charon and O. Hudry, 1993, 2009)
- Evaluation function: provided by the Lagrangean relaxation.
- Branching rule (J.-P. Barthélemy, A. Guénoche, O. Hudry, 1989; I. Charon, A. Guénoche, O. Hudry, F. Woirgard, 1996):

The root of the BB-tree contains all the permutations defined on X.

A node of the BB-tree contains the permutations sharing a given *beginning section* S (i.e. a permutation of a subset of X):

$$S(x_{j1}, x_{j2}, \ldots, x_{jp}) = x_{j1} >_\sigma x_{j2} >_\sigma \ldots >_\sigma x_{jp}.$$

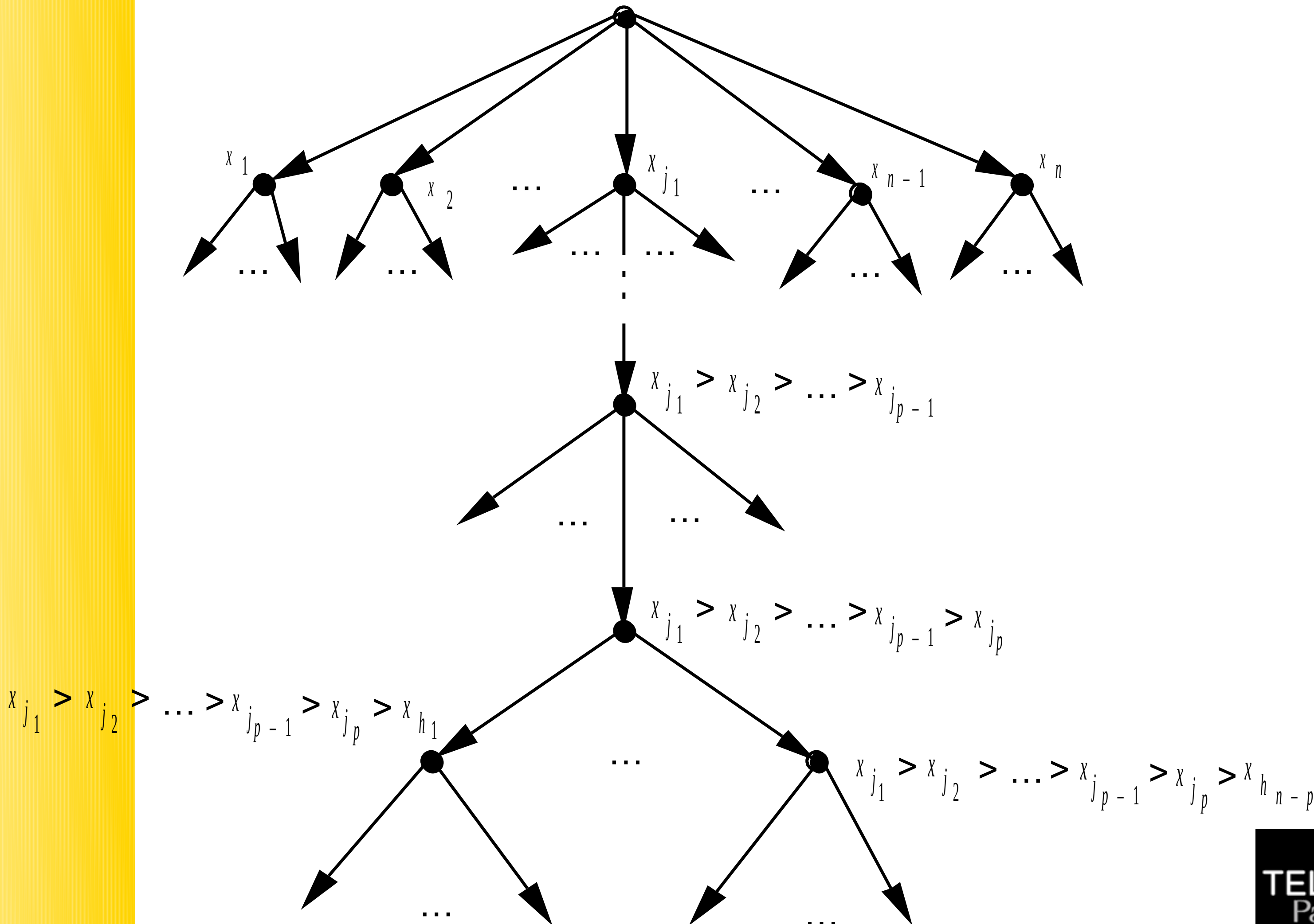
The branching principle consists in expanding this beginning section:

$$S(x_{j1}, x_{j2}, \ldots, x_{jp}, x) = x_{j1} >_\sigma x_{j2} >_\sigma \ldots x_{jp} >_\sigma x$$

with $x \notin \{x_{j1}, x_{j2}, \ldots, x_{jp}\}$.

TELECOM ParisTech

Shape of the BB-tree



Other components to prune the BB-tree

- **Hamiltonian permutations**.

* We may summarize a profile Π of permutations by a tournament T (weighted by $-m_{xy} > 0$): there is an arc (x, y) if a majority of voters prefer x to y (we assume that there is no tie).

* We say that a *permutation* σ *is Hamiltonian* if it induces a Hamiltonian path in T.

* **Theorem** (R. Remage and W.A. Thompson, 1966): a median permutation is Hamiltonian.

→ $x_{j1} >_\sigma x_{j2} >_\sigma \ldots >_\sigma x_{jp}$ is expanded into $x_{j1} >_\sigma \ldots >_\sigma x_{jp} >_\sigma x$ only if a majority of voters prefer x_{jp} to x.

Example

- $X = \{a, b, c, d, e, f\}$

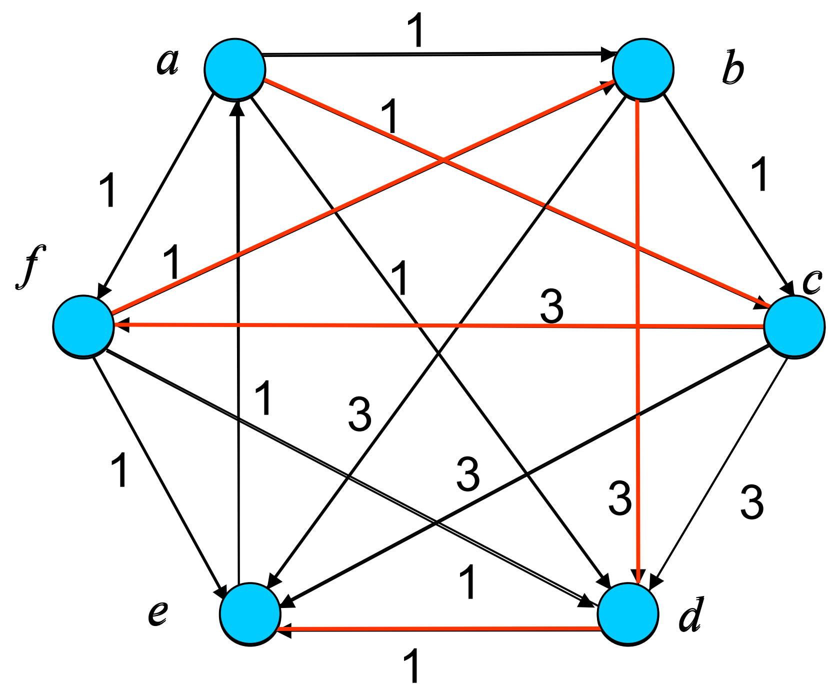
$\sigma_1 = a > b > c > f > d > e$

$\sigma_2 = a > c > f > b > d > e$

$\sigma_3 = e > d > a > f > b > c$

$\sigma_4 = b > c > d > e > f > a$

$\sigma_5 = c > f > b > e > a > d.$



Here, $a > c > f > b > d > e$ is a median permutation and induces a Hamiltonian path.

Other components to prune the BB-tree

- We compute the variation of ρ_Π when, from a permutation σ beginning with $S = x_{j1} >_\sigma x_{j2} >_\sigma \dots >_\sigma x_{jp}$, we take an interval $x_{jh} >_\sigma \dots >_\sigma x_{jp}$ $(1 \le h \le p)$ and we shift it at the end of σ, after the elements of $X - S$ (= OS = « out of section »):

$$\sigma = x_{j1} >_\sigma x_{j2} >_\sigma \dots x_{jh-1} >_\sigma x_{jh} >_\sigma \dots >_\sigma x_{jp} >_\sigma (OS)$$

becomes

$$\sigma' = x_{j1} >_{\sigma'} x_{j2} >_{\sigma'} \dots x_{jh-1} >_{\sigma'} (OS) >_{\sigma'} x_{jh} >_{\sigma'} \dots >_{\sigma'} x_{jp}.$$

If ρ_Π decreases, we do not keep the node associated with S.

OSmoves will count this kind of cuts.

Other components to prune the BB-tree (end)

- When we deal with a new beginning section

$$S = x_{j1} >_\sigma x_{j2} >_\sigma \ldots x_{jh-1} >_\sigma x_{jh} >_\sigma \ldots >_\sigma x_{jp} >_\sigma x,$$

we consider the beginning sections that we can get by moving, inside S, an "interval" of S including x, i.e., the beginning sections with the following shape:

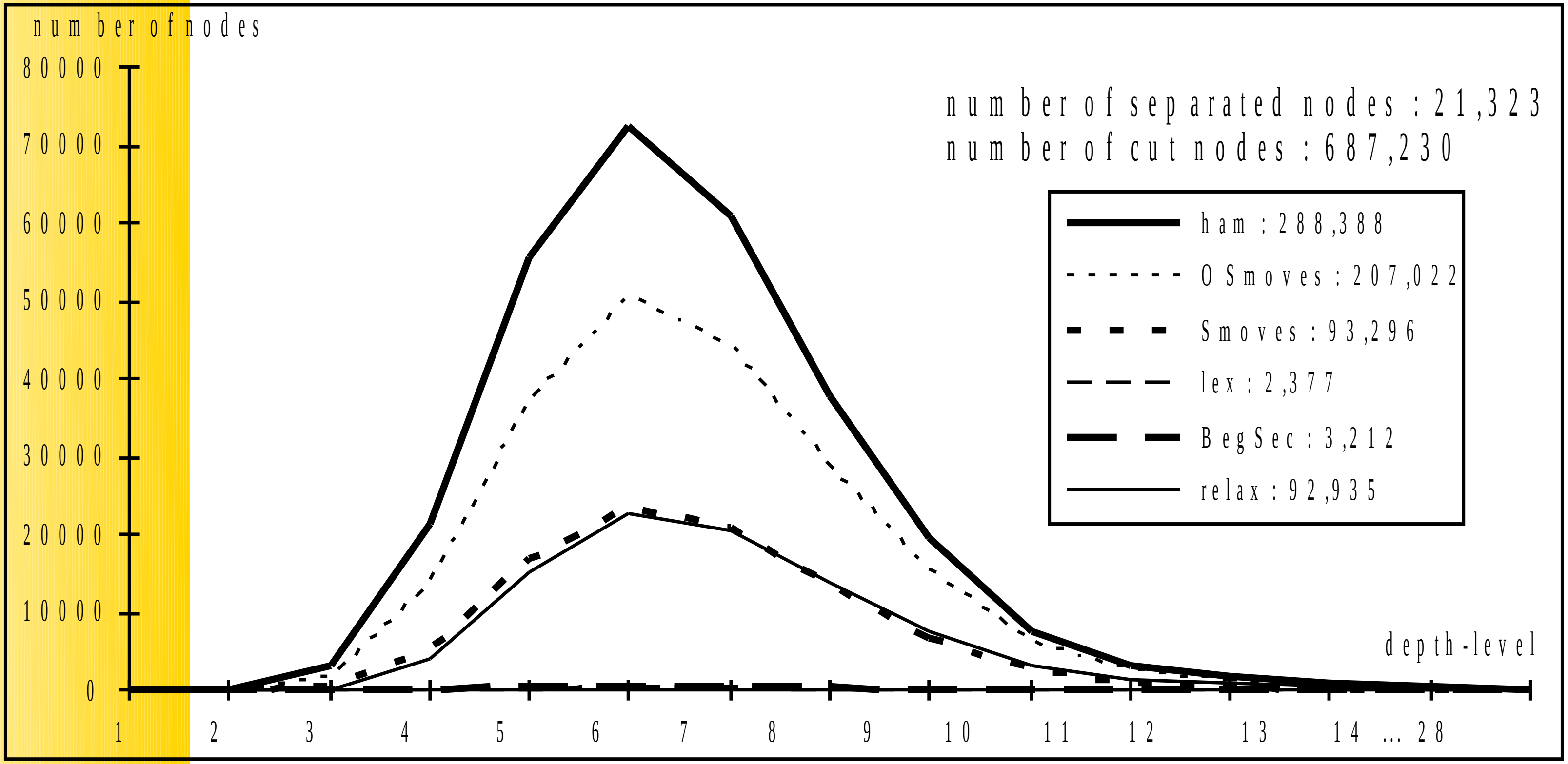
$$x_{jh} >_{\sigma'} \ldots >_{\sigma'} x_{jp} >_{\sigma'} x >_{\sigma'} x_{j1} >_{\sigma'} x_{j2} >_{\sigma'} \ldots x_{jh-1}.$$

If ρ_Π decreases, we do not keep the node associated with S.

Smoves will count this kind of cuts.

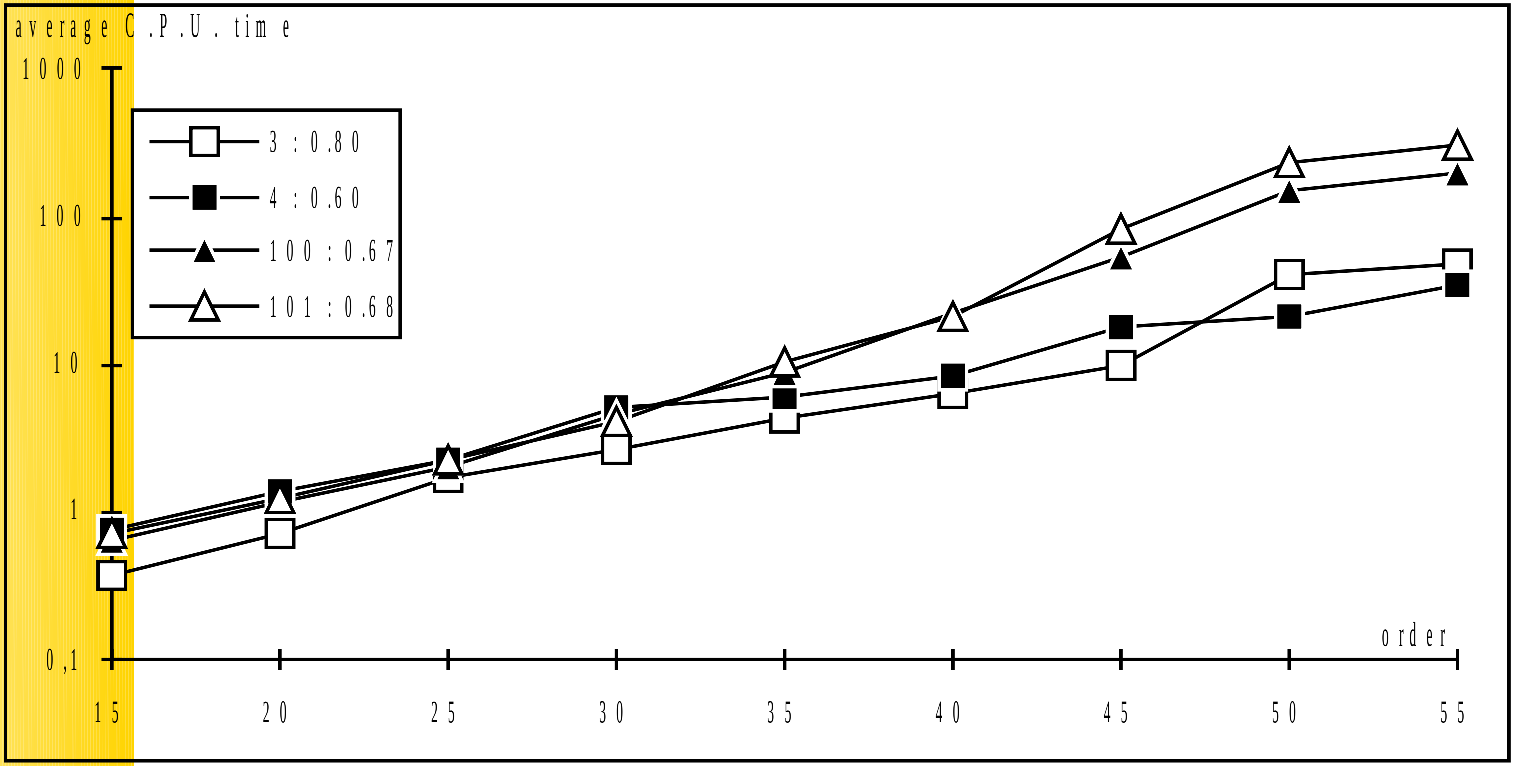
An experimental result on the efficiency of the branch and bound components

- Numbers of cuts for an instance on 39 candidates



CPU times for $m \in \{3, 4, 100, 101\}$

- CPU times in seconds (Rk: order = n).



Number of median permutations versus number of Hamiltonian permutations

- Let $M(n)$ and $H(n)$ denote respectively the maximum number of median permutations or of Hamiltonian permutations for instances on n candidates.
- If n is even with $n \geq 2$: $M(n) = n!$
- If n is odd: $M(n) \leq H(n)$.
- **Theorem** (N. Alon, 1990): $H(n) \leq (c \times n^{1.5} \times n!)/2^n$ where c is a constant.
- **Theorem** (I. Charon, O. Hudry, 2000): for $n = 3^k$,

$$3^{0.75(n-1)}/n^2 \leq M(n).$$

TELECOM ParisTech

Thank you for your attention!

☺