Commutative Combinatorial Hopf Algebras

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Combinatorial Hopf Algebras

Graded (connected) Hopf algebras based on combinatorial objects + some extra properties (yet unspecified!).

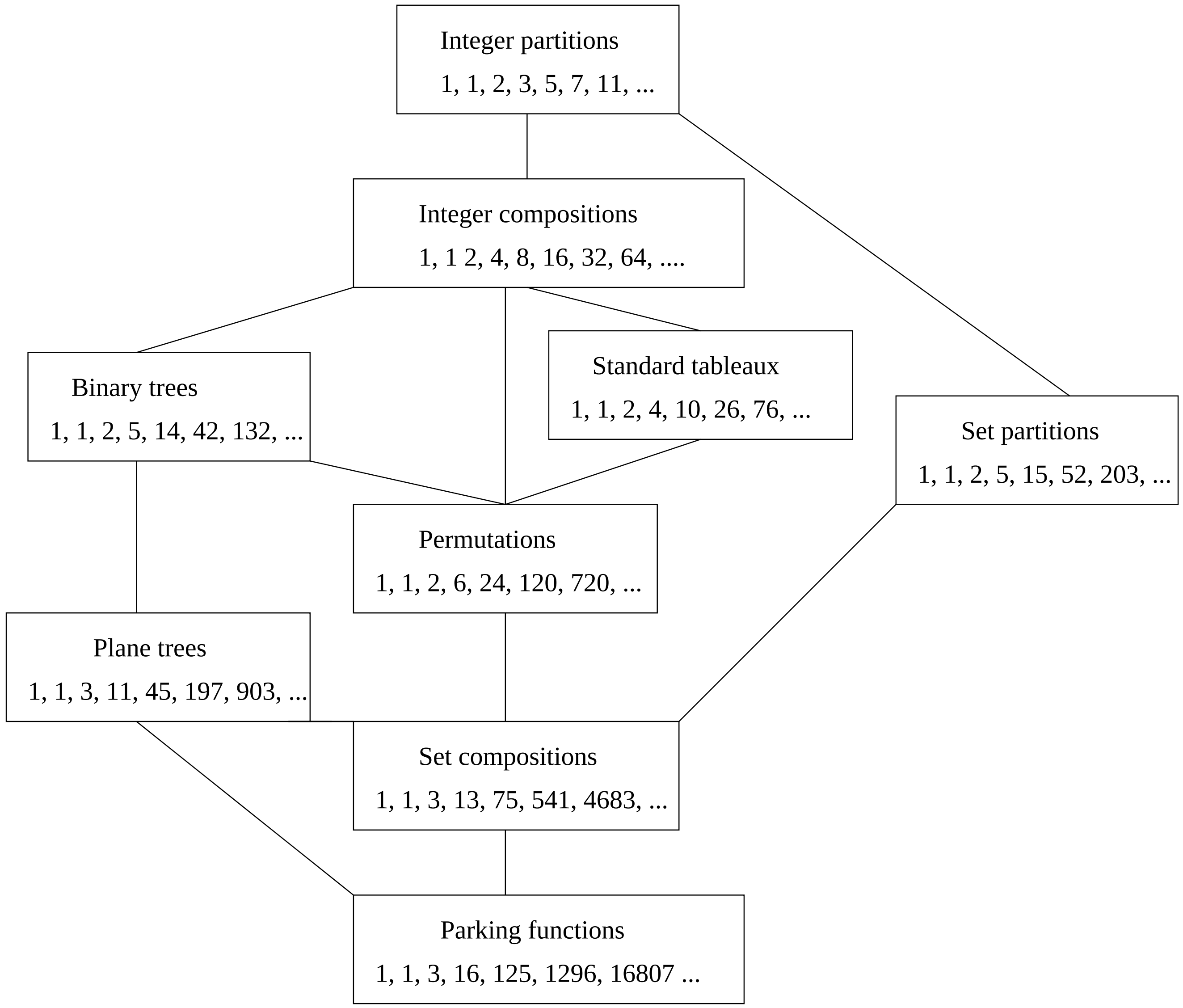
Several sources:

- Combinatorics (symmetric functions and generalizations)
- Differential equations, renormalization
- Representation theory
- Operads

One possible approach

Combinatorial Hopf algebras are generalizations of symmmetric functions.

- Many of them (next slide) arise from questions about symmetric functions
- Mimic to a certain extent λ-ring operations (coproducts $f(X+Y)$, $f(XY)$, transformations $X \to (1-q)X$, etc.)
- Internal products
- Analogs of the Robinson-Schensted correspondence


Integer partitions
1, 1, 2, 3, 5, 7, 11, ...
Integer compositions
1, 1 2, 4, 8, 16, 32, 64, ...
Binary trees
1, 1, 2, 5, 14, 42, 132, ...
Standard tableaux
1, 1, 2, 4, 10, 26, 76, ...
Set partitions
1, 1, 2, 5, 15, 52, 203, ...
Permutations
1, 1, 2, 6, 24, 120, 720, ...
Plane trees
1, 1, 3, 11, 45, 197, 903, ...
Set compositions
1, 1, 3, 13, 75, 541, 4683, ...
Parking functions
1, 1, 3, 16, 125, 1296, 16807 ...

Multiple structures

Many combinatorial sets admit *several* CHA structures. Most frequently

- A self dual one
- A cocommutative one (hence a pair $U(\mathfrak{g})$, $\mathcal{F}(G)$)
- One with an internal product

Examples

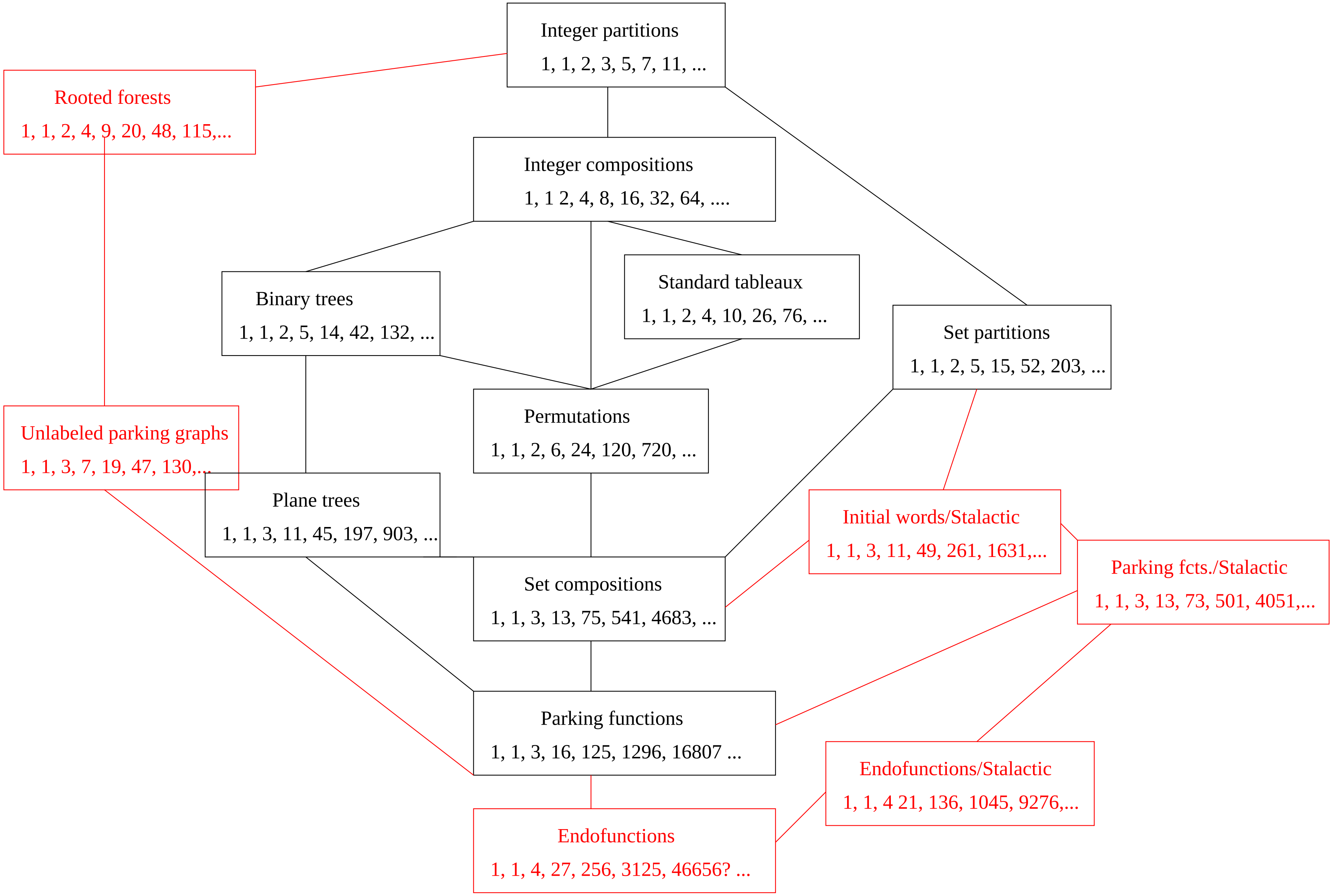
- Integer partitions: Sym (self-dual), also Faa-di-Bruno (function algebra)
- Integer compositions: $QSym$,**Sym** (ribbon Schur functions, descent algebras, free Lie algebras), also noncommutative formal diffeomorphisms
- Standard Young tableaux **FSym** (Littlewood-Richardson rule, plactic monoid)
- Binary trees: **PBT** (Loday-Ronco), also **CQSym** (cocommutative, internal product)

- Set partitions: **WSym** (cocommutative), also self-dual structure (M. Rey)
- Plane trees (Schröder sets): Free tridendriform (self-dual), also **SQSym** (neither self-dual nor cocommutative)
- Permutations: **FQSym** (Malvenuto-Reutenauer), also Heap-ordered trees (Grossman-Larson), cocommutative

$\boxed{\textbf{Purpose of this talk}}$

A simple and unified way to construct the (co)-commutative CHA structures on the previously mentioned combinatorial sets (and several new ones).

This leads to new results on several already known CHAs, in particular on the long neglected $\mathbf{WSym} = \varprojlim_{n} K\langle a_1, \ldots, a_n\rangle^{\mathfrak{S}_n}$.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1, 1, 2, 6, 24, 120, 720, ...
Unlabeled parking graphs
1, 1, 3, 7, 19, 47, 130,...
Plane trees
1, 1, 3, 11, 45, 197, 903, ...
Initial words/Stalactic
1, 1, 3, 11, 49, 261, 1631,...
Parking fcts./Stalactic
1, 1, 3, 13, 73, 501, 4051,...
Set compositions
1, 1, 3, 13, 75, 541, 4683, ...
Parking functions
1, 1, 3, 16, 125, 1296, 16807 ...
Endofunctions/Stalactic
1, 1, 4 21, 136, 1045, 9276,...
Endofunctions
1, 1, 4, 27, 256, 3125, 46656? ...

Hopf algebras of graphs

Recall the *quasi-symmetrization* operator:

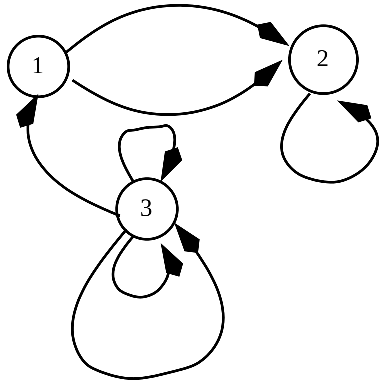
$$x^I = x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} \longrightarrow M_I = \sum_{k_1 < k_2 < \ldots < k_r} x_{k_1}^{i_1} x_{k_2}^{i_2} \cdots x_{k_r}^{i_r}$$

We can apply it to bi-indexed variables x_{ij}:

$$x^G = \prod_{i=1}^{r} \prod_{j=1}^{s} x_{ij}^{g_{ij}} \longrightarrow M_G = \sum_{k_1 < k_2 < \ldots < k_m} \prod_{i=1}^{r} \prod_{j=1}^{s} x_{k_i k_j}^{g_{ij}}$$

where m is the number of nonzero exponents g_{ij}.

Such monomials are conveniently represented by labeled directed graphs, e.g.,



encodes the monomial $x_{12}^2 x_{22}^1 x_{31}^1 x_{33}^3$, and its adjacency matrix is $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

The grading is the number of edges. We just have to forbid vertices of degree 0.

Theorem *The M_G span a subalgebra of $K[x_{ij}|i,j \geq 1]$. Moreover, the coproduct*

$$\Delta M_G = \sum_{G=G' \bullet G''} M_{G'} \otimes M_{G''} \,,$$

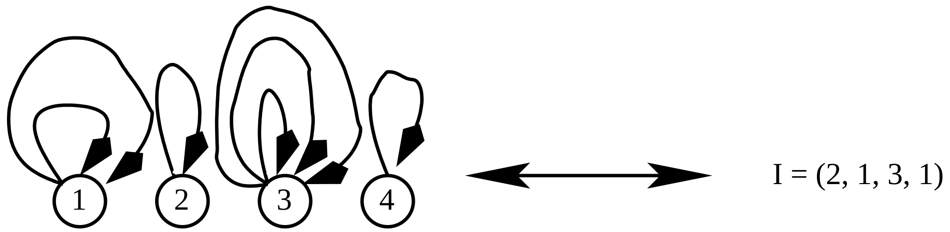
where $\bullet$ denotes shifted concatenation (direct sum of adjacency matrices), turns it into a Hopf algebra. Its dual is free over irreducible graphs.

Dimensions: 1, 1, 3, 39, 819, 23939 ... Large enough to accomodate a wealth of (old or new) CHAs.

Summing over permutations of the labels yield commutative and cocommutative Hopf algebras based on the unlabeled structures.

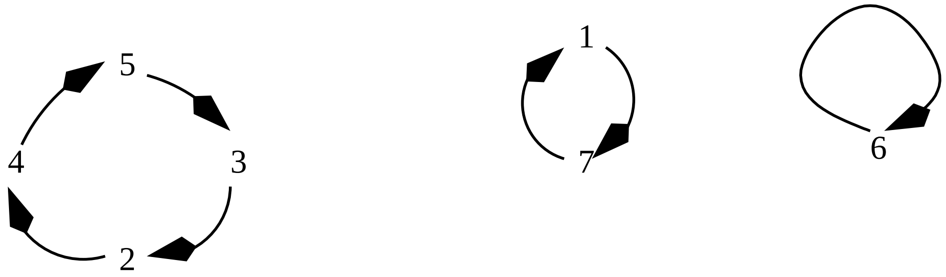
Examples

Compositions: $QSym$ is obtained from graphs of the form



The M_G labeled by such graphs span a Hopf subalgebra isomorphic to $QSym$ (alternatively, one can impose $x_{ij} = 0$ for $i \neq j$).

Permutations are graphs: labeled unions of cycles



σ = (2453)(17)(6)

The M_σ do not span a subalgebra (non-permutation graphs can appear in $M_\sigma M_\tau$), but it is easy to fix this by imposing

$$x_{ij}x_{ik} = 0 \;\forall\, i, j, k\,. \quad (*)$$

$\mathfrak{S}QSym$: subalgebra of $K[x_{ij}]/(*)$ spanned by the M_σ.

Commutative, noncocommutative Hopf algebra.

Dual $\mathfrak{S}\mathbf{Sym}$ free over connected permutations ($\mathbf{S}^\sigma = M_\sigma^*$ is multiplicative).

$\mathfrak{S}\mathbf{Sym} = U(\mathfrak{g})$, where $\mathfrak{g}$ is a free Lie algebra over a graded set labeled by connected permutations.

This determines completely the Hopf structure of $\mathfrak{S}\mathbf{Sym}$, and implies that it is isomorphic to other Hopf algebras, such as Heap Ordered Trees (Grossman-Larson), Aguiar-Sottile, Patras-Reutenauer, and a fourth one constructed directly from biwords.

The self-dual Hopf algebras of permutations (**FQSym**, or Malvenuto-Reutenauer) is the centralizer of $GL(V)$ in $T(V)$ (in the limit $\dim V \to \infty$), with the convolution of endomorphisms as multiplication.

Similarly, the commutative version $\mathfrak{S}QSym$ is a convolution algebra of certain endomorphisms of a Hopf algebra $H(V)$, where the analytic functor H is

$$H(V) = \bigoplus_{n\geq 0} V^{\otimes n} \otimes_{K\mathfrak{S}_n} M_n \,,$$

where M_n is the representation of $\mathfrak{S}_n$ on $K\mathfrak{S}_n$ by conjugation.

Endofunctions: The relations $(*)$ imply in fact that general endofunction graphs (edges $i \to f(i)$) build up a Hopf algebra $EQSym$

$$f: [n] \to [n] \quad \longleftrightarrow M_f = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1 i_{f(1)}} \cdots x_{i_n i_{f(n)}}$$

Various restrictions on endofunctions yield Hopf subalgebras. We have already seen bijective endofunctions. On can also impose $f^2 = f$ (idempotency), $f^2 = id$ (involutions or standard tableaux), and more generally $f^p = f^q$ (Burnside classes).

Algebras derived from $\mathfrak{S}QSym$

Dual of WSym. For $\sigma \in \mathfrak{S}_n$, let $\mathrm{supp}(\sigma)$ be the set partition whose blocks are the supports of the cycles of σ:

$$\mathrm{supp}((4182)(536)(7)) = \{\{1,2,4,8\},\{3,5,6\},\{7\}\}\,.$$

Then, the elements

$$U_\pi = \sum_{\mathrm{supp}(\sigma)=\pi} M_\sigma$$

span a Hopf subalgebra $\Pi QSym$ of $\mathfrak{S}QSym$, isomorphic to $\mathbf{WSym}^*$.

Quasi-symmetric functions (a new realization). Each set partition π has a canonical representative as a set composition $\Pi = (B_1 < B_2 < \ldots < B_r)$, where the order on blocks $B = \{i_1 < i_2 < \ldots < i_k\}$ is the lexicographic order on the words $i_1 i_2 \cdots i_k$. We set

$$K(\pi) = (|B_1|, \ldots, |B_r|)$$

and for an integer composition I,

$$U_I = \sum_{K(\pi)=I} U_\pi \,.$$

The U_I span a Hopf subalgebra isomorphic to $QSym$. By duality, this yields an epimorphism $\mathbf{WSym} \to \mathbf{Sym}$.

Symmetric functions. For an integer partition λ, let

$$u_\lambda = \sum_{I\downarrow=\lambda} U_I$$

Then, the u_λ span a Hopf subalgebra isomorphic to Sym. An explicit identification is given by

$$u_\lambda = p_\lambda^* = \frac{p_\lambda}{z_\lambda}$$

In terms of the matrix $X = (x_{ij})$, we end up with

$$p_n = \mathrm{tr}(X^n)$$

which is morally satisfactory ...

The stalactic congruence

On our way from $EQSym$ to Sym, we encountered $\Pi QSym$, which shed some light on **WSym**, the cocommutative CHA of set partitions.

There is also a self-dual CHA structure on set partitions, recently constructed by M. Rey, by means of an analog of the Robinson-Schensted correspondence, as in the case of **PBT** (binary trees) and **FSym** (standard tableaux).

Actually, **WSym** can also be obtained in a similar way, by means of a rather degenerate correspondence, which we call the *stalactic* correspondence.

The stalactic congruence on A^* is generated by the relations

$$awa \equiv aaw \quad \forall a \in A,\ \forall w \in A^*\,.$$

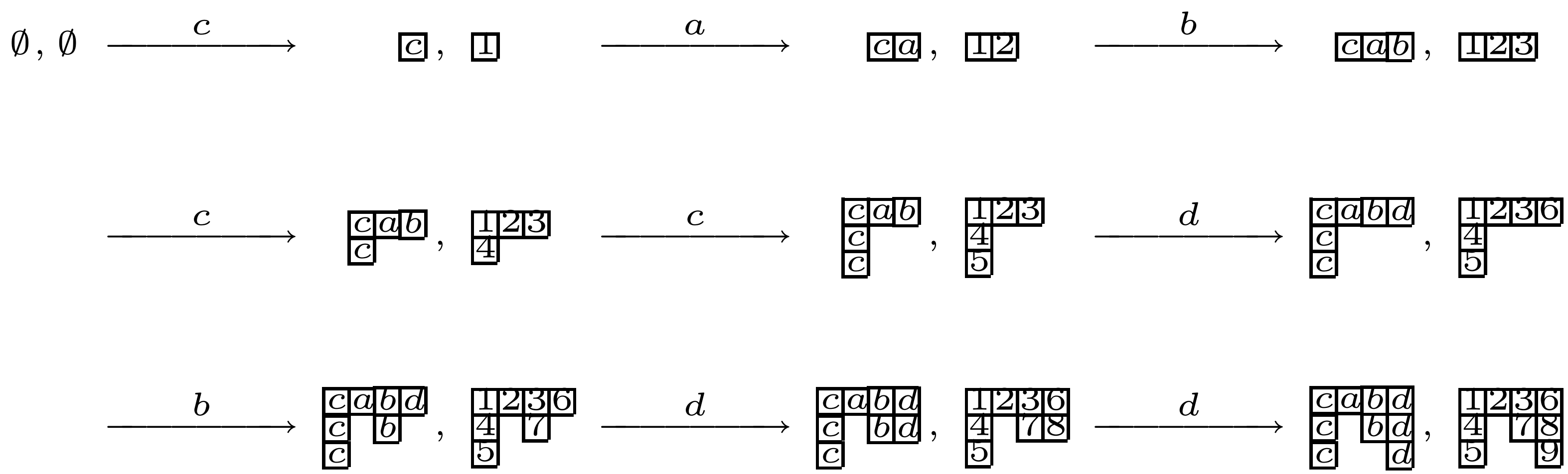
Canonical representatives are words of the form

$$a_1^{i_1} a_2^{i_2} \cdots a_r^{i_r} \text{ with } a_i \neq a_j \text{ for } i \neq j\,.$$

We can represent such a canonical word by a tableau-like planar diagram, *e.g.*,

$$c^3ad^3b^2 \longleftrightarrow \begin{array}{|c|c|c|c|}\hline c & a & d & b \\ \hline c & \multicolumn{1}{c|}{} & d & b \\ \cline{1-1}\cline{3-4} c & \multicolumn{1}{c|}{} & d & \multicolumn{1}{c}{} \\ \cline{1-1}\cline{3-3} \end{array}\,. \tag{1}$$

Analog of RS. Scan a word from left to right and arrange its identical letters in columns, creating a new column to the right when one scans a letter for the first time. Call $P(w)$ the resulting canonical word. The Q-symbol records the intermediate shapes. With $w = cabccdbdd$,



The Q-symbol can be interpreted as a set partition of $[n]$, whose blocks are the columns.

$$Q(cabccdbd) = \{\{1,4,5\},\ \{2\},\ \{3,7\},\ \{6,8,9\}\}.$$

The natural basis $\mathbf{M}_\pi$ of **WSym** can be characterized as

$$\mathbf{M}_\pi = \sum_{Q(w)=\pi} w.$$

This is completely similar to the definition of the bases R_I of **Sym** via the hypoplactic congruence, of $\mathbf{S}_t$ of **FSym** via the plactic congruence, and of $\mathbf{P}_T$ of **PBT** via the sylvester congruence.

Admittedly, the stalactic correspondence is rather trivial. Nevertheless, it yields interesting results when applied to special classes of words.

Equivalence classes yield subalgebras of $EQSym$:

Stalactic classes of endofunctions. This builds up a CHA with signature 1, 1, 4, 21, 136, 1045, 9276 [Sloane A052852], the EGF is $\frac{z}{1-z} \exp\left(\frac{z}{1-z}\right)$.

Stalactic classes of parking functions. Here the signature is 1, 1, 3, 13, 73, 501, 4051, ... [Sloane A000262], with EGF $\exp\left(\frac{z}{1-z}\right)$. This fills a gap in the diagram: these numbers count set partitions with an order inside each block but no order among the blocks ("sets of lists").

Stalactic classes of initial words. We get 1, 1, 3, 11, 49, 261, 1631, ... [Sloane A001339], EGF is $\frac{e^z}{(1-z)^2}$.

Parking functions and trees

For $\mathbf{a} \in \mathrm{PF}_n$, let

$$M_{\mathbf{a}} := \sum_{i_1 < \cdots < i_n} x_{i_1 \ i_{\mathbf{a}(1)}} \cdots x_{i_n \ i_{\mathbf{a}(n)}}.$$

Again, the $M_{\mathbf{a}}$ span a Hopf subalgebra $PQSym$ of $EQSym$.

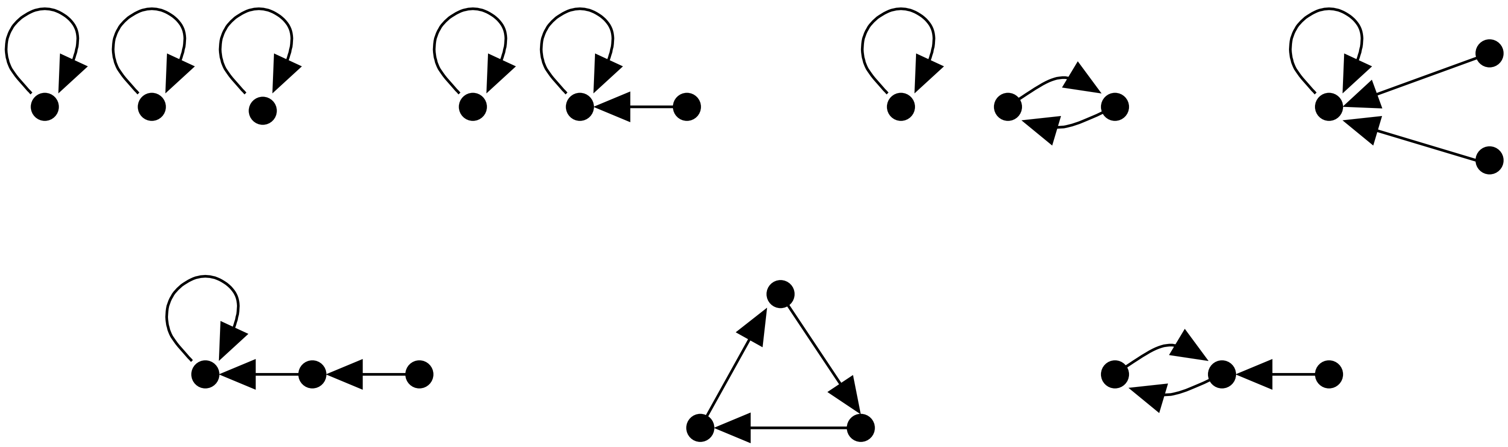
The non-commutative and non-cocommutative Hopf algebra of parking functions leads to two algebras of trees, a cocommutative Hopf algebra of planar binary trees by summing over the distinct permutations of parking functions, and an algebra of planar trees by summing over hypoplactic classes.

Let us now see whether similar constructions can be found for the commutative version $PQSym$.

Summing over permutations leads to the the unlabeled structures. Recall that the resulting CHA are always cocommutative.

Unlabeled objects

Unlabeled parking graphs. The signature is 1, 1, 3, 7, 19, 47, 130, 343, 951, 2615, ... [Sloane A001372].



We get a polynomial algebra on generators indexed by connected parking graphs.

Binary trees and nondecreasing parking functions. In $PQSym$, summing over parking functions having the same reordering does not lead to a subalgebra. But the subspace I of $PQSym$ spanned by the $M_{\mathbf{a}}$ where $\mathbf{a}$ is not nondecreasing is an ideal and a coideal, and $CQSym := PQSym/I$ is a commutative Hopf algebra with basis given by the classes $M_\pi = \overline{M_{\mathbf{a}}}$ labeled by nondecreasing parking functions.

$CQSym$ is also isomorphic to the image of $PQSym$ in the quotient

$$x_{ij}x_{kl} = 0, \text{ for all } i < k \text{ and } j > l.$$

The dual basis of M_π is $S^\pi = \sum_{\mathbf{a}} S^{\mathbf{a}}$, where the sum is taken over all permutations of π.

The dual $CQSym^*$ is free over the set S^π, where π runs over connected nondecreasing parking functions. The usual Cartier-Milnor-Moore argument then shows that

$$\mathbf{CQSym} \sim CQSym^*, \qquad\qquad CQSym \sim \mathbf{CQSym}^*.$$

Rooted forests. Nondecreasing parking functions correspond to parking graphs of a particular type: rooted forests with a particular labeling, the root being given by the loops in each connected component.

Taking sums over the allowed labelings of a given rooted forest, we get that the

$$M_F := \sum_{supp(\pi)=F} M_\pi,$$

span a commutative and cocommutative Hopf algebra of rooted forests (same signature 1, 1, 2, 4, 9, 20, 48, 115 ... as Connes-Kreimer).

Conclusion: quantization ?

When several Hopf algebra structures can be defined on the same class of combinatorial objects, it is tempting to try to interpolate between them. This can be done, for example with compositions: the algebra of quantum quasi-symmetric functions $QSym_q$ interpolates between quasi-symmetric functions and non-commutative symmetric functions.

However, the natural structure on $QSym_q$ is not exactly that of a Hopf algebra but rather of a *twisted Hopf algebra*.

This can be lifted to **FQSym**.

Let A' and A'' be q-commuting copies of the ordered alphabet A, *i.e.*, $a''a' = qa'a''$ for $a' \in A'$ and $a'' \in A''$. Then, the coproduct $\Delta_q f = f(A' \dot{+} A'')$ defines a twisted Hopf algebra structure on **FQSym**:

$$\Delta_q \mathbf{F}_\sigma = \sum_{\alpha\cdot\beta=\sigma} q^{\mathrm{inv}(\alpha,\beta)} \mathbf{F}_{\mathrm{Std}(\alpha)} \otimes \mathbf{F}_{\mathrm{Std}(\beta)}$$

One can also define a one-parameter family of ordinary Hopf algebra structures on **FQSym**, by restricting the formula for Δ_q to connected permutations σ, and requiring that Δ_q be an algebra homomorphism. Then, for $q = 0$, Δ_q becomes cocommutative, and the resulting

Hopf algebra is isomorphic to $\mathfrak{S}\mathbf{Sym}$.

Similar considerations apply to various examples, in particular to the Loday-Ronco algebra $\mathbf{PBT}$, whose commutative version can be quantized in the same way as $QSym$, by means of q-commuting variables.

One may expect that these are special cases of a general construction, and that a nice classification theorem will enventually be found.