# TEN YEARS OF NONCOMMUTATIVE SYMMETRIC FUNCTIONS

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# INTRODUCTION

# Brief history

- Started in 1993 after a suggestion of I. M. Gel'fand
- Original goal: find applications of the Gel'fand-Retakh *quasi-determinants*
- It was soon understood that part of the theory was related to Solomon's descent algebras, quasi-symmetric functions and the combinatorics of free Lie algebras
- Next, connections with Hecke algebras and quantum groups at $q = 0$ have been found
- Most of the classical applications of symmetric functions have now a noncommutative or a quasisymmetric analogue

# The starting point

Different notions of noncommutative elementary symmetric polynomials, depending on the property to be generalized.

- Roots of noncommutative polynomials
- Expansion of a product of linear factors
- Characteristic polynomials for matrices with noncommuting entries

# Roots of noncommutative polynomials

$c_1, \ldots, c_n \in R$ division ring. Find

$$g(x) = x^n - \Lambda_1 x^{n-1} + \Lambda_2 x^{n-2} - \cdots + (-1)^n \Lambda_n$$

such that the $c_k$ are right roots of $g(x)$, i.e.

$$c_k^n - \Lambda_1 c_k^{n-1} + \Lambda_2 c_k^{n-2} - \cdots + (-1)^n \Lambda_n = 0$$

**Answer:** (Bray and Whaples, Canad. J. Math **35** (1983), 509–515)

$$g(x) = \begin{vmatrix} 1 & \ldots & 1 & 1 \\ c_1 & \ldots & c_n & x \\ c_1^2 & \ldots & c_n^2 & x^2 \\ \vdots & & \vdots & \vdots \\ c_1^n & \ldots & c_n^n & \boxed{x^n} \end{vmatrix}$$

(where $x$ commutes with $R$) unique solution if $c_i$ pairwise non conjugate.
Quasi-determinants of a generic $n \times n$ matrix $A = (a_{ij})$

$$|A|_{ij} = (A^{-1})_{ji}^{-1}$$

(analogues of the ratio of a determinant and one of its principal minors).
Example: $\Lambda_1(x_1, x_2) = (x_2^2 - x_1^2)(x_2 - x_1)^{-1}$,
$\Lambda_2(x_1, x_2) = (x_2^2 - x_1 x_2)(x_1^{-1} x_2 - 1)^{-1}$ (symmetric).

# Products of linear factors

(Noncommutative variables)
$A = \{a_1 < \ldots < a_n\}$ ordered alphabet
Elementary symmetric functions

$$\lambda_t(A) = (1 + ta_n)(1 + ta_{n-1}) \cdots (1 + ta_1) = \sum_{k=0}^{n} \Lambda_k(A) t^k$$

where $t$ commutes with the $a_i$.

Symmetric, but for the plactic action of Lascoux and Schützenberger
(now generalized by Kashiwara's action of Weyl groups on crystal graphs).
Embedding of the usual ring of symmetric functions in the plactic algebra.

For $A$ infinite: formal noncommutative symmetric functions.

# Matrices over noncommutative rings

Analogues of the characteristic polynomial for a $n \times n$ matrix $A$ with noncommutative entries:
*$n$ quasi-characteristic polynomials*

$$|1 + tA|_{ii} = \lambda_t(\alpha^{(i)})$$

($n$ "virtual alphabets" of quasi-eigenvalues $\alpha^{(i)}$).

Example: $A = E_n - (n-1)I$

$$E_n = \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ e_{21} & e_{22} - 1 & \dots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \dots & e_{nn} - n + 1 \end{pmatrix}$$

(Capelli matrices),
$e_{ij}$ generators of $U(gl_n)$
$\longrightarrow$ new generators of its center $Z(gl_n)$

Generalized by A. Molev to the case of Sklyanin determinants in twisted Yangians

# Formal noncommutative symmetric functions

There are several reasonable but non equivalent notions of noncommutative elementary symmetric functions.

## Conclusions

- *Formal Noncommutative Symmetric Functions*

$$\mathbf{Sym} = \mathbf{K}\langle \Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots \rangle$$

  free associative algebra over an infinite sequence of noncommuting indeterminates $\Lambda_k, k \geq 1$, with $\Lambda_k$ of degree $k$,

- try to generalize as much as possible the classical constructions on commutative symmetric functions

$$Sym = \mathbf{K}[e_1, e_2, \ldots, e_n, \ldots]$$

- consider the previous examples as specializations.

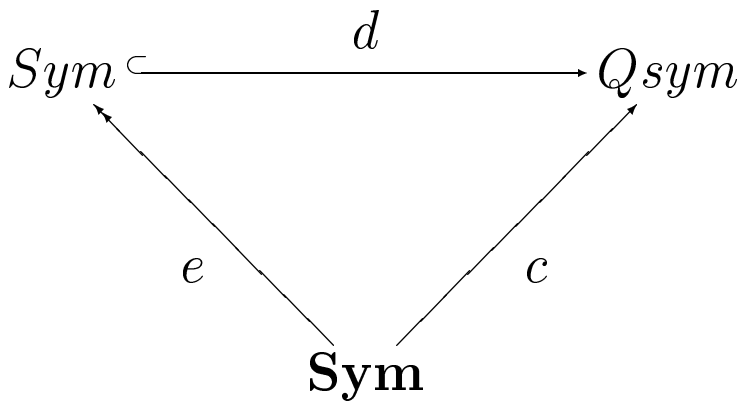
# The theory of symmetric functions

- **Special bases:** Schur, Hall-Littlewood, Jack, Macdonald ...
- **Operations:** Multiplications and comultiplications, scalar products, differential operators, plethysms ...

describe, among other things:

- $GL(n, \mathbb{C})$: Characters, tensor product multiplicities, weight multiplicities, branching rules ...
- $\mathfrak{S}_N$: Character table, induction, restriction, tensor products, branching rules ...
- $GL(n, \mathbb{F}_q)$: Character table, induction, restriction ...
- $GL(n, \mathbb{Q}_p)$: Spherical functions ...
- $\widehat{\mathfrak{sl}}_n$: Fock space representations, singular vectors for Virasoro and $W$-algebras ...
- Quantum analogues: Hecke algebra $H_N(q)$, quantized function algebras $F_q(GL_n)$, quantized enveloping algebras $U_q(\mathfrak{gl}_n)$ and their affine versions ...

The algebra of symmetric functions fits into a triangular diagram:




- $Sym$ = symmetric functions
- $QSym$ = *quasi-symmetric functions*
- **Sym** = *noncommutative symmetric functions*
- $d$ = inclusion
- $e$ = abelianization

**Our program:** find analogues of the previous slide in **Sym** and $Qsym$

# BACKGROUND

# Symmetric functions

$$Sym = \mathbf{K}[e_1, e_2, \ldots] \quad \deg(e_k) = k$$

$$\sum_{k\geq 0} t^k e_k(X) = \prod_{i\geq 1}(1+tx_i) = \lambda_t(X)$$

$$\sum_{k\geq 0} t^k h_k(X) = \prod_{i\geq 1}(1-tx_i)^{-1} = \sigma_t(X)$$

- Schur functions: $s_\lambda = \det(h_{\lambda_i - i+j})$
- Cauchy identity: $XY = \{x_i y_j,\ i,j \geq 1\}$

$$h_n(XY) = \sum_{\lambda\vdash n} s_\lambda(X) s_\lambda(Y)$$

- Frobenius map: $\mathrm{ch}\,(\chi^\lambda) = s_\lambda$
- Internal product: $s_\lambda * s_\mu = \mathrm{ch}\,(\chi^\lambda \chi^\mu)$
- Generalized Cauchy:

$$s_\lambda(XY) = \sum_\mu (s_\lambda * s_\mu)(X) s_\mu(Y)$$

# Descent algebras

$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_r} = \mathrm{ch}\left(\mathbf{1} \uparrow_{\mathfrak{S}_{\mu_1}\times\mathfrak{S}_{\mu_2}\times\cdots\times\mathfrak{S}_{\mu_r}}^{\mathfrak{S}_n}\right)$$

- Mackey's formula gives

$$h_\mu * h_\nu = \sum_{M\in \mathrm{Mat}(\mu,\nu)} h_M$$

$$h_M = \prod_{i,j} h_{m_{ij}} \quad \text{and} \quad \mathrm{Mat}(\mu,\nu) \simeq \mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\nu$$

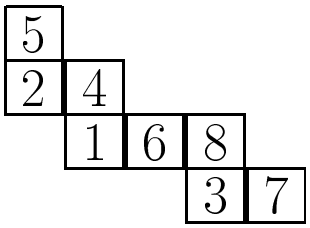
- Descent algebra: noncommutative lifting in $\mathfrak{S}_n$

$$H_I H_J = \sum_{M\in \mathrm{Mat}(I,J)} H_M$$

- $I$ and $J$ are now *compositions* of $n$, and $H_I$, $H_J$ certain elements of $\mathbb{Z}\mathfrak{S}_n$.

# Descents of permutations

$\sigma = \underline{5}2\underline{4}16\underline{8}37 \longrightarrow$




- Descent set $\mathrm{Des}\,(\sigma) = \{1, 3, 6\}$
- Descent composition $C(\sigma) = I = (1, 2, 3, 2)$
- $\mathrm{Des}\,(I) = \{1, 3, 6\}$
- $D_I = \displaystyle\sum_{C(\sigma)=I} \sigma \quad \in \quad \mathbb{Z}[\mathfrak{S}_n]$
- $H_I = \displaystyle\sum_{\mathrm{Des}\,(J)\subseteq \mathrm{Des}\,(I)} D_J$

**Descent algebra** (L. Solomon 1976) The $D_I$ or the $H_I$ span a $\mathbb{Z}$-subalgebra $\Sigma_n$

$$D_I D_J = \sum_{|K|=n} d_{IJ}^K \, D_K \,, \qquad d_{IJ}^K \in \mathbb{Z}$$

$$H_I H_J = \sum_{M \in \mathrm{Mat}(I,J)} H_M$$

(similar result for all Coxeter groups)

# Noncommutative symmetric functions

$$\mathbf{Sym} = \mathbf{K}\langle S_1, S_2, \ldots, S_n, \ldots \rangle$$

(or $\mathbf{K}\langle \Lambda_1, \Lambda_2, \ldots \rangle$) graded by $\deg(S_k) = \deg(\Lambda_k) = k$

- Linear bases: $S^I = S_{i_1} S_{i_2} \cdots S_{i_r}$, $\Lambda^I$
- Algebra morphism $\mathbf{Sym} \to Sym : \ S_k \mapsto h_k$
- Linear isomorphism $\alpha : \ \Sigma_n \to \mathbf{Sym}_n : \ H_I \mapsto S^I$
- Internal product $*$ on $\mathbf{Sym}$ defined by $\mathbf{Sym}_m * \mathbf{Sym}_n = 0$ for $m \neq n$ and $\alpha$ *anti*-isomorphism
- Polynomial realization: $A = \{a_1 < a_2 < \ldots\}$ totally ordered set of noncommuting variables

$$\sigma_t(A) = \prod_{i \geq 1}^{\rightarrow} (1 - t a_i)^{-1} = \sum_{n \geq 0} t^n S_n(A)$$

$$\lambda_t(A) = \sigma_{-t}(A)^{-1} = \prod_{i \geq 1}^{\leftarrow} (1 + t a_i) = \sum_{n \geq 0} t^n \Lambda_n(A)$$

# Quasi-symmetric functions

Analog of Cauchy formula:

- $X = \{x_1 < x_2 < \cdots\}$ totally ordered set of *commuting* variables
- Product alphabet $XA = \{x_i a_j\}$ with lexicographic order on $(i, j)$
- Noncommutative symmetric functions of $XA$ given by

$$\sigma_1(XA) = \sum_I M_I(X) S^I(A) = \sum_I F_I(X) R_I(A)$$

- $M_I(X)$, $F_I(X)$ are *quasi-symmetric* functions (Gessel, 1984)
- Quasi-monomial functions

$$M_K = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$$

- $R_I = \sum_{J \preceq I} (-1)^{l(I)-l(J)} S^J$ (ribbon Schur functions)
- Quasi-ribbons $F_I = \sum_{I \preceq J} M_J$
- $Qsym$ is a subalgebra of $\mathbf{K}[X]$ containing $Sym$

# The whole picture

- **Sym** and $Qsym$ are two mutually dual Hopf algebras
- Duality bracket: $\langle F_I, R_J \rangle = \langle M_I, S^J \rangle = \delta_{IJ}$
- Comultiplications and antipodes $\leftrightarrow$ natural operations on alphabets (ordered sum, opposite ...) $\leftrightarrow$ $\lambda$-ring formalism of the commutative case
- The internal product lives in **Sym** and comes from the descent algebras
- The internal products allows to handle products of alphabets

$$R_I(XA) = \sum_J F_J(X)(R_I * R_J)(A)$$

- In particular, the alphabet $X_q = \{1, q, q^2, \dots\} = 1/(1-q)$ [with its natural order $\cdots < q^n < \cdots q^2 < q < 1$] and its formal inverse $1 - q$ can be used in the same way as in the commutative case (Hall-Littlewood functions, Hecke algebras, etc.)
- A striking application: *Lie idempotents*

# SYNOPSIS OF ALGEBRAIC STRUCTURES

# Commutative theory

- $Sym = \mathbf{K}[e_1, e_2, \ldots]$ graded by $\deg(e_k) = k$
- Other generators: let $\lambda(t) = \sum_{k\geq 0} e_k t^k$.
  $\sigma(t) = \lambda(-t)^{-1} = \sum_{k\geq 0} h_k t^k$ (complete homogeneous functions), $\psi(t) = \sum_{k\geq 1} p_k t^{k-1} = \frac{d}{dt}\log\sigma(t)$ (power sums)
- Hopf algebra: coproduct $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$, antipode $\tilde{\omega}(p_k) = -p_k$
- Isomorphic to its (graded) dual
- $\dim Sym_n = |\mathrm{Part}\,(n)| = p(n)$.
- Linear bases $m_\lambda$ (monomial), $e_\lambda, h_\lambda, p_\lambda$ (products), $s_\lambda$ (Schur functions)
- scalar product $\langle s_\lambda, s_\mu\rangle = \langle m_\lambda, h_\mu\rangle = \langle p_\lambda, p^*_\mu\rangle = \delta_{\lambda\mu}$ ($p^*_\mu = p_\mu/z_\mu$).
- Cauchy type identities

$$\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu} \Leftrightarrow \prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(X) v_\lambda(Y)$$

- Second coproduct $\delta(f) = f(XY)$ $(u(X)v(Y) \equiv u \otimes v)$
- Internal product $\langle f * g, h\rangle = \langle f \otimes g, \delta(h)\rangle$
- Cycle index $\alpha : Z(\mathfrak{S}_n) \to Sym_n$
- Frobenius characteristic map $s_\lambda = \mathrm{ch}\,(\chi^\lambda)$ isometric isomorphism $R(\mathfrak{S}_n) \to Sym_n$ ($\chi^\lambda$ irreducible characters of $\mathfrak{S}_n$)
- $s_\lambda * s_\mu = \mathrm{ch}\,(\chi^\lambda \chi^\mu)$
- Frobenius character formula: $\chi^\lambda(\mu) = \langle s_\lambda, p_\mu\rangle$
- Induction $\mathfrak{S}_n \times \mathfrak{S}_n \leftrightarrow$ product $Sym_n \times Sym_m$
- Schur character formula: $s_\lambda(x_1, \ldots, x_n) = \mathrm{trace}_{V_\lambda}(X)$ ($V_\lambda$ irreducible polynomial representations of $GL(n, \mathbb{C})$, $X = \mathrm{diag}\,(x_1, \ldots, x_n)$.

  Also valid for the quantum analogues $U_q(gl_n)$, $F_q(GL_n)$. Plactic algebra
- Generic Hecke algebra $H_n(q)$: same characteristic map $\mathrm{ch}\,(\chi^\lambda) = s_\lambda$, modified character formula $\chi^\lambda_\mu = \langle s_\lambda,\, h_\mu((q-1)X)\rangle/(q-1)^{\ell(\mu)}$
- $q$ and $(q, t)$ analogues: Hall-Littlewood and Macdonald functions

# Noncommutative theory

- **Sym** $= \mathbf{K}\langle \Lambda_1, \Lambda_2, \ldots \rangle$ graded by $\deg(\Lambda_k) = k$
- Other generators: let $\lambda(t) = \sum_{k\geq 0} \Lambda_k t^k$.
  $\sigma(t) = \lambda(-t)^{-1} = \sum_{k\geq 0} S_k t^k$ (complete homogeneous functions), **several choices for power sums, e.g.**
  $\psi(t) = \sum_{k\geq 1} \Psi_k t^{k-1} = \lambda(-t)\sigma'(t)$ (first kind),
  $\Phi(t) = \log \sigma(t) = \sum_{k\geq 1} \Phi_k t^k/k$ (second kind)
- Hopf algebra: coproduct $\Delta(\Psi_k) = \Psi_k \otimes 1 + 1 \otimes \Psi_k$, antipode $\tilde{\omega}(\Psi_k) = -\Psi_k$. **Primitive Lie algebra = Power sums**
- **Not** isomorphic to its (graded) dual, which is $Qsym$ (**quasi-symmetric functions**)
- $\dim Sym_n = |\mathrm{Comp}\,(n)| = 2^{n-1}$.
- Linear bases $M_I \in Qsym$ (quasi-monomial), $\Lambda^I, S^I, \Psi^I, \Phi^I \in$ **Sym** (products), Schur functions $\rightarrow$ $R_I \in$ **Sym** (ribbons), $F_I \in Qsym$ (quasi-ribbons)
- No scalar product but a pairing $Qsym \times$ **Sym** $\rightarrow$ **K**
  $\langle F_I, R_J \rangle = \langle M_I, S^J \rangle = \ldots = \delta_{IJ}$
- Cauchy type identities

$$\langle U_I, V_J \rangle = \delta_{IJ} \Leftrightarrow \prod_i^{\rightarrow} \prod_j^{\rightarrow} (1 - x_i a_j)^{-1} = \sum_\lambda U_\lambda(X) V_\lambda(A)$$

- Second coproduct lives on $Qsym$ $\delta(f) = f(XY)$ $(u(X)v(Y) \equiv u \otimes v)$
- Internal product $\langle h, f * g \rangle = \langle \delta(h), f \otimes g \rangle$
- Correspondence with descent algebras $\alpha : \Sigma_n \to \mathbf{Sym}_n$
- Frobenius-like characteristic maps for $H_n(0)$ Grothendieck groups $G_0(H_n(0) \simeq Qsym_n$ (simple modules $\leftrightarrow F_I$, $K_0(H_n(0)) \simeq \mathbf{Sym}_n$ (indecomposable projectives $\leftrightarrow R_I$)
- $*$ is not the tensor product of $H_n(0)$ modules (induces a different operation on $Qsym$
- Character formula: $\chi^I(J) = \langle F_I, S^I((q-1)A) \rangle / (q-1)^{\ell(I)}|_{q=0}$
- Induction product $\leftrightarrow$ products of $Qsym$ and $\mathbf{Sym}$
- $F_I$ and $R_I$ are characters of simple and indecomposable polynomial modules of certain quantum groups at $q = 0$. Weyl and Demazure character formulas. Hypoplactic algebra.
- $q$ and $(q, t)$ analogues: Hall-Littlewood functions, Macdonald functions.

# LIE IDEMPOTENTS AND NONCOMMUTATIVE POWER SUMS

# Lie idempotents

- Right action of $\mathfrak{S}_n$ on $A^n$ (words of length $n$)

$$x_1 x_2 \cdots x_n \cdot \sigma = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

- Free Lie algebra $L(A) \subset \mathbf{K}\langle A \rangle$
- $\pi \in \mathbf{K}\mathfrak{S}_n$ is a *Lie idempotent* if it acts as a projector $\mathbf{K}_n\langle A \rangle \to L_n(A)$ (onto)
- Basic examples:

1. Dynkin (1947):

$$\theta_n = \frac{1}{n}[\ldots[1,2],3],\ldots],n] = \frac{1}{n}\sum_{k=0}^{n-1} D_{(1^k,n-k)}$$

2. Solomon (1968):

$$\phi_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\sigma \qquad (d(\sigma) = |\mathrm{Des}\,(\sigma)|)$$

3. Klyachko (1974):

$$\kappa_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \omega^{\mathrm{maj}\,(\sigma)}\sigma$$

$$\omega = e^{2\mathrm{i}\pi/n}, \qquad \mathrm{maj}\,(\sigma) = \sum_{j\in\mathrm{Des}\,(\sigma)} j\,.$$

# The Hausdorff series

$$H(a_1, a_2, \ldots, a_N) = \log\left(e^{a_1} e^{a_2} \cdots e^{a_N}\right) = \sum_{n \geq 0} H_n(A)$$

is a *Lie series* (BCH theorem)

Natural questions about $H_n \longleftrightarrow$ idempotents in $\mathbb{C}\mathfrak{S}_n$

**Q1.** Given

$$H_n(A) = \sum_{w \in A^n} c_w \, w$$

express it as a linear combination of commutators.

**Example:**

$H_3(a_1, a_2, a_3) = \frac{1}{12} a_1 a_1 a_2 + \frac{1}{12} a_1 a_1 a_3 - \frac{1}{6} a_1 a_2 a_1 +$
$\frac{1}{12} a_1 a_2 a_2 + \frac{1}{3} a_1 a_2 a_3 - \frac{1}{6} a_1 a_3 a_1 + \frac{1}{12} a_1 a_3 a_3 + \frac{1}{12} a_2 a_1 a_1 -$
$\frac{1}{6} a_2 a_1 a_2 - \frac{1}{6} a_2 a_1 a_3 + \frac{1}{12} a_2 a_2 a_1 + \frac{1}{12} a_2 a_2 a_3 - \frac{1}{6} a_2 a_3 a_1 -$
$\frac{1}{6} a_2 a_3 a_2 + \frac{1}{12} a_2 a_3 a_3 + \frac{1}{12} a_3 a_1 a_1 - \frac{1}{6} a_3 a_1 a_2 - \frac{1}{6} a_3 a_1 a_3 +$
$\frac{1}{3} a_3 a_2 a_1 + \frac{1}{12} a_3 a_2 a_2 - \frac{1}{6} a_3 a_2 a_3 + \frac{1}{12} a_3 a_3 a_1 + \frac{1}{12} a_3 a_3 a_2$
$= \frac{1}{12}[a_1, [a_1, a_2]] + \frac{1}{12}[[a_1, a_2], a_2] + \frac{1}{12}[a_1, [a_1, a_3]] +$
$\frac{1}{12}[[a_1, a_3], a_3] + \frac{1}{12}[a_2, [a_2, a_3]] + \frac{1}{12}[[a_2, a_3], a_3] +$
$\frac{1}{6}[a_1, [a_2, a_3]] + \frac{1}{6}[[a_1, a_2], a_3]$

**Answer 1.** (Dynkin 1947) $P \in \mathbf{K}_n(A)$ is in $L(A)$ iff

$$\Theta_n(P) = P$$

where

$$\Theta_n(x_1 x_2 \cdots x_n) = \frac{1}{n}[\ldots[[x_1, x_2], x_3], \ldots, x_n]$$

$$\Longrightarrow H_n(A) = \sum_{w \in A^n} c_w \Theta_n(w)$$

is a Lie expression.

$\Theta_n$ is a symmetrizer

$$\Theta_n(w) = w \cdot \theta_n$$

$$\theta_n = \frac{1}{n}[\ldots[[1, 2], 3], \ldots, n] \quad \in \quad \mathbb{Q}[\mathfrak{S}_n]$$

$$(x_1 x_2 \cdots x_n) \cdot \sigma = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} \qquad \sigma \in \mathfrak{S}_n$$

$$\text{Dynkin's theorem} \Longleftrightarrow \theta_n^2 = \theta_n$$

$\theta_n$ is a *Lie idempotent* in the *descent algebra*

$$2\theta_2 = \boxed{1}\boxed{2} - \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$3\theta_3 = \boxed{1}\boxed{2}\boxed{3} - \left( \begin{array}{|c|c|}\hline 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \right) + \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$4\theta_4 = \underbrace{\boxed{1}\boxed{2}\boxed{3}\boxed{4}}_{D_4} - \underbrace{\left( \begin{array}{|c|c|c|}\hline 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right)}_{D_{13}} - \underbrace{\left( \begin{array}{|c|c|c|}\hline 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 4 & \multicolumn{1}{c}{} \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \hline \end{array} \right)}_{D_{112}} - \underbrace{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}}_{D_{1111}}$$

Correspondence $\Sigma_n \leftrightarrow \mathbf{Sym}_n$

$$D_I \longleftrightarrow R_I$$

$$n\theta_n = \sum_{k=0}^{n-1} (-1)^k D_{1^k,n-k} \quad \longleftrightarrow \Psi_n \in \mathbf{Sym}$$

Newton type recurrence

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + S_1\Psi_{n-1} + \Psi_n$$

**Q2.** How to obtain the expansion

$$H_n(A) = \sum_{w \in A^n} c_w w$$

on words ?

**Answer 2.** (Goldberg 1956, Solomon 1968, Bialynicki-Birula, Mielnik and Plebański 1969) Let

$$e^{a_1} e^{a_2} \cdots e^{a_N} = \sum_{n \geq 0} E_n(A)\,.$$

Then,

$$H_n(A) = E_n(A) \cdot \phi_n$$

where

$$\phi_n = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\, \sigma \qquad (d(\sigma) = |\mathrm{Des}\,(\sigma)|)$$

is a Lie idempotent in $\Sigma_n$.

As an NCSF,

$$\phi_n \longleftrightarrow \frac{1}{n} \Phi_n$$

**Q3.** The Dynkin elements $\Theta_n(w)$ are not linearly independent. How to obtain an expansion on a *basis* of $L(A)$?

**Answer 3.** Use *Klyachko's basis* $k_w = w \cdot \kappa_n$ ($w$ Lyndon word), where

$$\kappa_n = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \omega^{\mathrm{maj}\,(\sigma)} \sigma$$

$$\omega = e^{2\mathrm{i}\pi/n}, \qquad \mathrm{maj}\,(\sigma) = \sum_{j \in \mathrm{Des}\,(\sigma)} j\,.$$

is Klyachko's Lie idempotent (clearly in $\Sigma_n$).

$n\kappa_n \leftrightarrow$ specialization $q = \omega$ of

$$K_n(q) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I = (q)_n S_n\left(\frac{A}{1-q}\right)$$

noncommutative analog of the Hall-Littlewood function $\tilde{Q}'_{1^n}(X;q)$.

$$\kappa_n \text{ Lie idempotent} \longleftrightarrow \tilde{Q}'_{1^n}(X;\omega) = p_n(X)$$

(simplest case of a known specialization property).

# Noncommutative power sums

- Commutative case: $\sigma_t(X) = \exp\left\{\sum_{k\geq 1} p_k(X)\frac{t^k}{k}\right\}$ equivalent to Newton's recursion
  $nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + h_1 p_{n-1} + p_n$
- Both expressions make sense in the noncommutative case but define different "power sums":
  1. $\sigma_t(A) = \exp\left\{\sum_{k\geq 1} \Phi_k(A)\frac{t^k}{k}\right\}$
  2. $nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + S_1\Psi_{n-1} + \Psi_n,$
- $\Phi(t) = \log \sigma_t$ where $\sigma_t$ is the solution of $\frac{d}{dt}\sigma_t = \sigma_t \psi(t)$ satisfying $\sigma_0 = 1$, and $\psi(t) = \sum_{k\geq 1} t^{k-1}\Psi_k$
- The expression of $\Phi(t)$ in terms of the $\Psi_k$ is known as the continuous Baker-Campbell-Hausdorff series
- A third possibility: $\sigma_t = e^{Z_1 t} e^{Z_2 \frac{t^2}{2}} e^{Z_3 \frac{t^3}{3}} \cdots$
- $\Phi(t) = H(Z_1 t, Z_2 t^2/2, Z_3 t^3/3, \ldots)$ where $H$ is the standard Hausdorff series
- All these elements ($\Phi_n$, $\Psi_n$ or $Z_n$) generate the same (free) Lie algebra $\mathcal{L}$ in **Sym** and have $p_n$ as commutative image
- Any homogeneous element $P_n \in \mathcal{L}_n$ having $p_n$ as commutative image is called a noncommutative power sum

# Lie idempotents in $\Sigma_n$

The fundamental splitting formula

$$(F_1F_2\cdots F_r) * G = \mu_r[(F_1 \otimes \cdots \otimes F_r) * \Delta^r G]$$

allows to prove:

- $\pi_n \in \Sigma_n$ is a Lie idempotent $\Leftrightarrow P_n = n \cdot \alpha(\pi_n)$ is a noncommutative power sum [NCSF1]
- The Lie idempotents of $\Sigma_n$ form an affine space, corresponding to $\frac{1}{n}\Psi_n + [\mathcal{L}, \mathcal{L}]$ in $\mathbf{Sym}_n$
- Any sequence $P_n(A)$ of noncommutative power sums can be obtained as $P_n(A) = \Psi_n(XA)/p_n(X)$, for $X$ a virtual ordered alphabet [NCSF3]
- Building blocks for other idempotents [NCSF2]:
  1. The $P^I$ are idempotent up to a nonzero scalar factor
  2. If $S_n = \sum_{|I|=n} c_I P^I$, define for a partition $\lambda$ of $n$

$$E_\lambda(P) = \sum_{I \in \mathfrak{S}(\lambda)} c_I P^I$$

The $E_\lambda(P)$ form a complete set of orthogonal idempotents of $\Sigma_n$ ($P_n = \Phi_n$: Garsia-Reutenauer 1989)

3. The $E_n^{[k]}(P) = \sum_{l(\lambda)=k} E_\lambda(P)$ form a complete set of orthogonal idempotents of an $n$-dimensional commutative subalgebra ($P_n = \Phi_n$: Eulerian idempotents)

# Sample calculation

## Dynkin's idempotent

We want to know whether $\Psi_n * \Psi_n = n\Psi_n$.

Generating function for the $*$-squares

$$\sum_{n\geq 1}(xy)^{n-1}(\Psi_n * \Psi_n) = \psi(x) * \psi(y)\,,$$

since $\Psi_i * \Psi_j = 0$ for $i \neq j$.

Now,

$$\psi(t) = \sum_{n\geq 1} t^{n-1}\Psi_n = \lambda(-t)\sigma'(t)$$

applying the splitting formula we get

$$\sum_{n\geq 1}(xy)^{n-1}(\Psi_n * \Psi_n) = \lambda(-x)\sigma'(x) * \psi(y)$$

$$= \mu\left[(\lambda(-x)\otimes\sigma'(x)) * (\psi(y)\otimes 1 + 1\otimes\psi(y))\right]$$

$$= \mu\left[(\lambda(-x) * 1)\otimes(\sigma'(x) * \psi(y))\right]\,,$$

(since $\sigma'(x)$ has no degree zero terms)

$$= \left(\sum_{n\geq 1} nx^{n-1}\,S_n\right) * \left(\sum_{n\geq 1} y^{n-1}\,\Psi_n\right) = \sum_{n\geq 1}(xy)^{n-1}\,n\,\Psi_n\,,$$

the last equality following from the fact that $S_n * F = F$ for $F \in \mathbf{Sym}_n$.

Hence, $\Psi_n * \Psi_n = n\Psi_n$.

# Example 1: Hall-Littlewood vs. Klyachko

- $\tilde{Q}'_\mu(X;q) = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q) s_\lambda(X)$
- Specialization formulas for $q = \omega^k = e^{2\mathrm{i}k\pi/n}$
- Explicit formula for $\mu = (1^n)$

$$\tilde{Q}'_{(1^n)}(X;q) = (q)_n h_n\left(\frac{X}{1-q}\right)$$

- Natural noncommutative analog

$$K_n(q) = (q)_n S_n\left(\frac{A}{1-q}\right) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I$$

- Particular case: $\tilde{Q}'_{(1^n)}(X;\omega) = p_n$
- Therefore, $K_n(\omega) \mapsto p_n$. Also, $K_n(\omega) \in \mathcal{L}$
- Hence, $K_n(\omega)$ is a noncommutative power sum.
- The corresponding Lie idempotent is Klyachko's
- $K_n(\omega^k)$ computable ($\Rightarrow \ell_n^{(j)} * \ell_n^{(k)}$ )
- Other noncommutative Hall-Littlewood functions (F. Hivert)

# Example 2: a projective line

- $P_n(A;q) = (1-q^n)\Psi_n\left(\frac{A}{1-q}\right)$ is a noncommutative power sum
- The corresponding Lie idempotent is

$$\varphi_n(q) = \frac{1}{n}\sum_{|I|=n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} q^{\mathrm{maj}\,(\sigma)-\binom{d(\sigma)+1}{2}}\sigma$$

- it specializes to

$$\varphi_n(0) = \theta_n\,, \qquad \varphi_n(1) = \phi_n \qquad \varphi_n(\omega) = \kappa_n$$

(and $\varphi_n(\infty) = \theta_n^*$).

- Nonlinear operator $E_q[H(t)]$, where $H(t) = \sum_{n\geq 1} H_n t^{n-1}$

$$E_q[H(t)] = \sum_I c_I(q) H^I t^{|I|}$$

Then,

$$E_1[H(t)] = \exp \int_0^t H(s)ds$$

while $E_0$ is Dyson's chronological exponential

$$E_0[H(t)] = T \exp \int_0^t H(s)ds$$

$$= 1 + \int_0^t dt_1 H(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 H(t_2)H(t_1) + \cdots$$

# Example 3: $q$-brackets

- $[a,b]_q = ab - qba$
- $[\ldots[a_1,a_2]_q,a_3]_q,\ldots]_q,a_n]_q = (a_1a_2\cdots a_n)\cdots\theta_n(q)$
- $\theta_n(q) \in \Sigma_n \;\leftrightarrow\; (1-q)^{-1}S_n((1-q)A) \in \mathbf{Sym}$
- For generic $q$, $\theta_n(q)$ is invertible and its inverse is

$$(1-q)S_n\left(\frac{A}{1-q}\right) = \frac{1-q}{(q)_n}K_n(q)$$

- $P \in \mathbf{K}_n\langle A\rangle$ is in $L(A)$ iff $P\theta_n(q) = [n]_qP$
- Eigenvalues of $\theta_n(q)$: $p_\lambda(1-q) = \prod_i (1-q^{\lambda_i})$
- Multiplicities $n!/z_\lambda \Rightarrow$ Krattenthaler's conjecture

$$\det\left[\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}\,(\sigma)}\sigma\right] = \prod_{i=1}^{n}(1-q^i)^{n!(i-1)/i}$$

- Spectral projectors $\pi_\lambda(q)$: complete set of orthogonal idempotents of $\Sigma_n$ constructed from $(\pi_n(q))_{n\geq 1}$
- $\pi_n(1) =$ Dynkin, $\pi_n(\omega) =$ Klyachko, $\pi_n(0) =$ Zassenhaus $(Z_n/n)$([Duchamp-Krob-Vassilieva], conjectured in [NCSF2]

- The associated "$q$-exponential" interpolates between Dyson's ($q = 1$) and

$$E_0[H(t)] = e^{H_1 t} e^{H_2 \frac{t^2}{2}} e^{H_3 \frac{t^3}{3}} \dots$$

# HALL-LITTLEWOOD FUNCTIONS

# Hecke algebras

- The classical Hall-Littlewood functions can be defined in terms of (affine) Hecke algebras
- $H_N(q)$ Hecke algebra of type $A_{N-1}$: $\mathbb{C}$ or $\mathbb{C}(q)$ algebra generated by $T_1, \ldots, T_{N-1}$

$$T_i T_j = T_j T_i \qquad |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad i = 1, \ldots, N-2$$

$$T_i^2 = (q-1) T_i + q \qquad i = 1, \ldots, N-1$$

- Faithful representation on $\mathbb{C}[x_1, \ldots, x_N]$:

$$T_i = (q-1)\pi_i + \sigma_i$$

$$\pi_i(f) = \frac{x_i f - \sigma_i(x_i f)}{x_i - x_{i+1}}, \quad \sigma_i = (i, i+1)$$

- Actually, a representation of the affine Hecke algebra $\hat{H}_N(q)$ (extra generators $y_i \mapsto x_i$)
- Full $q$-symmetrizer $S^{(N)} = \sum_{\sigma \in \mathfrak{S}_N} T_\sigma$
- $q$-symmetrized monomials $\simeq$ Hall-Littlewood functions

$$Q_\lambda(x_1, \ldots, x_N; q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}}!} S^{(N)}(x^\lambda)$$

- Hall basis

$$P_\lambda(X;q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q! \cdots [m_n]_q!} Q_\lambda(X;q)$$

  where $m_i$ is the multiplicity of the part $i$ in $\lambda$

- Dual basis for $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

$$Q'_\mu(X;q) = Q_\mu\left(\frac{X}{1-q};q\right) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X)$$

- Geometric dual basis

$$\tilde{Q}'_\mu(X;q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda(X) = q^{n(\mu)} Q'_\mu(X;q^{-1})$$

- $\tilde{K}_{\lambda\mu}(q)$ are Kazhdan-Lusztig polynomials and entries of the character table of $GL_n(\mathbb{F}_q)$
- Combinatorial interpretation: cocharge [Lascoux-Schützenberger 1978]
- $\pi_i$: Demazure operators (Demazure character formula: $s_\lambda = \pi_\omega(x^\lambda)$ and interpretation of $\pi_\sigma(x^\alpha)$
- $T_i \mapsto -\pi_i$ is a representation of $H_N(0)$

# Noncommutative Hall-Littlewood functions ?

- If analogs of the $Q'$ live in **Sym**, their dual basis in $Qsym$ should be analogs of the $P$ or $Q$ functions
- $P$ or $Q$ functions can be defined through an action of $H_N(q)$ on polynomials
- This action comes from the Demazure operators $\pi_i$
- The $\pi_i$ are defined in terms of the usual action of $\mathfrak{S}_N$

Action of $H_N(q)$ on polynomials such that the full $q$-symmetrizer $S^{(N)}$ becomes a *quasi*-symmetrizer ?

- If yes, try $S^{(N)}(x^I)$ (up to normalization) as a quasi-symmetric analog of the Hall basis
- The dual basis in **Sym** should be an analog of the $Q'$
- A test: $I = (1^n)$ should be $(q)_n \Lambda_n(A/(1-q))$

# Hivert's Hall-Littlewood functions

- Quasi-symmetric functions are in fact symmetric ... but for an exotic (quasi-symmetrizing) action of the symmetric group
- $X = \{x_1, \ldots, x_N\}$
- Support of a monomial $\mathbf{m} = x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N}$

$$A = \{i | m_i \neq 0\} \subseteq X$$

- $I$ composition obtained by removing the 0's from $(m_1, \ldots, m_N)$
- Notation: $\mathbf{m} = A^I$
- $\sigma \in \mathfrak{S}_N$ acts on $\mathbf{m}$ by

$$\sigma'(\mathbf{m}) = (\sigma(A))^I$$

- Examples: $\sigma'_1(x_1 x_2^3) = x_1 x_2^3$ and $\sigma'_2(x_1 x_2^3) = x_1 x_3^3$
- The quasi-symmetric polynomials are the invariants of this action
- quasi-monomial functions: orbit sums of $x^I = \{x_1, \ldots, x_r\}^{(i_1, \ldots, i_r)}$
- Jacobi-Weyl-Demazure formula for quasi-ribbons

$$F_I = \pi'_\omega(x^I)$$

where $\pi'_i = (x_i - x_{i+1})^{-1}(x_i - x_{i+1}\sigma'_i)$

- Defines $F_I$ when $I$ is a pseudo-composition: some of the $i_k$ may be 0
- Example: $F_{04003} = -F_{31111} - F_{22111} - F_{13111}$
- Hecke algebra: $\bar{\pi}_i = \pi'_i - 1$, $T_i = (1-q)\bar{\pi}_i + q\sigma'_i$
- Quasi-symmetric analogs of the $P$ functions

$$G_I(X;q) = \frac{1}{[r]_q![N-r]_q!} S^{(N)}(X^I) \quad (r = l(I))$$

- Explicit formula $G_I = \sum_{J \succeq I} (-1)^{l(I)-l(J)} q^{s(I,J)} F_J$
- Example: $G_{1121} = F_{1121} - q^3 F_{11111}$
- Dual basis $H_I \in \mathbf{Sym}$: analogs of $Q'$
- Explicit formula $H_K = \sum_{J \preceq K} q^{t(K,J)} R_J$
- $H_{(1^n)} = \sum_J q^{\mathrm{maj}\,(J^\sim)} R_J$ as required
- Example: $H_{321} = R_{321} + qR_{33} + q^2 R_{51} + q^3 R_6$

- Explicit formula for multiplication

$$H_I H_J = \sum_{J \geq K} q^{t(J,K)} (c(r, s-p) H_{IK} + c(r, s-p+1) H_{I \triangleright K})$$

- In the basis $H'_I = (1-q)^{l(I)} H_I$, the structure constants are in $\mathbb{N}[q]$
- Factorization at roots of unity: $\zeta$ primitive $k$th root of 1, $J_1, \ldots, J_m$ composition of length $k$, $J_{m+1}$ of length $r < k$, $I = J_1 J_2 \cdots J_{m+1}$

$$H_I(A; \zeta) = H_{J_1}(A; \zeta) H_{J_2}(A; \zeta) \cdots H_{J_{m+1}}(A; \zeta)$$

- Analogues of Macdonald functions have been found recently (Hivert-Lascoux-T.)
- Different version by Bergeron-Zabrocki

# 0-QUANTUM STRUCTURES

# Generic Hecke algebras

$H_N(q)$ Hecke algebra of type $A_{N-1}$:

$\mathbb{C}$ or $\mathbb{C}(q)$ algebra generated by $T_1, \ldots, T_{N-1}$

$$T_i T_j = T_j T_i \qquad |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad i = 1, \ldots, N-2$$

$$T_i^2 = (q-1) T_i + q \qquad i = 1, \ldots, N-1$$

- $H_N(q)$ is isomorphic to $\mathbb{C}[\mathfrak{S}_N]$ for generic $q \in \mathbb{C}$, but not for $q$ a root of unity of order $\leq N$ or for $q = 0$.
- $q$-Specht modules $V_\lambda(q)$, $\lambda \vdash N$
- Matrix elements in $\mathbb{Z}[q]$
- Characteristic map $R(H_N(q)) \longrightarrow Sym_N$

  $$\mathrm{ch}\left([V_\lambda(q)]\right) = s_\lambda$$

  (independent of $q$).

# Representation theory of $H_N(0)$

(P.N. Norton 1979, R.W. Carter 1985)

- Irreducible representations $\mathbf{D}_I$ are 1-dimensional. Parametrized by subsets $E \subseteq \{1, \ldots, N-1\}$ (or compositions $I = C(E)$)

$$\rho_I(T_j) = \begin{cases} -1 & \text{if } j \in E \\ 0 & \text{otherwise} \end{cases}$$

- Indecomposable projective modules $\mathbf{N}_I = H_N(0)\nu_I$

$$H_N(0) = \bigoplus_{|I|=N} \mathbf{N}_I$$

with $\dim \mathbf{N}_I = |\{\sigma | C(\sigma) = I\}$

- decomposition matrix

$$d_{\lambda I} = [V_\lambda(0) : \mathbf{D}_I] = |\{T \in \mathrm{STab}\,(\lambda) | C(T) = I\}|$$

# Grothendieck rings

$\mathcal{A}$ finite dimensional $\mathbb{C}$-algebra.

- $G_0(\mathcal{A}) = \mathbb{Z}\{\text{finitely generated left } \mathcal{A}\text{-modules}\}/\sim$ where $[Q] \sim [P] + [R] \Leftrightarrow 0 \to P \to Q \to R \to 0$
- $K_0(\mathcal{A}) =$ abelian group generated by isomorphism classes of f.g. projective left-$\mathcal{A}$-modules
- Duality between $G_0(\mathcal{A})$ and $K_0(\mathcal{A})$: $\langle N, P\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(P, N)$

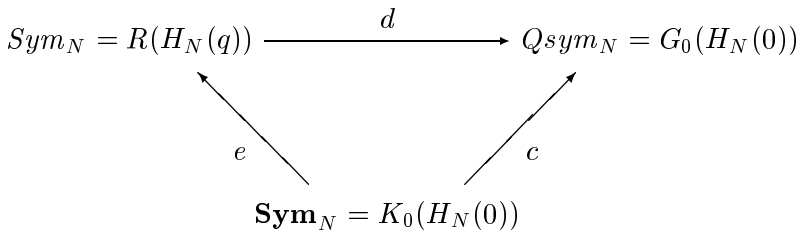
Let

$$\mathcal{G} = \bigoplus_{n\geq 0} G_0(H_n(0)) \qquad \mathcal{K} = \bigoplus_{n\geq 0} K_0(H_n(0))$$

Define ring structures on $\mathcal{G}$ and $\mathcal{K}$ by

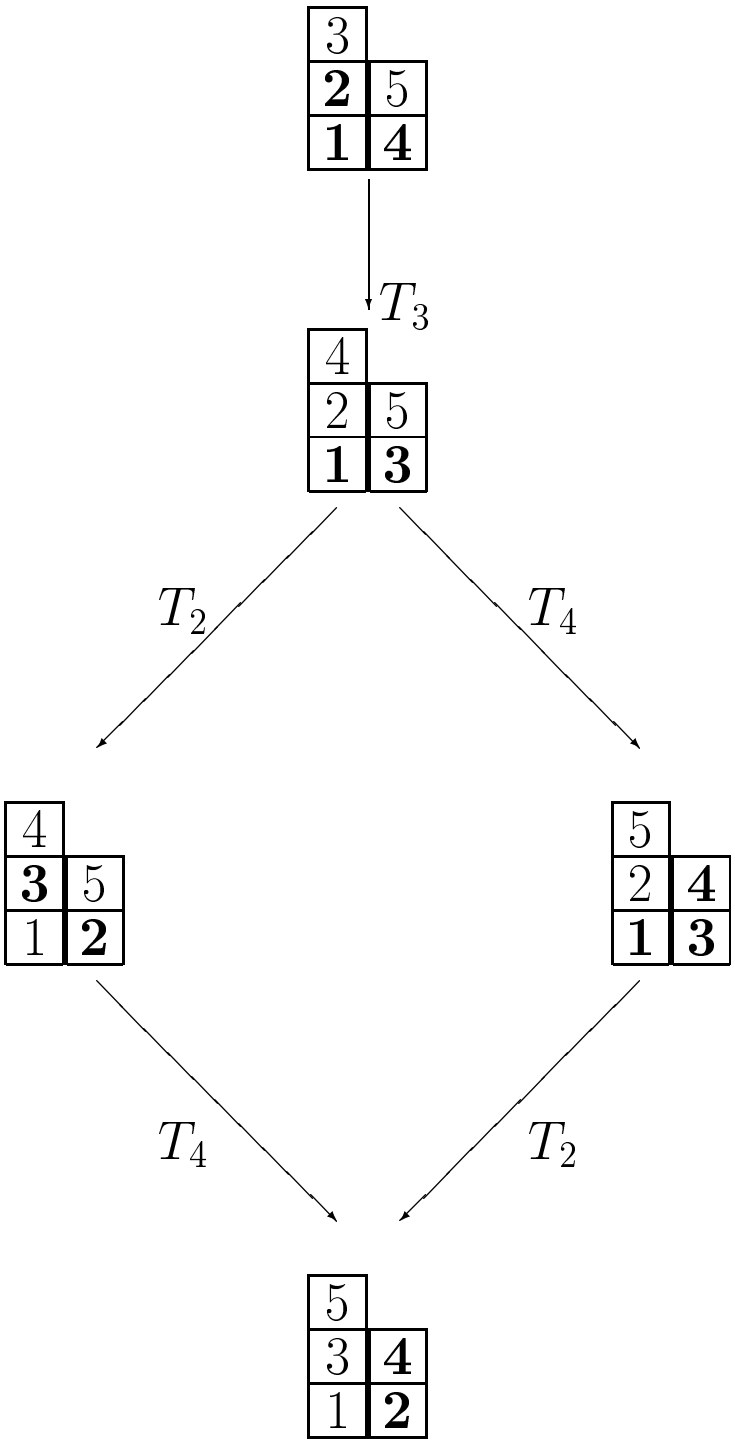
$$[M]\cdot[N] = [M \otimes_{\mathbb{C}} N] \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}$$

- The map $[\mathbf{D}_I] \mapsto F_I$ is a ring isomorphism $\mathcal{G} \to Qsym$
- The map $[\mathbf{N}_I] \mapsto R_I$ is a ring isomorphism $\mathcal{K} \to \mathbf{Sym}$
- Decomposition map $d : Sym \hookrightarrow Qsym$
- Cartan map $c : \mathbf{Sym} \longrightarrow Qsym$ given by the commutative image
- Character formula $\chi_M(T_{w_I}) = \langle [M], (q-1)^{-\ell(I)} S^I((q-1)A) \rangle|_{q=0}$
- Hopf algebra duality $\mathbf{Sym} \leftrightarrow Qsym$ special case of $G_0(\mathcal{A}) \leftrightarrow K_0(\mathcal{A})$




The Cartan-Brauer triangle of $H_N(0)$

*Remark.* For $q = \omega$ a primitive $n$th root of unity, $\mathcal{G}$ is a quotient of $Sym$ and $\mathcal{K}$ a subring. In this case the decomposition map is surjective and the Cartan map injective.




The Specht module $V_{221}(0)$

Composition factors of $V_{221}(0)$ given by

$$s_{221} = F_{1121} + F_{122} + F_{212} + F_{1211} + F_{221}$$

More precisely

$$s_{221} \sim F_{1121} + qF_{122} + q^2(F_{212} + F_{1211}) + q^3 F_{221}$$

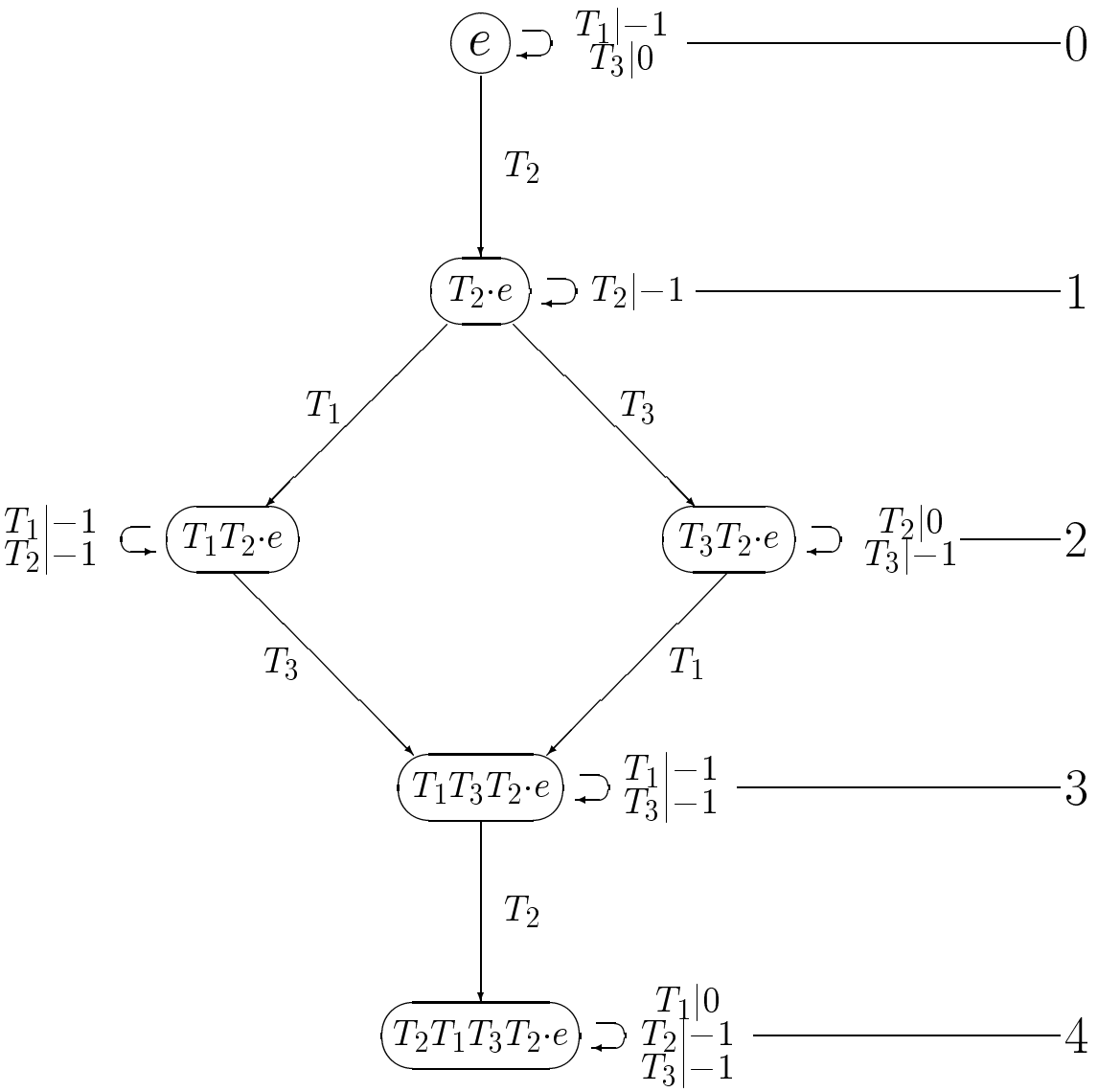
taking into account the filtration of $H_N(0)$

$$H_n^{(k)} = \bigoplus_{\ell(w)\geq k} \mathbb{C}T_w$$

| $\lambda \backslash I$ | 4 | 31 | 22 | 211 | 13 | 121 | 112 | 1111 |
|---|---|---|---|---|---|---|---|---|
| 4 | 1 | | | | | | | |
| 31 | | 1 | $q$ | | | $q^2$ | | |
| 22 | | | $q$ | | | 1 | | |
| 211 | | | | $q^2$ | | $q$ | 1 | |
| 1111 | | | | | | | | 1 |

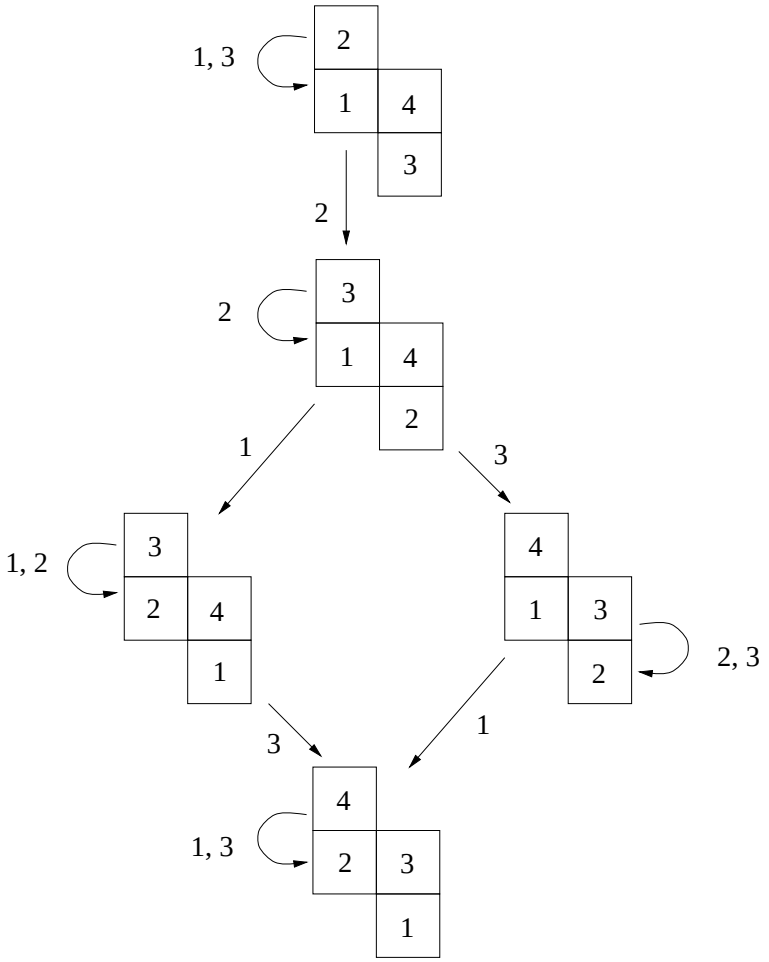
$q$-Decomposition matrix of $H_4(0)$

## Induced module $M_{(11),(2)} = \mathbf{D}_{11} \otimes \mathbf{D}_2 \uparrow^{H_4(0)}$




$$F_{11} \times_q F_2 = F_{13} + q\,F_{22} + q^2\,(F_{112} + F_{31}) + q^3\,F_{121} + q^4\,F_{211}$$

$$21 \odot_q 34 = 2134 + q\,2314 + q^2\,2341 + q^2\,3214 + q^3\,3241 + q^4\,3421$$




The indecomposable projective $H_4(0)$-module $\mathbf{N}_{121}$

$$\hat{R}_{121} = q^2(F_{121} + qF_{22} + q^2F_{112} + q^2F_{211} + q^3F_{121})$$

$q$-Cartan invariants

$$c_{IJ}(q) = \sum_{C(\sigma)=I, C(\sigma^{-1})=J} q^{\ell(\sigma)}$$

# Quantum linear groups at $q = 0$

- Dipper-Donkin version of the quantized function algebra $A_q(n)$

$$\begin{aligned} x_{jk}\,x_{il} &= q\,x_{il}\,x_{jk} && \text{for } i<j,\ k\le l \\ x_{ik}\,x_{il} &= x_{il}\,x_{ik} && \text{for all } i,k,l \\ x_{jl}\,x_{ik} - x_{ik}\,x_{jl} &= (q-1)\,x_{il}\,x_{jk} && \text{for } i<j,\ k<l \end{aligned}$$

- Hopf algebra with coproduct

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}\ .$$

- Vector corepresentation:

$$V = \bigoplus_{i=1}^{n} \mathbb{C}(q)\,e_i$$

- Coaction: $\delta(e_i) = \sum_{j=1}^{n} x_{ij} \otimes e_j$

- Action of $H_N(q)$ on $\mathbf{v} = v_{k_1} \otimes \cdots \otimes v_{k_N} i \in V^{\otimes n}$

$$\begin{cases} \mathbf{v}\cdot T_i = \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \\ \mathbf{v}\cdot T_i = q\mathbf{v} & \text{if } k_i = k_{i+1} \\ \mathbf{v}\cdot T_i = q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \end{cases}$$

- Commutation: for $h \in H_N(q)$, $(1 \otimes h) \circ \delta^{\otimes N} = \delta^{\otimes N} \circ h$

# Characters

- $M$ finite dimensional $A_q(n)$-comodule
- $(m_i)_{i=1,m}$ basis of $M$.
- coaction: $\delta(m_i) = \sum_{j=1}^{m} a(i,j) \otimes m_j$, $a(i,j) \in A_q(n)$
- Character of $M$

$$\chi(M) = \sum_{i=1}^{m} a(i,i) \in A_q(M)$$

- independent of the basis $(m_i)$.
- $0 \to M' \to M \to M'' \to 0 \Rightarrow \chi(M) = \chi(M') + \chi(M'')$
- $\chi(M \otimes N) = \chi(M)\,\chi(N)$
- $M \simeq N \Rightarrow \chi(M) = \chi(N)$

# Diagonal algebra (or quantum torus)

- $\Delta_q(n) = \langle x_{ii},\, i = 1, \ldots, n \rangle \subset A_q(n)$
- $\chi(M) \in \Delta_q(n)$ (always true, even for $q = 0$)
- Follows from the identity

$$\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}_q \stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\, x_{1\sigma(1)} \ldots x_{n\sigma(n)}$$

$$= \frac{1}{(1-q)^{n-1}} \left[x_{nn}, [\ldots, [x_{22}, x_{11}]_q \ldots]_q\right]_q$$

$$[a, b]_q = a\, b - q\, b\, a$$

# The hypoplactic algebra

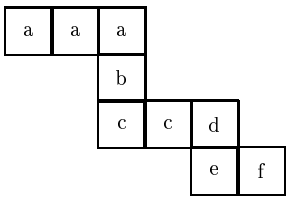
- For $q = 0$, $\Delta_0(n)$ is a quotient of the plactic algebra: $a, b, c, d \in A = \{x_{11} < x_{22} < \ldots < x_{nn}\}$ satisfy
- the plactic relations

$$\begin{cases} aba = baa \ , \ bba = bab & \text{for } a < b \\ acb = cab \ , \ bca = bac & \text{for } a < b < c \end{cases}$$

- and the quartic Hypoplactic relations

$$\begin{cases} baba = abab \ , \ baca = abac & \text{for } a < b < c \\ cacb = acbc \ , \ cbab = bacb & \text{for } a < b < c \\ badc = dbca \ , \ acbd = cdab & \text{for } a < b < c < d \end{cases}$$

- Irreducible $A_0(n)$ comodules $\mathbf{D}_I = V^{\otimes N} \cdot \eta_I$ where $H_N(0)\eta_I$ are minimal left ideals
- Quasi-ribbon words

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- Characters are hypoplactic quasi-symmetric functions

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} w \otimes w^* = F_I(x_{11}, \ldots, x_{nn})$$

(analogues of the plactic Schur functions of Lascoux-Schützenberger)

- The hypoplactic $F_I$ span a commutative subalgebra
- Indecomposable summands of $V^{\otimes N}$ $\mathbf{N}_I = V^{\otimes N}\nu_I$, where $H_N(0)\nu_I$ are the indecomposable projective modules
- $\chi(\mathbf{N}_I) = R_I(x_{11}, \ldots, x_{nn})$

Example

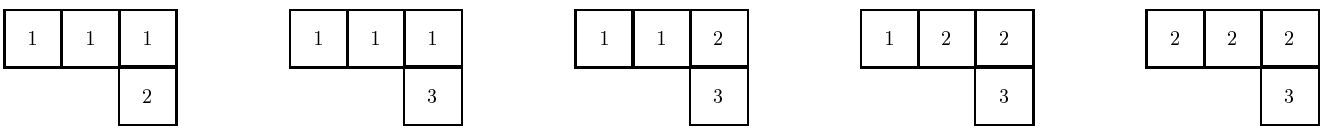
$$\eta_{31} = T_3\, T_2\, T_1\, (1 + T_2)\, (1 + T_3)\, (1 + T_2)$$

$$\mathbf{D}_{31} = V^{\otimes 4} \cdot \eta_{31}.$$

$$\mathbf{D}_{31} = \mathbb{C}\, a_1 a_2 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_2 a_2 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_1 a_1 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C}\, a_1 a_1 a_3 a_1 \cdot \eta_{31} \oplus \mathbb{C}\, a_1 a_1 a_2 a_1 \cdot \eta_{31}\ .$$

$$\chi(\mathbf{D}_{31}) = x_{22}x_{11}x_{11}x_{11} + +x_{33}x_{11}x_{11}x_{11} + x_{33}x_{22}x_{22}x_{22} + x_{33}x_{11}x_{11}x_{22} + x_{33}x_{11}x_{22}x_{22}$$

$$= x_{11}x_{11}x_{22}x_{11} + +x_{11}x_{11}x_{33}x_{11} + x_{22}x_{22}x_{33}x_{22} + x_{11}x_{11}x_{33}x_{22} + x_{11}x_{22}x_{33}x_{22}$$




$$\chi(\mathbf{D}_{31}) = F_{31}(x_{11}, x_{22}, x_{33})$$

Indecomposable summands of $V^{\otimes N}$

$$\mathbf{N}_I = V^{\otimes N}\nu_I$$

$H_N(0)\nu_I$ indecomposable projective modules

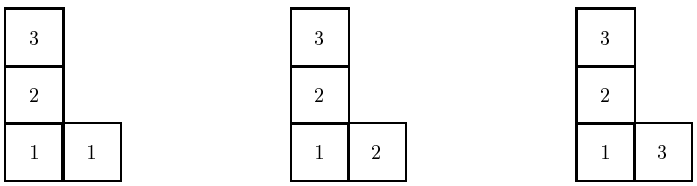
$$\chi(\mathbf{N}_I) = R_I(x_{11}, \ldots, x_{nn})$$

Example

$$\nu_{112} = T_1\, T_2\, T_1\, (1 + T_3)$$
$$\mathbf{N}_{112} = V^{\otimes 4} \cdot \nu_{112}.$$

$$\mathbf{N}_{112} = \mathbb{C}a_3a_2a_1a_1 \cdot \nu_{112} \oplus \mathbb{C}a_3a_2a_1a_2 \cdot \nu_{112} \oplus \mathbb{C}a_3a_2a_1a_3 \cdot \nu_{112}$$

$$\chi(\mathbf{N}_{112}) = x_{33}\, x_{22}\, x_{11}\, x_{11} + x_{33}\, x_{22}\, x_{11}\, x_{22} + x_{33}\, x_{22}\, x_{11}\, x_{33}\ .$$




$$\chi(\mathbf{N}_{112}) = R_{112}(x_{11}, x_{22}, x_{33})$$

# The 0-enveloping algebra $\mathcal{U}_0(gl_n)$

Takeuchis's 2-parameter algebra $U_{r,s}(gl_n)$ for $(r,s) = (q,1)$

$$k_i\, k_j \;=\; k_j\, k_i \qquad \text{for } 1 \leq i,j \leq n$$

$$\begin{cases} q\, k_i\, e_{i-1} \;=\; e_{i-1}\, k_i & \text{for } 2 \leq i \leq n-1 \\ k_i\, e_i \;=\; q\, e_i\, k_i & \text{for } 1 \leq i \leq n-1 \\ k_i\, e_j \;=\; e_j\, k_i & \text{for } j \neq i-1, i \end{cases}$$

$$\begin{cases} k_i\, f_{i-1} \;=\; q\, f_{i-1}\, k_i & \text{for } 2 \leq i \leq n-1 \\ q\, k_i\, f_i \;=\; f_i\, k_i & \text{for } 1 \leq i \leq n-1 \\ k_i\, f_j \;=\; f_j\, k_i & \text{for } j \neq i-1, i \end{cases}$$

$$[e_i, f_j] \;=\; \delta_{ij}\, \frac{k_i - k_{i+1}}{q-1}$$

$$\begin{cases} q\, e_{i+1}\, e_i^2 - (1+q)\, e_i\, e_{i+1}\, e_i + e_i^2\, e_{i+1} \;=\; 0 \\ q\, e_{i+1}^2\, e_i - (1+q)\, e_{i+1}\, e_i\, e_{i+1} + e_i\, e_{i+1}^2 \;=\; 0 \end{cases}$$

$$\begin{cases} f_{i+1}\, f_i^2 - (1+q)\, f_i\, f_{i+1}\, f_i + q\, f_i^2\, f_{i+1} \;=\; 0 \\ f_{i+1}^2\, f_i - (1+q)\, f_{i+1}\, f_i\, f_{i+1} + q\, f_i\, f_{i+1}^2 \;=\; 0 \end{cases}$$

$$\begin{cases} [e_i, e_j] \;=\; 0 & \text{for } |i-j| > 1 \\ [f_i, f_j] \;=\; 0 & \text{for } |i-j| > 1 \end{cases}$$

$$\begin{cases} e_i\, e_{i+1}\, k_{i+1} \;=\; k_{i+1}\, e_i\, e_{i+1} & \text{for } 1 \leq i \leq n-2 \\ f_{i+1}\, f_i\, k_{i+1} \;=\; k_{i+1}\, f_{i+1}\, f_i & \text{for } 1 \leq i \leq n-2 \end{cases}$$

- Bialgebra structure

$$\begin{cases} \Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i, & \varepsilon(e_i) = 0 \\ \Delta(f_i) = k_{i+1} \otimes f_i + f_i \otimes 1, & \varepsilon(f_i) = 0 \\ \Delta(k_i) = k_i \otimes k_i, & \varepsilon(k_i) = 1 \end{cases}$$

- Vector representation

$$\begin{cases} \rho_V(e_i) = E_{i,i+1} \\ \rho_V(f_i) = E_{i+1,i} \\ \rho_V(k_i) = q\,E_{i,i} + \sum_{j \neq i} E_{j,j} \end{cases}$$

- Duality with $A_q(n)$

$$\rho_V(g) = (a_{ij}(g))_{1 \leq i,j \leq n}$$

satisfy the defining relations of $A_q(n) \Rightarrow$ homomorphism $\Psi(x_{ij}) = a_{ij}$

$$\langle x, g \rangle = \Psi(x)(g)$$

$x \in A_q(N)$ and $g \in \mathcal{U}_q(gl_N)$.

# Characters

$M$ polynomial $\mathcal{U}_q(gl_n)$ module, $(m_i)$ basis of $M$

$$g \cdot m_i = \sum_{j=1}^{m} \mu_{ij}(g)\, m_j \ .$$

$$c(M) = \sum_{i=1}^{m} \mu_{ii} \in \mathcal{U}_q(gl_N)^*$$

independent of basis

$$g \cdot \xi^I = \sum_{J \in [1,N]^n} a_{JI}(g)\, \xi^J$$

$$c(M) = \sum_{I,J \in [1,N]^n} p_{IJ}(q)\, a_{IJ} \ .$$

character:

$$\chi(M) = \sum_{I,J \in [1,N]^n} p_{IJ}(q)\, x_{IJ} \quad \in \Delta_q(n)$$

# Polynomial $\mathcal{U}_0(gl_n)$ modules

- Same constructions as for $A_0(n)$
- Irreducible polynomial modules:

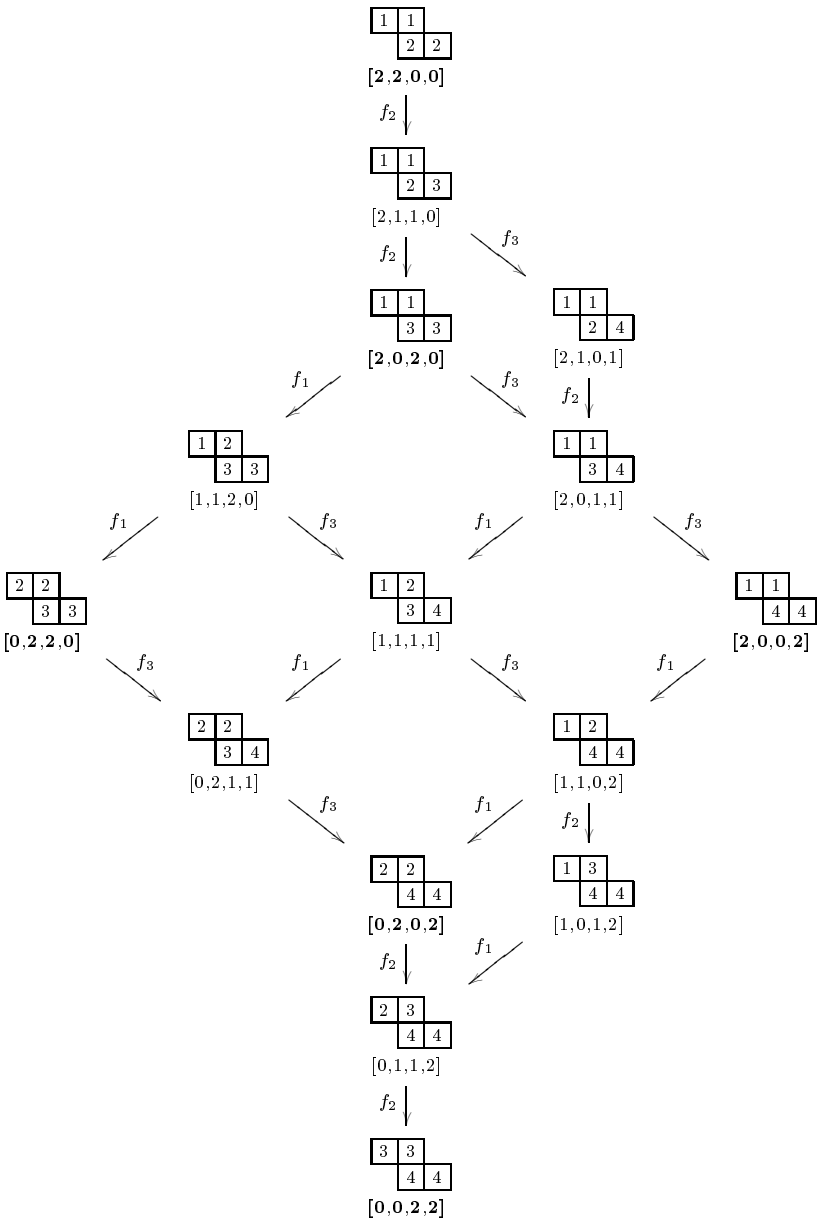
$$\mathbf{D}_I = V^{\otimes N} \cdot \eta_I$$

- character $F_I(x_{11}, \ldots, x_{nn})$
- Indecomposable direct summands of $V^{\otimes N}$:

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I$$

  character $R_I(x_{11}, \ldots, x_{nn})$
- Irreducible representations: quasi-crystal graphs

# Example: $D_{22}$ for $n = 4$



# Classical characters

- Homomorphism $\Delta_0(n) \to \mathbb{C}[x_1, \ldots, x_n]$:

$$x_{ii} \longrightarrow x_i \quad \text{commutative}$$

- Weyl and Demazure character formulas: Hivert's quasi-symmetrizations
- Weyl character formula:

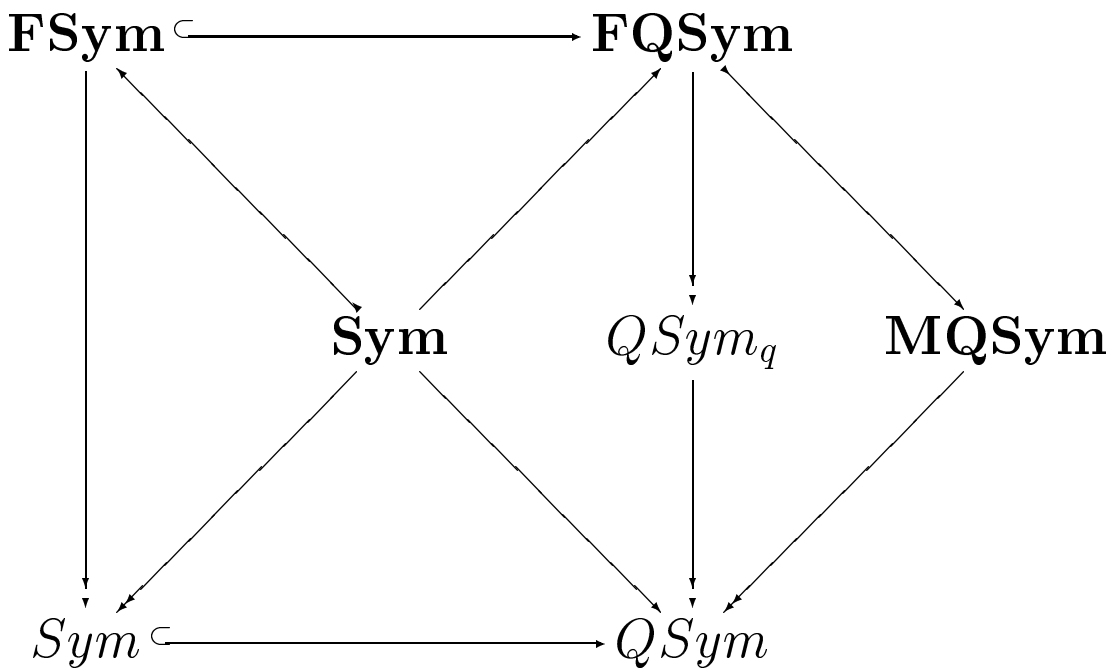
$$\mathrm{ch}\,(\mathbf{D}_I) = F_I = \pi'_\omega(x^I)$$

- Characters of Demazure modules: $M_{\sigma,I} = \mathcal{U}_0(\mathfrak{b}^+)\xi$, where $\xi$ is an extremal vector, i.e. a weight vector of extremal weight $\sigma'(I)$

$$\mathrm{ch}\,(M_{\sigma,I}) = \pi'_\sigma(x^I)$$

# MORE ALGEBRAS

# Overview




- $Sym$: Partitions $\lambda$
- **Sym** and $QSym$: Compositions $I$
- $QSym_q$: Compositions $I$ ($\mathbb{C}(q)$ basis)
- **FSym**: Standard Young tableaux $\mathbf{t}$
- **FQSym**: Permutations $\sigma$
- **MQSym**: Packed integer matrices $M$ (double cosets $\mathfrak{S}_I \backslash \mathfrak{S}_n / \mathfrak{S}_J$)

# Free quasi-symmetric functions

- Free quasi-ribbons:

$$\mathbb{F}_\sigma = \sum_{\mathrm{std}\,(w)=\sigma^{-1}} w \quad \in \mathbb{Z}\langle A\rangle$$

- Commutative image: $\mathbb{F}_\sigma \mapsto F_I$, where $I = C(\sigma)$
- $q$-commutative image ($x_j x_i = q x_i x_j$ for $i < j$): $q^{l(\sigma)} F_I \in QSym_q$
- Multiplication rule: $\alpha \in \mathfrak{S}_k$, $\beta \in \mathfrak{S}_l$,

$$\mathbb{F}_\alpha \mathbb{F}_\beta = \sum_{\sigma \in \alpha \,\text{ш}\, \beta[k]} \mathbb{F}_\sigma$$

- Scalar product

$$\langle \mathbb{F}_\sigma \,,\, \mathbb{F}_\tau \rangle = \delta_{\sigma^{-1},\tau}$$

- Dual basis $\mathbb{G}_\sigma = \mathbb{F}_{\sigma^{-1}}$
- **FQSym** is isomorphic to the convolution algebra of Reutenauer:

$$\mathbb{G}_\alpha \mathbb{G}_\beta = \mathbb{G}_{\alpha \star \beta}$$

# Examples

- Free symmetric functions: $\mathbb{S}_{\mathbf{t}} = \sum_{P(\sigma)=t} \mathbb{F}_\sigma$
- Proof of the Littlewood-Richardson rule reduces to

$$\mathbb{S}_{\mathbf{t}'}\mathbb{S}_{\mathbf{t}''} = \sum_{\substack{P(\sigma')=\mathbf{t}' \\ P(\sigma'')=\mathbf{t}''}} \mathbb{F}_{\sigma'}\mathbb{F}_{\sigma''} = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} \sum_{P(w)=\mathbf{t}} \mathbb{F}_\sigma = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} \mathbb{S}_{\mathbf{t}}$$

- Free quasisymmetric function $\mathbf{\Gamma}(P) = \sum_{\sigma\in L(P)} \mathbb{F}_\sigma$ associated to a poset
- Operation $P_1 \wedge P_2$ on posets $\to$ algebra of planar binary trees [Loday-Ronco]
- Stanley's $QS(x)$-distribution on symmetric groups: $\mathrm{Prob}(\sigma) = \mathbb{G}_\sigma(x)$
- **FQSym** is a kind of Grothendieck (induction) ring of a category $\mathcal{N}$ of 0-Hecke modules, containing simple, projective and skew Specht modules
- If $M = N_\sigma \hat{\otimes} N_\tau$, one has

$$0 \to N_\beta \to M \to N_\alpha \to 0$$

- In particular: indecomposable summands of induction products of simple modules

# Free symmetric functions

$\mathbf{t}$ standard tableau of shape $\lambda$

$$\mathbb{S}_{\mathbf{t}} = \sum_{P(\sigma)=\mathbf{t}} \mathbb{F}_\sigma = \sum_{Q(w)=\mathbf{t}} w\,,$$

$w \mapsto (P(w), Q(w))$ usual Robinson-Schensted map
LRS rule:

$$\mathbb{S}_{\mathbf{t}'}\mathbb{S}_{\mathbf{t}''} = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} \mathbb{S}_{\mathbf{t}}$$

$\mathrm{Sh}\,(\mathbf{t}', \mathbf{t}'')$ set of standard tableaux in the shuffle of $\mathbf{t}'$ with the plactic class of $\mathbf{t}''[k]$.
Plactic congruence compatible with restriction to intervals:
$w \in A^*$, $I = [a_i, a_j]$ interval of $A$
$w|_I$ word obtained by erasing in $w$ the letters not in $I$
plactic relations $xzy \equiv zxy$ $(z \leq y < z$ and
$yxz \equiv yzx$ $(x < y \leq z)$ reduce to equalities after erasing $x$ or $z$, $w \equiv w' \Rightarrow w|_I \equiv w'|_I$

$$\mathbb{S}_{\mathbf{t}'}\mathbb{S}_{\mathbf{t}''} = \sum_{P(\sigma')=\mathbf{t},P(\sigma'')=\mathbf{t}''} \mathbb{F}_{\sigma'}\mathbb{F}_{\sigma''} = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} \sum_{P(w)=\mathbf{t}} \mathbb{F}_\sigma$$

since $\{\sigma' \,\text{ш}\, \sigma''[k] \,|P(\sigma') = t, P(\sigma'') = t''\}$ is a union of plactic classes
Each class contains a unique tableau

The restriction of a tableau to an initial segment has to be a tableau

Such a tableau can appear only in the shuffles $\mathbf{t}' \sqcup\!\sqcup \sigma''[k]$.

Example:

$$\mathbf{t}' = \mathbf{t}'' = \begin{array}{|c|c|}\hline 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array},$$

$\mathbb{S}_{\mathbf{t}'}\mathbb{S}_{\mathbf{t}''}$ is equal to $\sum_{\mathbf{t}} \mathbb{S}_{\mathbf{t}}$ where $\mathbf{t}$ ranges over:

$$\begin{array}{|c|c|c|c|}\hline 3 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & 4 & 6 \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 6 & \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{3}{c}{} \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 4 & \multicolumn{2}{c}{} \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 6 & \multicolumn{2}{c}{} \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 4 & 6 \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 6 & \multicolumn{2}{c}{} \\ \hline 4 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 6 & \multicolumn{1}{c}{} \\ \hline 4 & \multicolumn{1}{c}{} \\ \hline 3 & 5 \\ \hline 1 & 2 \\ \hline \end{array}$$

$\mathbf{Sym}(A)$ is a subalgebra of $\mathbf{FSym}$, since

$$R_I(A) = \sum_{\mathrm{Rec}\,(\mathbf{t})=\mathrm{Des}\,(I)} \mathbb{S}_{\mathbf{t}}\,,$$

$\mathrm{Rec}\,(\mathbf{t})$ recoil (or descent) set of $\mathbf{t}$.

# The $QS$-distribution on symmetric groups

A probability distribution on symmetric groups (R.P. Stanley, Generalized riffle shuffles and quasi-symmetric functions)
$x = (x_i)_{i \geq 1}$ probability distribution on $A = \{a_1, a_2, \ldots\}$
$\rightarrow$ probability distribution $QS(x)$ on each $\mathfrak{S}_n$

$$\mathrm{Prob}\,(\sigma) = \mathbb{G}_\sigma(x)\ .$$

$\mathbb{G}_1^n = \sum_{\sigma \in \mathfrak{S}_n} \mathbb{G}_\sigma$.
Stanley's Theorem 2.1: $\mathrm{Prob}\,(\sigma) = F_{C(\sigma^{-1})}(x)$
follows from $\mathbb{G}_\sigma = \mathbb{F}_{\sigma^{-1}}$ and $\mathbb{F}_\sigma(x) = F_{C(\sigma)}(x)$.
Next, Stanley introduces the operator

$$\Gamma_n(x) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{Prob}\,(\sigma)\sigma \ \in\ \mathbb{C}\mathfrak{S}_n\ .$$

$\Gamma_n(x) \in \Sigma_n \leftrightarrow S_n(xA)$
Eigenvalues of $\Gamma_n(x)$ are $p_\lambda(x)$, with multiplicities $n!/z_\lambda$ [NCSF2]
$\Gamma_n(x)\Gamma_n(y)$ given by $S_n(xA) * S_n(yA) = S_n(xyA)$
Another result of Stanley: the probability $M_n(k)$ that a QS-random permutation has $k$ inversions is equal to the probability that it has major index $k$ (Theorem 3.2).
This is equivalent to the identity

$$\sum_{\sigma \in \mathfrak{S}_n} q^{l(\sigma)} \mathbb{G}_\sigma(x) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}\,(\sigma)} \mathbb{G}_\sigma(x)$$

The right-hand side is equal to

$$\sum_{|I|=n} q^{\mathrm{maj}\,(I)} r_I(x) = \sum_{I,J} q^{\mathrm{maj}\,(I)} \langle r_I, r_J \rangle F_J(x)$$

$$= \sum_J \left( \sum_{C(\sigma^{-1})=J} q^{\mathrm{maj}\,(\sigma)} \right) F_J(x)$$

$$= \sum_J \left( \sum_{C(\sigma^{-1})=J} q^{l(\sigma)} \right) F_J(x) \sum_{\sigma \in \mathfrak{S}_n} q^{l(\sigma)} \mathbb{G}_\sigma(x)$$

according to Foata-Schützenberger formula

The specialization $\mathbb{S}_t(x)$ is the probability that a $QS$-random permutation has $t$ as insertion tableau, and $R_I(x)$ is the probability that a random permutation has shape $I$ (Stanley's Theorems 3.4 and 3.6).

# Posets, $P$-partitions, and the like

Partial order $P$ on $[n] = \{1, 2, \ldots, n\}$
$P$-partition (Stanley) $f : [n] \to X$ for some totally ordered set $X$, such that

$$i <_P j \Rightarrow \ f(i) \leq f(j)$$

$$i <_P j \text{ and } i > j \ \Rightarrow f(i) < f(j)\,.$$

Gessel associates to $P$

$$\Gamma(P) = \sum_{f \in \mathcal{A}(P)} f(1) f(2) \cdots f(n)$$

($\mathcal{A}(P)$ set of all $P$-partitions), a quasi-symmetric function:

$$\Gamma(P) = \sum_{\sigma \in L(P)} \Gamma(\sigma) = \sum_{\sigma \in L(P)} F_{C(\sigma)}$$

$L(P)$ set of linear extensions of $P$ = permutations $\sigma \in \mathfrak{S}_n$ such that $i <_P j \Rightarrow \sigma^{-1}(i) < \sigma^{-1}(j)$
$P$-partition $f \leftrightarrow$ word $w_f = a_{f(1)} a_{f(2)} \cdots a_{f(n)}$
Free quasi-symmetric generating function

$$\mathbf{\Gamma}(P) = \sum_{\sigma \in L(P)} \mathbb{F}_\sigma$$

Satisfies

$$\mathbf{\Gamma}(P_1 \sqcup P_2) = \mathbf{\Gamma}(P_1) \mathbf{\Gamma}(P_2)$$

# Planar binary trees

$P = P_1 \wedge P_2$ poset obtained by adjoining a maximal element to the juxtapostion of $P_1$ and $P_2$
Maximal element $h = k + 1$ if $P_1$ is a poset on $[k]$
Linear extensions of $P$

$$L(P_1 \wedge P_2) = (L(P_1) \sqcup\!\sqcup L(P_2[h])) \cdot h\,.$$

Posets generated from $\bullet = [1]$ by $\wedge \leftrightarrow$ binary trees
The $\mathbb{F}(T) = \mathbf{\Gamma}(T)$ span a subalgebra of **FQSym**, the Hopf algebra of planar binary trees introduced by Loday and Ronco
$T = T_1 \wedge T_2$ and $T' = T'_1 \wedge T'_2$ two binary trees

$$L(T \sqcup T') = L(T) \sqcup\!\sqcup L(T'[n]) =$$

$$\sum_{\alpha \in L(T_1)} \sum_{\beta \in L(T_2)} \sum_{\alpha' \in L(T'_1[n])} \sum_{\beta' \in L(T'_2[n])} [(\alpha \sqcup\!\sqcup \beta[h])h] \sqcup\!\sqcup [(\alpha' \sqcup\!\sqcup \beta'[h'])h']$$

$$= \sum [(\alpha \sqcup\!\sqcup \beta[h]) \sqcup\!\sqcup (\alpha' \sqcup\!\sqcup \beta'[h'])h']\, h$$

$$+ \sum [((\alpha \sqcup\!\sqcup \beta[h])h) \sqcup\!\sqcup (\alpha' \sqcup\!\sqcup \beta'[h'])]\, h'$$

(using the formula $(ua) \sqcup\!\sqcup (vb) = (u \sqcup\!\sqcup vb)a + (ua \sqcup\!\sqcup v)b$, valid for $a, b \in A$ and $u, v \in A^*$). Therefore,

$$L(T \sqcup T') = L(T_1 \wedge (T_2 \sqcup T')) + L((T \sqcup T'_1) \wedge T_2)$$

which proves that $\mathbb{F}(T)\mathbb{F}(T')$ is a sum of elements $\mathbb{F}(T'')$ given by the above recursion.

$\mathbb{F}(T) = \sum_{\mathbf{T}(\sigma)=T} \mathbb{G}_\sigma$
where $\mathbf{T}(\sigma)$ is the underlying binary tree of $\sigma$
defined as follows:
if $n = 0$ ($\sigma$ is the empty word), $\mathbf{T}(\sigma) = \bullet$
otherwise,
write $\sigma = unv$, $\alpha = \mathrm{std}\,(u)$, $\beta = \mathrm{std}\,(v)$
Then, $\mathbf{T}(\sigma) = \mathbf{T}(\alpha) \wedge \mathbf{T}(\beta)$, where $T_1 \wedge T_2$ is the binary tree having $T_1$ as left subtree and $T_2$ as right subtree.
The Loday-Ronco algebra can now be constructed by means of a commutation monoïd (the *sylvester* monoïd). Construction analogous to the plactic and hypoplactic definitions of **FSym** and **Sym**.

# Posets as 0-Hecke modules

There is a striking similarity between
the behaviour of the quasi-symmetric generating functions of posets under concatenation
and the characteristic quasi-symetric functions of 0-Hecke modules under induction product
Actually, the former is a special case of the latter, since the set of linear extensions $L(P)$ of a finite poset can be regarded as the basis of a (right) 0-Hecke module $M_P$ satisfying $\mathrm{ch}\,(M_P) = \Gamma(P)$
This is done as follows
For $\sigma \in L(P)$, one sets $\sigma T_i$ equal to $\sigma\sigma_i$ if $i \notin \mathrm{Des}\,(\sigma)$ and $\sigma\sigma_i \in L(P)$, 0 if $i \notin \mathrm{Des}\,(\sigma)$ but $\sigma\sigma_i \notin L(P)$, and $-\sigma$ is $i \in \mathrm{Des}\,(\sigma)$.

# Matrix quasi-symmetric functions

$X = \{x_1 < \cdots < x_n\}$ (commuting indeterminates)
ideal $\mathbb{C}[X]^+$ of polynomials without constant term
$\mathbb{C}\{X\} = T(\mathbb{C}[X]^+)$ its tensor algebra.
Natural basis of $\mathbb{C}\{X\}$: tensor products of monomials (*multiwords*)
$\leftrightarrow$ nonnegative integer matrices $M = (m_{ij})$
$m_{ij}$ is the exponent of $x_i$ in the $j$th factor
No zero column: *horizontally packed*
$A$ be the *support* of a multiword $\mathbf{m}$: the set of $x_i$ such that the $i$th row of $M$ is non zero
$P =$ matrix obtained form $M$ by removing the null rows
We set $\mathbf{m} = A^P$
$P$ without zero rows or columns: *packed*.
Example: $\mathbf{m} = a \otimes ab^3e^5 \otimes a^2d$

$$\begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix} \quad A = \{a, b, d, e\} \; P = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}.$$

**MQSym**$(X)$ subspace of $\mathbb{C}\{X\}$ spanned by

$$\mathbf{MS}_M = \sum_{A \in \mathcal{P}_k(X)} A^M$$

$\mathcal{P}_k(X)$ set of $k$-element subsets of $X$
$M$ runs over packed matrices of height $h(m) < n$
**MQSym** is a subalgebra of $\mathbb{C}\{X\}$. Actually,

$$\mathbf{MS}_P \mathbf{MS}_Q = \sum_{R \in \underline{\uplus}(P,Q)} \mathbf{MS}_R$$

$\underline{\uplus}(P,Q)$: $r$ an integer between $\max(p,q)$ and $p+q$, where $p = h(P)$ and $q = h(Q)$
Insert null rows in $P$ and $Q \to \tilde{P}$ and $\tilde{Q}$ of height $r$
$R = (\tilde{P}, \tilde{Q})$
$\underline{\uplus}(P,Q)$ = all matrices without null rows obtained in this way

$Y = \{y_1 < \cdots < y_n\}$ (same cardinality as $X$)
$\mathbf{MQSym}(X) \otimes \mathbf{MQSym}(X)$ identified with
$\mathbf{MQSym}(X \oplus Y)$
$X \oplus Y$ ordered sum

$$\Delta : \mathbf{MQSym}(X) \to \mathbf{MQSym}(X \oplus Y)$$

$\Delta(\mathbf{MS}_M(X)) = \mathbf{MS}_M(X \oplus Y)$
defines a comultiplication
We suppose now that $X$ is infinite
$\mu : f \otimes g \mapsto fg$ multiplication of $\mathbf{MQSym}$ (induced by the multiplication of the tensor algebra)i
$e$ restriction to $\mathbf{MQSym}$ of the augmentation of $T(\mathbb{C}[X]^+)$
Grading $\deg(\mathbf{MS}_M) = \sum m_{ij}$
$\mathbf{MQSym}_d$ homogeneous component of degree $d$
$(\mathbf{MQSym}, \mu, 1, \Delta, e)$ is a self dual graded bialgebra, the duality pairing being given by $\langle \mathbf{MS}_P, \mathbf{MS}_Q \rangle = \delta_{P,{}^tQ}$.

Hilbert series of **MQSym**

$$\dim(\mathbf{MQSym}_d) = \sum_{l>0,\ h>0} \binom{d+lh-1}{lh-1} \left(\frac{1}{2}\right)^{l+h+2}$$

$$= \sum_{|I|=|J|=d} 2^{2d-\ell(I)-\ell(J)} \langle r_I, r_J \rangle$$

which yields

$$\sum_{d\geq 0} \dim(\mathbf{MQSym}_d)\, t^d = 1 + t + 5t^2 + 33t^3 + 281t^4$$

$$+2961t^5 + 37277t^6 + 546193t^7 + 9132865t^8 + \cdots$$

Packed matrix $P \to$ two compositions $\mathrm{Row}(P)$ and $\mathrm{Col}(P)$ (row-sums and column-sums of $P$)
Linear map

$$\begin{aligned} \mathrm{Ev} \ : \ \mathbf{MQSym} &\longrightarrow QSym \\ \mathbf{MS}_P &\longmapsto \mathrm{Ev}(\mathbf{MS}_P) = M_{\mathrm{Row}(P)} \end{aligned}$$

Ev is an epimorphism of bialgebras.
Dually, the transposed map

$$\begin{aligned} {}^t\mathrm{Ev} \ : \ \mathbf{Sym} &\to \mathbf{MQSym} \\ \mathbf{MS}_P &\longmapsto {}^t\mathrm{Ev}(S^I) = \textstyle\sum_{\mathrm{Col}(P)=I} \mathbf{MS}_P \end{aligned}$$

is a monomorphism of bialgebras
**MQSym** admits $QSym$ as a quotient and **Sym** as a subalgebra
$\mathbf{MS}_P$ simultaneous generalization of the dual bases $M_I$ and $S^I$
$\mathbb{C}\langle X \rangle$ is naturally a subalgebra of $\mathbb{C}\{X\}$
words $\leftrightarrow$ multiwords with exponent matrix having only one 1 in each column
This embedding maps $\mathbf{FQSym}(X)$ to a subspace of **MQSym**.

$P$ packed matrix of height $h$
Composition $K = (k_1, \ldots, k_p)$ of $h \to$ matrix $P\rangle K$
$R_1, R_2, \ldots, R_h$ rows of $P$.
The first row of $P\rangle K$ is the sum of the first $k_1$ rows of $P$ the second row is the sum of the next $k_2$ rows of $P$, and so on
We end with a matrix of height $p$
Example

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 5 \\ 1 & 3 & 1 & 0 \end{bmatrix} \Big\rangle (3,2) = \begin{bmatrix} 2 & 5 & 2 & 3 \\ 1 & 6 & 2 & 5 \end{bmatrix}.$$

Then, we set

$$\phi\mathbf{S}_P = \sum_{|K|=h} \frac{1}{K!} \mathbf{MS}_{P\rangle K}$$

where $K! = k_1! k_2! \cdots k_p!$
$\{\phi\mathbf{S}_P\}$ ($P$ packed matrix) homogeneous basis of **MQSym**
A packed matrix $A$ is *connected* if it cannot be written in block diagonal form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where $B$ and $C$ are not necessarily square matrices.
**MQSym** is freely generated by the family $\{\phi\mathbf{S}_A\}$, where $A$ runs over the set of connected packed matrices.

Interpretation of **MQSym** in terms of invariant theory
$V$ $n$-dimensional vector space
$T(S^+(V))$ tensor algebra of the augmentation ideal of the symmetric algebra $S(V)$.
$\rho$ natural representation of $GL(V)$ in $\mathrm{End}(T(S^+(V)))$
The exists a canonical homomorphism

$$\phi \ : \ \mathbf{MQSym} \longrightarrow \mathrm{End}(T(S^+(V)))$$

from **MQSym** to $\mathrm{End}(T(S^+(V)))$ regarded as a convolution algebra,
such that for all $d$, $\phi(\mathbf{MQSym}_d)$ is the commutant $\mathrm{End}_{GL(V)}(T(S^+(V))_d)$ of $\rho(GL(V))$ in the homogeneous component of degree $d$ of $\mathrm{End}(T(S^+(V)))$
Moreover, $\phi$ is one-to-one for $d \leq n$

# Noncommutative cyclic characters

Already encountered: specialization of the Hall-Littlewood function $\tilde{Q}'_{1^n}(X;q)$ at $q=\omega_n$, a primitive $n$-th root of unity.
Specializations $q=\omega_n^k$ at other roots of unity amount to

$$\tilde{Q}'_{1^n}(X;q) \mod 1-q^n = \sum_{k=0}^{n-1} q^k \ell_n^{(k)}$$

where $\ell_n^{(k)}$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ induced by the character $\gamma \mapsto \omega_n^k$ of the cyclic subgroup generated by a $n$-cycle $\gamma$:

$$\ell_n^{(k)} = \frac{1}{n} \sum_{d|n} c(k,d) p_d^{n/d},$$

where $c(k,d)$ is the sum of the $k$-th powers of the primitive $d$-th roots of unity (Ramanujan sums)
$\tilde{Q}'_{1^n}(X;q)$ is the commutative image of $K_n(q)$:

$$\ell_n^{(k)} = \sum_{\mathrm{maj}\,(I)\equiv k \mod n} r_I \ ,$$

(equivalent to the combinatorial description of Kraskiewicz and Weyman)

Noncommutative analogues:

$$L_n^{(k)} := \sum_{\mathrm{maj}\,(I)\equiv k \mod n} R_I \ .$$

Another basis of the subspace spanned by the $L_n^{(k)}$:

$$K_n^{(k)} := K_n(\omega^k) = \sum_i \omega^{ik} L_n^{(i)} \ ,$$

When $\omega^k$ is a primitive $n$-th root of unity $\frac{1}{n}K_n^{(k)}$ is an idempotent corresponding to Klyachko's
In particular, $K_n^{(k)}$ is primitive for $\Delta$.
When $\omega^k$ is not primitive, $K_n^{(k)}$ is a product of primitive elements
Let $\zeta$ be any $r$-th root of unity
Set $n = ar + b$ with $a, b \in \mathbb{N}, b < r$
Then

$$K_n(\zeta) = K_r(\zeta)^a K_b(\zeta)\,.$$

From this, one can show that the $K_n^{(m)}$ span a subalgebra, with the multiplication rule

$$K_n^{(k)} * K_n^{(l)} = \begin{cases} (n/d)!\, d^{n/d} K_n^{(l)} & \text{if} n \wedge k = n \wedge l = d \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$\mathcal{C}_n := \langle\langle K_n^{(k)} \mid k = 1, \ldots n\rangle\rangle = \langle\langle L_n^{(k)} \mid k = 1, \ldots n\rangle\rangle$$

is a subalgebra of **Sym** with respect to the internal product $*$. It is noncommutative for $n \geq 3$.

From this, one obtains

$$L_n^{(k)} * L_n^{(l)} = \sum_{m=1}^{n} \langle \ell_n^{(k)}, \ell_n^{(m-l)} \rangle L_n^{(m)} .$$

In particular, the commutative image

$$\ell_n^{(k)} * \ell_n^{(l)} = \sum_{m=1}^{n} \langle \ell_n^{(k)}, \ell_n^{(m-l)} \rangle \ell_n^{(m)}$$

amounts to an identity on Ramanujan sums

$$\frac{1}{d} \sum_{m=1}^{d} c(\ell - m, d) c(m, d) = c(\ell, d) .$$

It gives the decomposition of the product of two cyclic characters into a sum of characters of the same type

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