NONCOMMUTATIVE SYMMETRIC FUNCTIONS, QUASI-SYMMETRIC FUNCTIONS AND DESCENT ALGEBRAS

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INTRODUCTION

COMBINATORICS OF LIE TYPE A

$$\simeq$$

SYMMETRIC FUNCTIONS

- **Special bases:** Schur, Hall-Littlewood, Jack, Macdonald ...
- **Operations:** Multplications and comultiplications, scalar products, differential operators, plethysms ...

describe

- $GL(n, C)$: Characters, tensor product multiplicities, weight multiplicities, plethysms ...
- S_N: Character table, induction, restriction, tensor products, plethysms ...
- $GL(n, F_q)$: Character table, induction, restriction ...
- $GL(n, Q_p)$: Spherical functions ...
- $\widehat{sl}_n$: Fock space representations, singular vectors for Virasoro and W-algebras ...
- Quantum analogues: Hecke algebra $H_N(q)$, quantized function algebras $F_q(GL_n)$, quantized enveloping algebras $U_q(gl_n)$ and their affine versions ...

Symmetric functions can be better understood in terms of two other algebras
Sym $^{d}Qsym$

ec
Sym

- Sym = symmetric functions
- $QSym$ = *quasi-symmetric functions*
- **Sym** = *noncommutative symmetric functions*
- d = inclusion
- e = abelianization

All algebras are closely related to L. Solomon's *descent algebras* (for type A).

BACKGROUND

Symmetric functions

$$Sym = \mathbf{K}[e_1, e_2, \ldots] \quad \deg(e_k) = k$$

$$\sum_{k\geq 0} t^k e_k(X) = \prod_{i\geq 1}(1 + tx_i) = \lambda_t(X)$$

$$\sum_{k\geq 0} t^k h_k(X) = \prod_{i\geq 1}(1 - tx_i)^{-1} = \sigma_t(X)$$

- Schur functions: $s_\lambda = \det(h_{\lambda_i - i + j})$
- Cauchy identity: $XY = \{x_i y_j,\ i, j \geq 1\}$

$$h_n(XY) = \sum_{\lambda\vdash n} s_\lambda(X) s_\lambda(Y)$$

- Frobenius map: $\mathrm{ch}\,(\chi^\lambda) = s_\lambda$
- Internal product: $s_\lambda * s_\mu = \mathrm{ch}\,(\chi^\lambda \chi^\mu)$
- Generalized Cauchy:

$$s_\lambda(XY) = \sum_\mu (s_\lambda * s_\mu)(X) s_\mu(Y)$$

$\boxed{\text{Descent algebras}}$

$$h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_r} = \mathrm{ch}\left(\mathbf{1} \uparrow_{S_{\mu_1}\times S_{\mu_2}\times\cdots\times S_{\mu_r}}^{S_n} \right)$$

- Mackey's formula gives

$$h_\mu * h_\nu = \sum_{M \in \mathrm{Mat}(\mu,\nu)} h_M$$

$$h_M = \prod_{i,j} h_{m_{ij}} \quad \text{and} \quad \mathrm{Mat}(\mu,\nu) \simeq S_\mu \backslash S_n / S_\nu$$

- Descent algebra: noncommutative lifting in S_n

$$H_I H_J = \sum_{M \in \mathrm{Mat}(I,J)} H_M$$

- I and J are now *compositions* of n, and H_I, H_J certain elements of ZS_n.

$\boxed{\text{Descents of permutations}}$

$$\sigma = \underline{5}2\underline{4}16\underline{8}37 \longrightarrow 524 \quad 168 \qquad 37$$

- Descent set $\mathrm{Des}\,(\sigma) = \{1,3,6\}$
- Descent composition $C(\sigma) = I = (1,2,3,2)$
- $\mathrm{Des}\,(I) = \{1,3,6\}$
- $D_I = \sum_{C(\sigma)=I} \sigma \quad \in \quad Z[S_n]$
- $H_I = \sum_{\mathrm{Des}\,(J)\subseteq \mathrm{Des}\,(I)} D_J$

Descent algebra (L. Solomon 1976) The D_I or the H_I span a Z-subalgebra Σ_n

$$D_I D_J = \sum_{|K|=n} d_{IJ}^K \, D_K \,, \qquad d_{IJ}^K \in Z$$

$$H_I H_J = \sum_{M \in \mathrm{Mat}(I,J)} H_M$$

(similar result for all Coxeter groups)

Noncommutative symmetric functions

$$\mathbf{Sym} = \mathbf{K}\langle S_1, S_2, \ldots, S_n, \ldots \rangle$$

(or $\mathbf{K}\langle \Lambda_1, \Lambda_2, \ldots \rangle$) graded by $\deg(S_k) = \deg(\Lambda_k) = k$

- Linear bases: $S^I = S_{i_1} S_{i_2} \cdots S_{i_r}$, Λ^I
- Algebra morphism $\mathbf{Sym} \to Sym$: $S_k \mapsto h_k$
- Linear isomorphism $\Sigma_n @ > \alpha >> \mathbf{Sym}_n$: $H_I \mapsto S^I$
- Internal product $*$ on $\mathbf{Sym}$ defined by $\mathbf{Sym}_m * \mathbf{Sym}_n = 0$ for $m \neq n$ and α *anti*-isomorphism
- Polynomial realization: $A = \{a_1 < a_2 < \ldots\}$ totally ordered set of noncommuting variables

$$\sigma_t(A) = \prod_{i \geq 1}^{\rightarrow} (1 - t a_i)^{-1} = \sum_{n \geq 0} t^n S_n(A)$$

$$\lambda_t(A) = \sigma_{-t}(A)^{-1} = \prod_{i \geq 1}^{\leftarrow} (1 + t a_i) = \sum_{n \geq 0} t^n \Lambda_n(A)$$

$\boxed{\text{Quasi-symmetric functions}}$

Analog of Cauchy formula:

- $X = \{x_1 < x_2 < \cdots\}$ totally ordered set of *commuting* variables
- Product alphabet $XA = \{x_i a_j\}$ with lexicographic order on (i, j)
- Noncommutative symmetric functions of XA given by

$$\sigma_1(XA) = \sum_I M_I(X) S^I(A) = \sum_I F_I(X) R_I(A)$$

- $M_I(X)$, $F_I(X)$ are *quasi-symmetric* functions (Gessel, 1984)
- Quasi-monomial functions

$$M_K = \sum_{i_1 < i_2 < \ldots < i_r} x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$$

- $R_I = \sum_{J \preceq I} (-1)^{l(I)-l(J)} S^J$ (ribbon Schur functions)
- Quasi-ribbons $F_I = \displaystyle\sum_{I \preceq J} M_J$
- $Qsym$ is a subalgebra of $\mathbf{K}[X]$ containing Sym

The whole picture

- **Sym** and $Qsym$ are two mutually dual Hopf algebras
- Duality bracket: $\langle F_I, R_J \rangle = \langle M_I, S^J \rangle = \delta_{IJ}$
- Comultiplications and antipodes $\leftrightarrow$ natural operations on alphabets (ordered sum, opposite ...) $\leftrightarrow$ λ-ring formalism of the commutative case
- The internal product lives in **Sym** and comes from the descent algebras
- The internal products allows to handle products of alphabets

$$R_I(XA) = \sum_J F_J(X)(R_I * R_J)(A)$$

- In particular, the alphabet $X_q = \{1, q, q^2, \dots\} = 1/(1-q)$ [with its natural order $\cdots < q^n < \cdots q^2 < q < 1$] and its formal inverse $1-q$ can be used in the same way as in the commutative case (Hall-Littlewood functions, Hecke algebras, etc.)
- A striking application: *Lie idempotents*

LIE IDEMPOTENTS AND NONCOMMUTATIVE POWER SUMS

Lie idempotents

- Right action of S_n on A^n (words of length n)

$$x_1 x_2 \cdots x_n \cdot \sigma = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

- Free Lie algebra $L(A) \subset \mathbf{K}\langle A \rangle$
- $\pi \in \mathbf{K}S_n$ is a *Lie idempotent* if it acts as a projector $\mathbf{K}_n\langle A \rangle \to L_n(A)$ (onto)
- Basic examples:

1. Dynkin (1947):

$$\theta_n = \frac{1}{n}[\ldots[1,2],3],\ldots],n] = \frac{1}{n}\sum_{k=0}^{n-1} D_{(1^k,n-k)}$$

2. Solomon (1968):

$$\phi_n = \frac{1}{n}\sum_{\sigma\in S_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\sigma \qquad (d(\sigma) = |\mathrm{Des}\,(\sigma)|)$$

3. Klyachko (1974):

$$\kappa_n = \frac{1}{n}\sum_{\sigma\in S_n} \omega^{\mathrm{maj}\,(\sigma)}\sigma$$

$$\omega = e^{2i\pi/n}, \qquad \mathrm{maj}\,(\sigma) = \sum_{j\in \mathrm{Des}\,(\sigma)} j\,.$$

Noncommutative power sums

- Commutative case: $\sigma_t(X) = \exp\left\{\sum_{k\geq 1} p_k(X)\frac{t^k}{k}\right\}$ equivalent to Newton's recursion $nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + h_1 p_{n-1} + p_n$
- Both expressions make sense in the noncommutative case but define different "power sums":
  1. $\sigma_t(A) = \exp\left\{\sum_{k\geq 1} \Phi_k(A)\frac{t^k}{k}\right\}$
  2. $nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + S_1\Psi_{n-1} + \Psi_n$,
- $\Phi(t) = \log \sigma_t$ where σ_t is the solution of $\frac{d}{dt}\sigma_t = \sigma_t \psi(t)$ satisfying $\sigma_0 = 1$, and $\psi(t) = \sum_{k\geq 1} t^{k-1}\Psi_k$
- The expression of $\Phi(t)$ in terms of the Ψ_k is known as the continuous Baker-Campbell-Hausdorff series
- A third possibility: $\sigma_t = e^{Z_1 t} e^{Z_2 \frac{t^2}{2}} e^{Z_3 \frac{t^3}{3}} \cdots$
- $\Phi(t) = H(Z_1 t, Z_2 t^2/2, Z_3 t^3/3, \ldots)$ where H is the standard Hausdorff series
- All these elements (Φ_n, Ψ_n or Z_n) generate the same (free) Lie algebra $\mathcal{L}$ in **Sym** and have p_n as commutative image
- Any homogeneous element $P_n \in \mathcal{L}_n$ having p_n as commutative image is called a noncommutative power sum

Lie idempotents in Σ_n

- $\pi_n \in \Sigma_n$ is a Lie idempotent $\Leftrightarrow P_n = n \cdot \alpha(\pi_n)$ is a noncommutative power sum [NCSF1]
- The Lie idempotents of Σ_n form an affine space, corresponding to $\frac{1}{n}\Psi_n + [\mathcal{L}, \mathcal{L}]$ in $\mathbf{Sym}_n$
- Any sequence $P_n(A)$ of noncommutative power sums can be obtained as $P_n(A) = \Psi_n(XA)/p_n(X)$, for X a virtual ordered alphabet [NCSF3]
- Building blocks for other idempotents [NCSF2]:
  1. The P^I are idempotent up to a nonzero scalar factor
  2. If $S_n = \sum_{|I|=n} c_I P^I$, define for a partition λ of n
  $$E_\lambda(P) = \sum_{I \in S(\lambda)} c_I P^I$$
  The $E_\lambda(P)$ form a complete set of orthogonal idempotents of Σ_n ($P_n = \Phi_n$: Garsia-Reutenauer 1989)
  3. The $E_n^{[k]}(P) = \sum_{l(\lambda)=k} E_\lambda(P)$ form a complete set of orthogonal idempotents of an n-dimensional commutative subalgebra ($P_n = \Phi_n$: Eulerian idempotents)

Example 1: Hall-Littlewood vs. Klyachko

- $\tilde{Q}'_\mu(X;q) = \sum_{\lambda \vdash n} \tilde{K}_{\lambda\mu}(q) s_\lambda(X)$
- Specialization formulas for $q = \omega^k = e^{2ik\pi/n}$
- Explicit formula for $\mu = (1^n)$

$$\tilde{Q}'_{(1^n)}(X;q) = (q)_n h_n \left(\frac{X}{1-q}\right)$$

- Natural noncommutative analog

$$K_n(q) = (q)_n S_n \left(\frac{A}{1-q}\right) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I$$

- Particular case: $\tilde{Q}'_{(1^n)}(X;\omega) = p_n$
- Therefore, $K_n(\omega) \mapsto p_n$. Also, $K_n(\omega) \in \mathcal{L}$
- Hence, $K_n(\omega)$ is a noncommutative power sum.
- The corresponding Lie idempotent is Klyachko's
- $K_n(\omega^k)$ computable ($\Rightarrow \ell_n^{(j)} * \ell_n^{(k)}$)
- Other noncommutative Hall-Littlewood functions (F. Hivert)

Example 2: a projective line

- $P_n(A;q) = (1-q^n)\Psi_n\left(\frac{A}{1-q}\right)$ is a noncommutative power sum
- The corresponding Lie idempotent is

$$\varphi_n(q) = \frac{1}{n}\sum_{|I|=n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma)\end{bmatrix}_q} q^{\mathrm{maj}\,(\sigma)-\binom{d(\sigma)+1}{2}}\sigma$$

- it specializes to

$$\varphi_n(0)=\theta_n\,, \qquad \varphi_n(1)=\phi_n \qquad \varphi_n(\omega)=\kappa_n$$

(and $\varphi_n(\infty)=\theta_n^*$).

- Nonlinear operator $E_q[H(t)]$, where $H(t) = \sum_{n\geq 1} H_n t^{n-1}$

$$E_q[H(t)] = \sum_I c_I(q) H^I t^{|I|}$$

Then,

$$E_1[H(t)] = \exp \int_0^t H(s)ds$$

while E_0 is Dyson's chronological exponential

$$E_0[H(t)] = T \exp \int_0^t H(s)ds$$

$$= 1 + \int_0^t dt_1 H(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 H(t_2)H(t_1) + \cdots$$

Example 3: q-brackets

- $[a,b]_q = ab - qba$
- $[\dots[a_1,a_2]_q,a_3]_q,\dots]_q,a_n]_q = (a_1a_2\cdots a_n)\cdots\theta_n(q)$
- $\theta_n(q)\in\Sigma_n \;\leftrightarrow\; (1-q)^{-1}S_n((1-q)A)\in\mathbf{Sym}$
- For generic q, $\theta_n(q)$ is invertible and its inverse is

$$(1-q)S_n\left(\frac{A}{1-q}\right) = \frac{1-q}{(q)_n}K_n(q)$$

- $P\in\mathbf{K}_n\langle A\rangle$ is in $L(A)$ iff $P\theta_n(q) = [n]_qP$
- Eigenvalues of $\theta_n(q)$: $p_\lambda(1-q) = \prod_i(1-q^{\lambda_i})$
- Multiplicities $n!/z_\lambda \Rightarrow$ Krattenthaler's conjecture

$$\det\left[\sum_{\sigma\in S_n} q^{\mathrm{maj}\,(\sigma)}\sigma\right] = \prod_{i=1}^{n}(1-q^i)^{n!(i-1)/i}$$

- Spectral projectors $\pi_\lambda(q)$: complete set of orthogonal idempotents of Σ_n constructed from $(\pi_n(q))_{n\geq1}$
- $\pi_n(1)$ = Dynkin, $\pi_n(\omega)$ = Klyachko, $\pi_n(0)$ = Zassenhaus (Z_n/n)([Duchamp-Krob-Vassilieva], conjectured in [NCSF2]

- The associated "q-exponential" interpolates between Dyson's ($q = 1$) and

$$E_0[H(t)] = e^{H_1 t} e^{H_2 \frac{t^2}{2}} e^{H_3 \frac{t^3}{3}} \cdots$$

ON THE TRACK OF HALL-LITTLEWOOD FUNCTIONS

Hecke algebras

- The classical Hall-Littlewood functions can be defined in terms of (affine) Hecke algebras
- $H_N(q)$ Hecke algebra of type A_{N-1}: C or $C(q)$ algebra generated by $T_1, \ldots, T_{N-1}$

$$T_iT_j = T_jT_i \qquad |i-j| > 1$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad i = 1, \ldots, N-2$$
$$T_i^2 = (q-1)T_i + q \qquad i = 1, \ldots, N-1$$

- Faithful representation on $C[x_1, \ldots, x_N]$:

$$T_i = (q-1)\pi_i + \sigma_i$$

$$\pi_i(f) = \frac{x_if - \sigma_i(x_if)}{x_i - x_{i+1}}, \quad \sigma_i = (i, i+1)$$

- Actually, a representation of the affine Hecke algebra $\hat{H}_N(q)$ (extra generators $y_i \mapsto x_i$)
- Full q-symmetrizer $S^{(N)} = \sum_{\sigma \in S_N} T_\sigma$
- q-symmetrized monomials $\simeq$ Hall-Littlewood functions

$$Q_\lambda(x_1, \ldots, x_N; q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}}!} S^{(N)}(x^\lambda)$$

- Hall basis

$$P_\lambda(X;q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q! \cdots [m_n]_q!} Q_\lambda(X;q)$$

where m_i is the multiplicity of the part i in λ

- Dual basis for $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

$$Q'_\mu(X;q) = Q_\mu\left(\frac{X}{1-q};q\right) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda(X)$$

- Geometric dual basis

$$\tilde{Q}'_\mu(X;q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda(X) = q^{n(\mu)} Q'_\mu(X;q^{-1})$$

- $\tilde{K}_{\lambda\mu}(q)$ are Kazhdan-Lusztig polynomials and entries of the character table of $GL_n(F_q)$
- Combinatorial interpretation: cocharge [Lascoux-Schützenberger 1978]
- π_i: Demazure operators (Demazure character formula: $s_\lambda = \pi_\omega(x^\lambda)$ and interpretation of $\pi_\sigma(x^\alpha)$
- $T_i \mapsto -\pi_i$ is a representation of $H_N(0)$

Noncommutative Hall-Littewood functions ?

- If analogs of the Q' live in **Sym**, their dual basis in $Qsym$ should be analogs of the P or Q functions
- P or Q functions can be defined through an action of $H_N(q)$ on polynomials
- This action comes from the Demazure operators π_i
- The π_i are defined in terms of the usual action of S_N

Can we find an action of $H_N(q)$ on polynomials such that the full q-symmetrizer $S^{(N)}$ becomes a *quasi*-symmetrizer ?

- If yes, one can try $S^{(N)}(x^I)$ (up to a normalization factor) as a quasi-symmetric analog of the Hall basis
- The dual basis in **Sym** should be an analog of the Q'
- A test: $I = (1^n)$ should be $(q)_n \Lambda_n(A/(1-q))$

Florent Hivert's thesis

- Quasi-symmetric functions are in fact symmetric ... but for an exotic (quasi-symmetrizing) action of the symmetric group
- $X = \{x_1, \ldots, x_N\}$
- Support of a monomial $\mathbf{m} = x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N}$
$$A = \{i | m_i \neq 0\} \subseteq X$$
- I composition obtained by removing the 0's from $(m_1, \ldots, m_N)$
- Notation: $\mathbf{m} = A^I$
- $\sigma \in S_N$ acts on $\mathbf{m}$ by
$$\sigma'(\mathbf{m}) = (\sigma(A))^I$$
- Examples: $\sigma_1'(x_1 x_2^3) = x_1 x_2^3$ and $\sigma_2'(x_1 x_2^3) = x_1 x_3^3$
- The quasi-symmetric polynomials are the invariants of this action
- quasi-monomial functions: orbit sums of $x^I = \{x_1, \ldots, x_r\}^{(i_1, \ldots, i_r)}$
- Jacobi-Weyl-Demazure formula for quasi-ribbons
$$F_I = \pi_\omega'(x^I)$$
where $\pi_i' = (x_i - x_{i+1})^{-1}(x_i - x_{i+1}\sigma_i')$

- Defines F_I when I is a pseudo-composition: some of the i_k may be 0
- Example: $F_{04003} = -F_{31111} - F_{22111} - F_{13111}$
- Hecke algebra: $\bar{\pi}_i = \pi'_i - 1$, $T_i = (1-q)\bar{\pi}_i + q\sigma'_i$
- Quasi-symmetric analogs of the P functions

$$G_I(X;q) = \frac{1}{[r]_q![N-r]_q!} S^{(N)}(X^I) \quad (r = l(I))$$

- Explicit formula $G_I = \sum_{J \succeq I} (-1)^{l(I)-l(J)} q^{s(I,J)} F_J$
- Example: $G_{1121} = F_{1121} - q^3 F_{11111}$
- Dual basis $H_I \in \mathbf{Sym}$: analogs of Q'
- Explicit formula $H_K = \sum_{J \preceq K} q^{t(K,J)} R_J$
- $H_{(1^n)} = \sum_J q^{\mathrm{maj}\,(J^\sim)} R_J$ as required
- Example: $H_{321} = R_{321} + qR_{33} + q^2R_{51} + q^3R_6$

- Explicit formula for multiplication

$$H_I H_J = \sum_{J \succeq K} q^{t(J,K)} (c(r, s-p) H_{IK} + c(r, s-p+1) H_{I \triangleright K})$$

- In the basis $H'_I = (1-q)^{l(I)} H_I$, the structure constants are in $N[q]$
- Factorization at roots of unity: ζ primitive kth root of 1, $J_1, \ldots, J_m$ composition of length k, J_{m+1} of length $r < k$, $I = J_1 J_2 \cdots J_{m+1}$

$$H_I(A;\zeta) = H_{J_1}(A;\zeta) H_{J_2}(A;\zeta) \cdots H_{J_{m+1}}(A;\zeta)$$

- Work in progress: Interpretation as a Hall algebra ? Analogues of Macdonald funtions ?

0-QUANTUM STRUCTURES

Generic Hecke algebras

- $H_N(q)$ is isomorphic to $C[S_N]$ for generic $q \in C$, but not for q a root of unity of order $\leq N$ or for $q = 0$.
- q-Specht modules $V_\lambda(q)$, $\lambda \vdash N$
- Matrix elements in $Z[q]$
- Characteristic map $R(H_N(q)) \longrightarrow Sym_N$

$$\mathrm{ch}\,([V_\lambda(q)]) = s_\lambda$$

(independent of q).

Representation theory of $H_N(0)$

(P.N. Norton 1979, R.W. Carter 1985)

- Irreducible representations $\mathbf{D}_I$ are 1-dimensional. Parametrized by subsets $E \subseteq \{1, \ldots, N-1\}$ (or compositions $I = C(E)$)

$$\rho_I(T_j) = \begin{cases} -1 & \text{if } j \in E \\ 0 & \text{otherwise} \end{cases}$$

- Indecomposable projective modules $\mathbf{N}_I = H_N(0)\nu_I$

$$H_N(0) = \bigoplus_{|I|=N} \mathbf{N}_I$$

with $\dim \mathbf{N}_I = |\{\sigma | C(\sigma) = I\}$

- decomposition matrix

$$d_{\lambda I} = [V_\lambda(0) : \mathbf{D}_I] = |\{T \in \mathrm{STab}\,(\lambda) | C(T) = I\}|$$

$\boxed{\text{Grothendieck rings}}$

$\mathcal{A}$ finite dimensional C-algebra.

- $G_0(\mathcal{A}) = Z\{\text{finitely generated left } \mathcal{A}\text{-modules}\}/\sim$ where $[Q] \sim [P] + [R] \Leftrightarrow 0 \to P \to Q \to R \to 0$
- $K_0(\mathcal{A})$ = abelian group generated by isomorphism classes of f.g. projective left-$\mathcal{A}$-modules
- Duality between $G_0(\mathcal{A})$ and $K_0(\mathcal{A})$: $\langle N, P\rangle = \dim \operatorname{Hom}_{\mathcal{A}}(P, N)$

Let

$$\mathcal{G} = \bigoplus_{n\geq 0} G_0(H_n(0)) \qquad \mathcal{K} = \bigoplus_{n\geq 0} K_0(H_n(0))$$

Define ring structures on $\mathcal{G}$ and $\mathcal{K}$ by

$$[M]\cdot[N] = [M \otimes_C N] \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}$$

- The map $[\mathbf{D}_I] \mapsto F_I$ is a ring isomorphism $\mathcal{G} \to Qsym$
- The map $[\mathbf{N}_I] \mapsto R_I$ is a ring isomorphism $\mathcal{K} \to \mathbf{Sym}$
- Decomposition map $d : Sym \hookrightarrow Qsym$
- Cartan map $c : \mathbf{Sym} \longrightarrow Qsym$ given by the commutative image
- Character formula $\chi_M(T_{w_I}) = \langle [M], (q-1)^{-\ell(I)} S^I((q-1)A)\rangle|_{q=0}$
- Hopf algebra duality $\mathbf{Sym} \leftrightarrow Qsym$ special case of $G_0(\mathcal{A}) \leftrightarrow K_0(\mathcal{A})$

$\mathrm{Sym}_N = R(H_N(q))^d Qsym_N = G_0(H_N(0))$

ec

$\mathbf{Sym}_N = K_0(H_N(0))$

The Cartan-Brauer triangle of $H_n(0)$

Remark. For $q = \omega$ a primitive nth root of unity, $\mathcal{G}$ is a quotient of Sym and $\mathcal{K}$ a subring. In this case the decomposition map is surjective and the Cartan map injective.

Quantum linear groups at $q = 0$

- Dipper-Donkin version of the quantized function algebra $A_q(n)$

$$\begin{array}{rcll} x_{jk}\,x_{il} &=& q\,x_{il}\,x_{jk} & \text{for } i<j,\ k\le l \\ x_{ik}\,x_{il} &=& x_{il}\,x_{ik} & \text{for all } i,k,l \\ x_{jl}\,x_{ik} - x_{ik}\,x_{jl} &=& (q-1)\,x_{il}\,x_{jk} & \text{for } i<j,\ k<l \end{array}$$

- Hopf algebra with coproduct

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik}\otimes x_{kj}\ .$$

- Vector corepresentation:

$$V = \bigoplus_{i=1}^{n} C(q)\,e_i$$

- Coaction: $\delta(e_i) = \displaystyle\sum_{j=1}^{n} x_{ij}\otimes e_j$

- Action of $H_N(q)$ on $\mathbf{v} = v_{k_1}\otimes\cdots\otimes v_{k_N} i \in V^{\otimes n}$

$$\begin{cases} \mathbf{v}\cdot T_i = \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \\ \mathbf{v}\cdot T_i = q\mathbf{v} & \text{if } k_i = k_{i+1} \\ \mathbf{v}\cdot T_i = q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \end{cases}$$

- Commutation: for $h\in H_N(q)$, $(1\otimes h)\circ\delta^{\otimes N} = \delta^{\otimes N}\circ h$

$\boxed{\text{Characters}}$

- M finite dimensional $A_q(n)$-comodule
- $(m_i)_{i=1,m}$ basis of M.
- coaction: $\delta(m_i) = \sum_{j=1}^{m} a(i,j) \otimes m_j$, $a(i,j) \in A_q(n)$
- Character of M

$$\chi(M) = \sum_{i=1}^{m} a(i,i) \in A_q(M)$$

- independent of the basis (m_i).
- $0 \to M' \to M \to M'' \to 0 \Rightarrow \chi(M) = \chi(M') + \chi(M'')$
- $\chi(M \otimes N) = \chi(M)\,\chi(N)$
- $M \simeq N \Rightarrow \chi(M) = \chi(N)$

Diagonal algebra (or quantum torus)

- $\Delta_q(n) = \langle x_{ii},\ i = 1, \ldots, n \rangle \subset A_q(n)$
- $\chi(M) \in \Delta_q(n)$ (always true, even for $q = 0$)
- Follows from the identity

$$\left| \begin{matrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{matrix} \right|_q \overset{\text{def}}{=} \sum_{\sigma \in S_n} \varepsilon(\sigma)\, x_{1\sigma(1)} \ldots x_{n\sigma(n)}$$

$$= \frac{1}{(1-q)^{n-1}} \left[x_{nn}, [\ldots, [x_{22}, x_{11}]_q \ldots]_q\right]$$

$$[a, b]_q = a\, b - q\, b\, a$$

The hypoplactic algebra

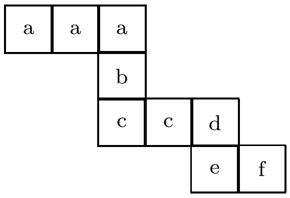
- For $q = 0$, $\Delta_0(n)$ is a quotient of the plactic algebra: $a, b, c, d \in A = \{x_{11} < x_{22} < \ldots < x_{nn}\}$ satisfy
- the plactic relations

$$\begin{cases} aba = baa \ , \ bba = bab & \text{for } a < b \\ acb = cab \ , \ bca = bac & \text{for } a < b < c \end{cases}$$

- and the quartic Hypoplactic relations

$$\begin{cases} baba = abab \ , \ baca = abac & \text{for } a < b < c \\ cacb = acbc \ , \ cbab = bacb & \text{for } a < b < c \\ badc = dbca \ , \ acbd = cdab & \text{for } a < b < c < d \end{cases}$$

- Irreducible $A_0(n)$ comodules $\mathbf{D}_I = V^{\otimes N} \cdot \eta_I$ where $H_N(0)\eta_I$ are minimal left ideals
- Quasi-ribbon words



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- Characters are hypoplactic quasi-symmetric functions

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} w \otimes w^* = F_I(x_{11}, \ldots, x_{nn})$$

(analogues of the plactic Schur functions of Lascoux-Schützenberger)

- The hypoplactic F_I span a commutative subalgebra
- Indecomposable summands of $V^{\otimes N}$ $\mathbf{N}_I = V^{\otimes N}\nu_I$, where $H_N(0)\nu_I$ are the indecomposable projective modules
- $\chi(\mathbf{N}_I) = R_I(x_{11}, \ldots, x_{nn})$

$\boxed{\text{The 0-enveloping algebra } \mathcal{U}_0(gl_n)}$

Takeuchis's 2-parameter algebra $U_{r,s}(gl_n)$ for $(r,s)=(q,1)$

$$k_i\,k_j \;=\; k_j\,k_i \qquad \text{for } 1\le i,j\le n$$

$$\left\{\begin{array}{rcll} q\,k_i\,e_{i-1} &=& e_{i-1}\,k_i & \text{for } 2\le i\le n-1\\ k_i\,e_i &=& q\,e_i\,k_i & \text{for } 1\le i\le n-1\\ k_i\,e_j &=& e_j\,k_i & \text{for } j\ne i-1,i \end{array}\right.$$

$$\left\{\begin{array}{rcll} k_i\,f_{i-1} &=& q\,f_{i-1}\,k_i & \text{for } 2\le i\le n-1\\ q\,k_i\,f_i &=& f_i\,k_i & \text{for } 1\le i\le n-1\\ k_i\,f_j &=& f_j\,k_i & \text{for } j\ne i-1,i \end{array}\right.$$

$$[e_i,f_j] \;=\; \delta_{ij}\,\frac{k_i-k_{i+1}}{q-1}$$

$$\left\{\begin{array}{rcl} q\,e_{i+1}\,e_i^2-(1+q)\,e_i\,e_{i+1}\,e_i+e_i^2\,e_{i+1} &=& 0\\ q\,e_{i+1}^2\,e_i-(1+q)\,e_{i+1}\,e_i\,e_{i+1}+e_i\,e_{i+1}^2 &=& 0 \end{array}\right.$$

$$\left\{\begin{array}{rcl} f_{i+1}\,f_i^2-(1+q)\,f_i\,f_{i+1}\,f_i+q\,f_i^2\,f_{i+1} &=& 0\\ f_{i+1}^2\,f_i-(1+q)\,f_{i+1}\,f_i\,f_{i+1}+q\,f_i\,f_{i+1}^2 &=& 0 \end{array}\right.$$

$$\left\{\begin{array}{rcll} [e_i,e_j] &=& 0 & \text{for } |i-j|>1\\ [f_i,f_j] &=& 0 & \text{for } |i-j|>1 \end{array}\right.$$

$$\left\{\begin{array}{rcll} e_i\,e_{i+1}\,k_{i+1} &=& k_{i+1}\,e_i\,e_{i+1} & \text{for } 1\le i\le n-2\\ f_{i+1}\,f_i\,k_{i+1} &=& k_{i+1}\,f_{i+1}\,f_i & \text{for } 1\le i\le n-2 \end{array}\right.$$

- Bialgebra structure

$$\left\{\begin{array}{rclrcl} \Delta(e_i) &=& 1\otimes e_i + e_i\otimes k_i, & \varepsilon(e_i) &=& 0\\ \Delta(f_i) &=& k_{i+1}\otimes f_i + f_i\otimes 1, & \varepsilon(f_i) &=& 0\\ \Delta(k_i) &=& k_i\otimes k_i, & \varepsilon(k_i) &=& 1 \end{array}\right.$$

- Vector representation

$$\left\{\begin{array}{rcl} \rho_V(e_i) &=& E_{i,i+1}\\ \rho_V(f_i) &=& E_{i+1,i}\\ \rho_V(k_i) &=& q\,E_{i,i} + \displaystyle\sum_{j\neq i} E_{j,j} \end{array}\right.$$

- Duality with $A_q(n)$

$$\rho_V(g) = (a_{ij}(g))_{1\le i,j\le n}$$

satisfy the defining relations of $A_q(n) \Rightarrow$ homomorphism $\Psi(x_{ij}) = a_{ij}$

$$\langle x, g\rangle = \Psi(x)(g)$$

$x\in A_q(N)$ and $g\in\mathcal{U}_q(gl_N)$.

Polynomial $\mathcal{U}_0(gl_n)$ modules

- Same constructions as for $A_0(n)$
- Irreducible polynomial modules:

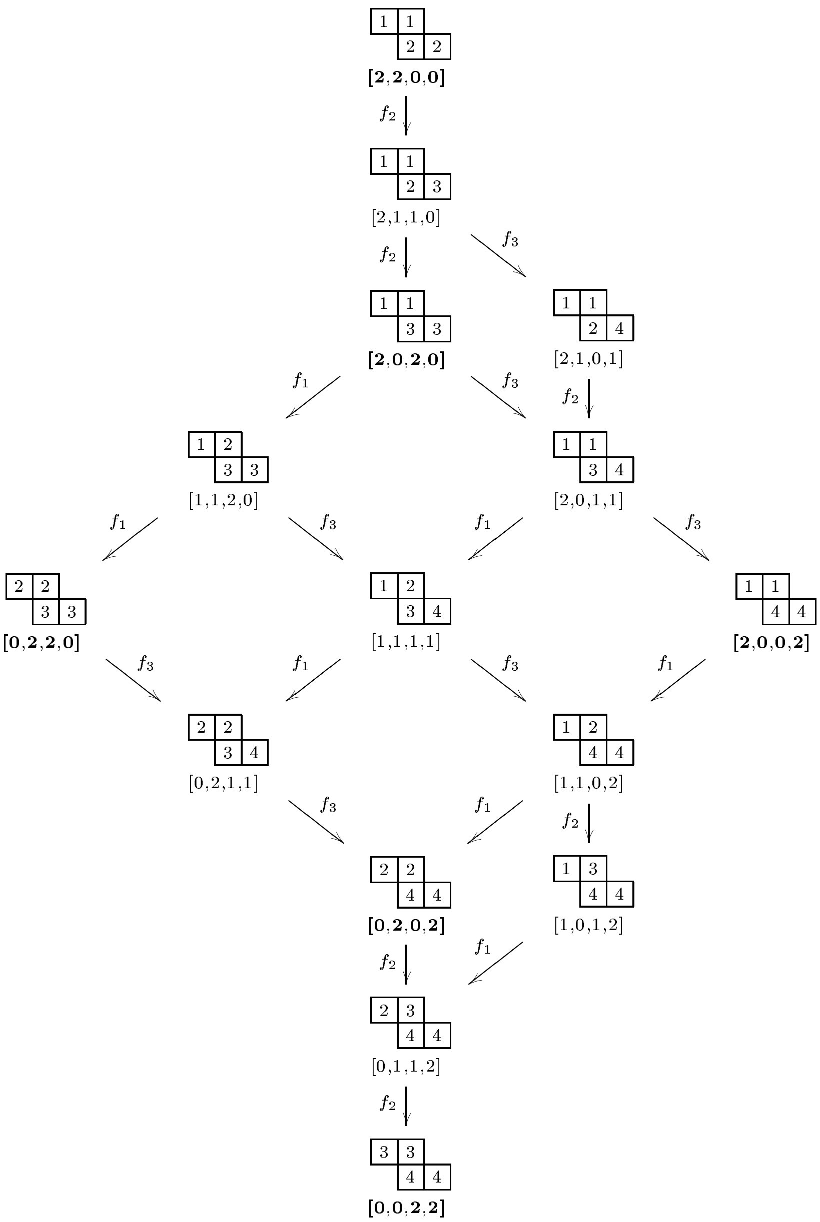
$$\mathbf{D}_I = V^{\otimes N} \cdot \eta_I$$

- character $F_I(x_{11}, \ldots, x_{nn})$
- Indecomposable direct summands of $V^{\otimes N}$:

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I$$

  character $R_I(x_{11}, \ldots, x_{nn})$
- Irreducible representations: quasi-crystal graphs

Example: D_{22} for $n = 4$



Classical characters

- Homomorphism $\Delta_0(n) \to C[x_1, \ldots, x_n]$:

$$x_{ii} \longrightarrow x_i \quad \text{commutative}$$

- Weyl and Demazure character formulas: Hivert's quasi-symmetrizations
- Weyl character formula:

$$\mathrm{ch}\,(\mathbf{D}_I) = F_I = \pi'_\omega(x^I)$$

- Characters of Demazure modules: $M_{\sigma,I} = \mathcal{U}_0(b^+)\xi$, where ξ is an extremal vector, i.e. a weight vector of extremal weight $\sigma'(I)$

$$\mathrm{ch}\,(M_{\sigma,I}) = \pi'_\sigma(x^I)$$

MORE ALGEBRAS

Overview

FSymFQSymSym$QSym_q$**MQSym**$SymQSym$

- Sym: Partitions λ
- **Sym** and $QSym$: Compositions I
- $QSym_q$: Compositions I ($C(q)$ basis)
- **FSym**: Standard Young tableaux $\mathbf{t}$
- **FQSym**: Permutations σ
- **MQSym**: Packed integer matrices M (double cosets $S_I\backslash S_n/S_J$)

Free quasi-symmetric functions

- Free quasi-ribbons:

$$F_\sigma = \sum_{\mathrm{std}\,(w)=\sigma^{-1}} w \quad \in Z\langle A\rangle$$

- Commutative image: $F_\sigma \mapsto F_I$, where $I = C(\sigma)$
- q-commutative image ($x_j x_i = q x_i x_j$ for $i < j$): $q^{l(\sigma)} F_I \in QSym_q$
- Multiplication rule: $\alpha \in S_k$, $\beta \in S_l$,

$$F_\alpha F_\beta = \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta[k]} F_\sigma$$

- Scalar product

$$\langle F_\sigma \, , \, F_\tau \rangle = \delta_{\sigma^{-1},\tau}$$

- Dual basis $G_\sigma = F_{\sigma^{-1}}$
- **FQSym** is isomorphic to the convolution algebra of Reutenauer:

$$G_\alpha G_\beta = G_{\alpha \star \beta}$$

Examples

- Free symmetric functions: $S_{\mathbf{t}} = \sum_{P(\sigma)=t} F_\sigma$
- Proof of the Littlewood-Richardson rule reduces to

$$S_{\mathbf{t}'} S_{\mathbf{t}''} = \sum_{\substack{P(\sigma')=\mathbf{t}' \\ P(\sigma'')=\mathbf{t}''}} F_{\sigma'} F_{\sigma''} = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} \sum_{P(w)=\mathbf{t}} F_\sigma = \sum_{\mathbf{t}\in \mathrm{Sh}\,(\mathbf{t}',\mathbf{t}'')} S_{\mathbf{t}}$$

- Free quasisymmetric function $\mathbf{\Gamma}(P) = \sum_{\sigma\in L(P)} F_\sigma$ associated to a poset
- Operation $P_1 \wedge P_2$ on posets $\longrightarrow$ algebra of planar binary trees [Loday-Ronco]
- Stanley's $QS(x)$-distribution on symmetric groups: $\mathrm{Prob}(\sigma) = G_\sigma(x)$
- **FQSym** is the Grothendieck (induction) ring of a category $\mathcal{N}$ of 0-Hecke modules, containing simple, projective and skew Specht modules
- It gives the indecomposable summands of induction products of all modules in $\mathcal{N}$

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