LIE IDEMPOTENTS IN DESCENT ALGEBRAS

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Abstract. Lie idempotents are symmetrizers which project the tensor algebra onto the free Lie algebra. Almost all known examples turn out to belong to the descent algebras of symmetric groups. This makes it possible to analyze them in terms of noncommutative symmetric functions. By extending various classical techniques of the theory of ordinary symmetric functions, it is then possible to produce many new examples, and in some sense, to classify all the possibilities.

1. Introduction

Let V be a vector space over some field $\mathbb{K}$ of characteristic 0. Let $T(V)$ be its tensor algebra, and $L(V)$ the free Lie algebra generated by V. We denote by $L_n(V) = L(V) \cap V^{\otimes n}$ its homogeneous component of degree n. The group algebra $\mathbb{K}\mathfrak{S}_n$ acts on the right on $V^{\otimes n}$ by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \,. \tag{1}$$

This action commutes with the left action of $GL(V)$, and when $\dim V \geq n$, which we shall usually assume, these actions are the commutant of each other (Schur-Weyl duality).

Any $GL(V)$-equivariant projector $\Pi_n : V^{\otimes n} \to L_n(V)$ can therefore be regarded as an idempotent π_n of $\mathbb{K}\mathfrak{S}_n$: $\Pi_n(\mathbf{v}) = \mathbf{v} \cdot \pi_n$. By definition (cf. [25]), such an element is called a *Lie idempotent* whenever its image is $L_n(V)$. Then, a homogeneous element $P_n \in V^{\otimes n}$ is in $L_n(V)$ if and only if $P_n \pi_n = P_n$.

From now on, we fix a basis $A = \{a_1, a_2, \ldots\}$ of V. We identify $T(V)$ with the free associative algebra $\mathbb{K}\langle A\rangle$, and $L(V)$ with the free Lie algebra $L(A)$.

2. The Hausdorff series

Lie idempotents arise naturally in the investigation of the Hausdorff series

$$H(a_1, a_2, \ldots, a_N) = \log\left(e^{a_1} e^{a_2} \cdots e^{a_N}\right) = \sum_{n\geq 0} H_n(A) \tag{2}$$

which is known to be a *Lie series*, i.e., each homogeneous component $H_n(A) \in L_n(A)$. This is known as the Baker-Campbell-Hausdorff (BCH) "formula". It follows immediately from the characterization of $L(A)$ as the space of primitive elements of the standard comultiplication of $\mathbb{K}\langle A\rangle$ (Friedrich's criterion[1]).

[1]This criterion, which appears only as a footnote [11, p. 203] in a text on quantum field theory, is for us, together with its converse (the Milnor-Moore theorem, due to Cartier) the "founding act" of combinatorial Hopf algebra theory, which has actually little to do with Hopf's original motivations.

Actually, the BCH formula is not a formula at all, only a (very important) property, which is the basis of the correspondence between Lie groups and Lie algebras. However, various applications (including real-world ones) require explicit calculation of the polynomials $H_n(A)$. That is where Lie idempotents come into play.

For small values of n and N, this can be done by hand, and one obtains for example

$$\begin{aligned}
H_3(a_1,a_2,a_3) \ &= \ \frac{1}{12}a_1a_1a_2 + \frac{1}{12}a_1a_1a_3 - \frac{1}{6}a_1a_2a_1 + \frac{1}{12}a_1a_2a_2 + \frac{1}{3}a_1a_2a_3 \\
&\quad -\frac{1}{6}a_1a_3a_1 + \frac{1}{12}a_1a_3a_3 + \frac{1}{12}a_2a_1a_1 - \frac{1}{6}a_2a_1a_2 \\
&\quad -\frac{1}{6}a_2a_1a_3 + \frac{1}{12}a_2a_2a_1 + \frac{1}{12}a_2a_2a_3 - \frac{1}{6}a_2a_3a_1 - \frac{1}{6}a_2a_3a_2 \\
&\quad +\frac{1}{12}a_2a_3a_3 + \frac{1}{12}a_3a_1a_1 - \frac{1}{6}a_3a_1a_2 - \frac{1}{6}a_3a_1a_3 + \frac{1}{3}a_3a_2a_1 \\
&\quad +\frac{1}{12}a_3a_2a_2 - \frac{1}{6}a_3a_2a_3 + \frac{1}{12}a_3a_3a_1 + \frac{1}{12}a_3a_3a_2
\end{aligned}$$

Already, it might not be obvious, at first sight, that this is indeed a Lie polynomial, which can be rewritten in the form

$$\begin{aligned}
H_3(a_1,a_2,a_3) \ &= \ \frac{1}{12}[a_1,[a_1,a_2]] + \frac{1}{12}[[a_1,a_2],a_2] + \frac{1}{12}[a_1,[a_1,a_3]] \\
&\quad +\frac{1}{12}[[a_1,a_3],a_3] + \frac{1}{12}[a_2,[a_2,a_3]] + \frac{1}{12}[[a_2,a_3],a_3] \\
&\quad +\frac{1}{6}[a_1,[a_2,a_3]] + \frac{1}{6}[[a_1,a_2],a_3]\,.
\end{aligned}$$

The problem of finding a systematic procedure for expressing the Hausdorff series as a linear combination of commutators was raised at the Gelfand seminar in the 1940's, and Dynkin [9] came up with the following solution (also discovered independently by Specht [29] and Wever [30]):

Theorem 2.1.

$$\theta_n = \frac{1}{n}[\ldots[[[1,2],3],\ldots,],n] \tag{3}$$

is a Lie idempotent. Therefore, expanding H_n as a linear combination of words, and writing $H_n = H_n\theta_n$, gives the required expression.

In order to apply this recipe, we need a reasonably efficient way to find the expansion on words

$$H_n(A) = \sum_{w\in A^n} c_w w\,. \tag{4}$$

One can show that $H_n(A)$ is the image of the homogeneous component $E_n(A)$ of the product of exponentials

$$E(A) = e^{a_1}e^{a_2}\cdots e^{a_N} = \sum_{n\geq 0} E_n(A)\,. \tag{5}$$

under an element of $\mathbb{K}\mathfrak{S}_n$

$$H_n(A) = E_n(A) \cdot \phi_n\,, \tag{6}$$

where

$$\phi_n = \frac{1}{n} \sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\,\sigma \qquad (d(\sigma) = |\,\mathrm{Des}(\sigma)|)\,. \tag{7}$$

Here, $\mathrm{Des}(\sigma) = \{i | \sigma(i) > \sigma(i+1)\}$ denotes the *descent set* of σ. This formula is due to Solomon [26] and independently to Bialynicki-Birula, Mielnik and Plebański [2]. It can be shown, although no one of these properties is clearly apparent on the expression (7), that ϕ_n is a Lie idempotent. It has been rediscovered many times, and is also known as the (first) Eulerian idempotent.

We can now write the Hausdorff polynomials

$$H_n(A) = E_n(A)\phi_n\theta_n = \sum_{w\in A^n} c_w \cdot w\theta_n \tag{8}$$

as linear combinations of commutators $w\theta_n$. However, these commutators are far from being linearly independent, and one would be interested in an expansion of H_n on a *basis* of the free Lie algebra.

One way to achieve this is to use *Klyachko's basis* of the free Lie algebra. This little known basis is obtained from a third Lie idempotent, discovered by A. Klyachko [15], and originally introduced as the solution of a different problem.

This problem was the following. The character of $GL(V)$ on $L_n(V)$ is known (Witt has given formulas for the dimensions of its weight spaces), and its expression shows that as a $GL(V)$-module, $L_n(V)$ is isomorphic to the space $\Gamma_n(V)$, image of the idempotent

$$\gamma_n = \frac{1}{n} \sum_{k=0}^{n-1} \omega^k c^k\,, \tag{9}$$

where $c = (12\ldots n)$ is an n-cycle of $\mathfrak{S}_n$ and ω a primitive nth root of unity. Klyachko's idempotent κ_n is an intertwiner between these two isomorphic representations. This means that for any word $w \in A^n$, (i) $w\kappa_n = 0$ is w is not primitive (i.e., $w = v^d$ for some non-trivial divisor d of n), and (ii) if w is primitive, $wc\kappa_n = \omega w\kappa_n$.

Hence, applying κ_n to some set of representatives of circular classes of primitive words, for example to Lyndon words (words which are lexicographically minimal among their circular shifts), we obtain a basis of $L_n(A)$.

There is a closed formula:

$$\kappa_n = \frac{1}{n} \sum_{\sigma\in\mathfrak{S}_n} \omega^{\mathrm{maj}\,(\sigma)}\sigma \tag{10}$$

where $\omega = e^{2i\pi/n}$ and $\mathrm{maj}\,(\sigma) = \sum_{j\in\mathrm{Des}(\sigma)} j$ is the *major index* of σ.

Now, applying κ_n to the Lyndon words appearing in the expansion of $E_n(A)$ obtained from the Eulerian idempotent gives an expansion of$H_n(A)$ on a basis of $L_n(A)$, which can be further expanded in terms of commutators by means of Dynkin's idempotent if needed.

At this point, we are facing three idempotents, which, admittedly, are given by rather different formulas.

It may therefore come as a surprise that the element

$$\varphi_n(q) = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} \, q^{\mathrm{maj}\,(\sigma) - \binom{d(\sigma)+1}{2}} \, \sigma \tag{11}$$

is a Lie idempotent, interpolating between our three examples.

Apart from the from the fact that $\varphi_n(1) = \phi_n$, none of these properties is evident. Nevertheless, one can show that $\varphi_n(0) = \theta_n$ and $\varphi_n(\omega) = \kappa_n$.

The explanation of this strange fact starts with the observation that the three idempotents do have something in common: they all belong to the *descent algebra* of the symmetric group.

3. Solomon's descent algebra

The descent algebras have been introduced by Solomon [27] for general finite Coxeter groups in the following way. Let (W, S) be a Coxeter system. One says that $w \in W$ has a descent at $s \in S$ if w has a reduced word ending by s. For $W = \mathfrak{S}_n$ and $s_i = (i, i+1)$, this means that $w(i) > w(i+1)$, whence the terminology. In this case, we rather say that i is a descent of w. Let as above $\mathrm{Des}(w)$ denote the descent set of w, and for a subset $E \subseteq S$, set

$$D_E = \sum_{\mathrm{Des}(w)=E} w \quad \in \mathbb{Z}W\,. \tag{12}$$

Solomon has shown that the D_E span a $\mathbb{Z}$-subalgebra of $\mathbb{Z}W$. Moreover

$$D_{E'} D_{E''} = \sum_E c^E_{E'E''} D_E \tag{13}$$

where the coefficients $c^E_{E'E''}$ are nonnegative integers.

In the case of $\mathfrak{S}_n$, it is convenient to encode descent sets by compositions of n. If $E = \{d_1, \ldots, d_{r-1}\}$, we set $d_0 = 0$, $d_r = n$ and $I = C(E) = (i_1, \ldots, i_r)$, where $i_k = d_k - d_{k-1}$. We also say that E is the descent set of I. From now on, we shall write D_I instead of D_E.

It is clear from the defintions that ϕ_n and κ_n are in the descent algebra of $\mathfrak{S}_n$. For θ_n, this is also quite easy to see:

$$2\theta_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} - \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$3\theta_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} - \left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array} \right) + \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$$

$$4\theta_4 = \underbrace{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}}_{D_4} - \underbrace{\left(\begin{array}{|c|c|c|}\cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 4 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-1} 3 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 4 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 3 \\ \hline\end{array} \right)}_{D_{13}} - \underbrace{\left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline\end{array} \right)}_{D_{112}} - \underbrace{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}}_{D_{1111}}$$

That is,

$$n\theta_n = \sum_{k=0}^{n-1} (-1)^k D_{1^k\, n-k} \,. \tag{14}$$

4. Noncommutative symmetric functions

The algebra of ordinary symmetric functions *Sym* can be regarded as the free associative and commutative algebra over an infinite sequence $(h_n)_{n\geq 1}$ of homogeneous generators (h_n is of degree n), so that its linear bases in degree n are naturally labelled by partitions of n (e.g., products $h_\mu = h_{\mu_1} \cdots h_{\mu_r}$ of complete homogeneous functions).

Similarly, the algebra **Sym** of noncommutative symmetric functions is the free associative (but noncommutative) algebra over an infinite sequence $(S_n)_{n\geq 1}$ of homogeneous generators, endowed with a natural homomorphism $S_n \mapsto h_n$ (commutative image).

Thus, linear bases of the homogeneous component $\mathbf{Sym}_n$ of **Sym** are labelled by compositions of n, exactly as those of the descent algebra Σ_n of $\mathfrak{S}_n$. It is then tempting to define a correspondence between them. There is a natural way to do this. Noncommutative symmetric functions can be realized in terms of an auxiliary (ordered) alphabet $A = \{a_1, a_2, \ldots\}$ by setting

$$S_n(A) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} a_{i_1} a_{i_2} \cdots a_{i_n} \,, \tag{15}$$

that is, the sum of nondecreasing words, or, otherwise said, words with no descent. Then, obviously,

$$S^I = S_{i_1} S_{i_2} \cdots S_{i_r} \tag{16}$$

is the sum of words whose descent set is contained in $\mathrm{Des}(I)$ (the descents of a word are defined as for permutations as those i such that $w_i > w_{i+1}$, and are similarly encoded as compositions $C(w)$ of n). Introducing the *noncommutative ribbon Schur functions*

$$R_I(A) = \sum_{C(w)=I} w \tag{17}$$

whose commutative image are indeed the skew Schur functions indexed by ribbon diagrams, we have

$$S^I = \sum_{J \le I} R_J \tag{18}$$

where $J \le I$ is the *reverse refinement order*, which means that $\mathrm{Des}(J) \subseteq \mathrm{Des}(I)$.

The map $\alpha : D_I \to R_I$ appears therefore as a natural choice for a correspondence $\Sigma_n \to \mathbf{Sym}_n$.

This choice is not only natural, it is *canonical*. Indeed, Solomon had proved that the structure constants of his descent algebra were the same as the decompositon coefficients of certain tensor products of representations of $\mathfrak{S}_n$. Precisely, if we set

$$B^I = \sum_{J \le I} D_I \ \in \Sigma_n \tag{19}$$

and

$$B^I B^J = \sum_K b_K^{IJ} B^K \,, \tag{20}$$

then, the Kronecker products of the characters β^I of $\mathfrak{S}_n$, induced by the trivial representations of the parabolic subgroups $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r}$, decompose as

$$\beta^I \beta^J = \sum_K b_K^{IJ} \beta^K \,. \tag{21}$$

Such products of induced characters can be calculated by Mackey's formula, whence the title of Solomon's paper [27].

So, why is the above correspondence canonical? This is because it is compatible with the Frobenius characteristic map, from $\mathfrak{S}_n$-characters to symmetric functions. Indeed, $\mathrm{ch}(\beta^I) = h_I$, the commutative image of S^I.

The Frobenius characteristic map allows one to define the *internal product* $*$ on symmetric functions, by setting $h_\lambda * h_\mu = \mathrm{ch}(\beta^\lambda \beta^\mu)$. We can now do the same on noncommutative symmetric functions, using the descent algebras instead of the character rings.

For technical reasons, we want our correspondence to be an *anti-isomorphism.* We set

$$S^I * S^J = \sum_K b_K^{JI} S^K \,. \tag{22}$$

This is because we want to interpret permutations as endomorphisms of tensor algebras: if $f_\sigma(w) = w\sigma$, then $f_\sigma \circ f_\tau = f_{\tau\sigma}$.

5. Lie idempotents as noncommutative symmetric functions

The first really interesting question about noncommutative symmetric functions is perhaps "what are the noncommutative power sums?". Indeed, the answer to this question is far from being unique.

If one starts from the classical expression

$$\sigma_t(X) = \sum_{n\geq 0} h_n(X) t^n = \exp\left\{ \sum_{k\geq 1} p_k \frac{t^k}{k} \right\} , \tag{23}$$

one can choose to define noncommutative power sums Φ_k by the same formula

$$\sigma_t(A) = \sum_{n\geq 0} S_n(A) t^n = \exp\left\{ \sum_{k\geq 1} \Phi_k \frac{t^k}{k} \right\} , \tag{24}$$

but a noncommutative version of the Newton formulas

$$nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + p_n \tag{25}$$

which are derived by taking the logarithmic derivative of (23) leads to different noncommutative power-sums Ψ_k inductively defined by

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + \Psi_n \, . \tag{26}$$

A bit of computation reveals then that

$$\Psi_n = R_n - R_{1,n-1} + R_{1,1,n-2} - \cdots = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k} \, , \tag{27}$$

which is analogous to the classical expression of p_n as the alternating sum of hook Schur functions. Therefore, in the descent algebra, Ψ_n correponds to Dynkin's element, $n\theta_n$.

The Φ_n can also be expressed on the ribbon basis without much difficulty, and one finds

$$\Phi_n = \sum_{|I|=n} \frac{(-1)^{l(I)-1}}{\binom{n-1}{l(I)-1}} R_I \tag{28}$$

so that Φ_n corresponds to $n\phi_n$.

The case of Klyachko's idempotent is even more interesting, but to explain it, we shall need the $(1-q)$-tranform, to be defined below.

6. The $(1-q)$-transform

In its classical (commutative) version, the $(1-q)$-transform ϑ_q is the algebra endomorphism of Sym defined on the power sums by $\vartheta_q(p_n) = (1-q^n)p_n$. In λ-ring notation, which is particularly convenient for dealing with such transformations, it reads $f(X) \mapsto f((1-q)X)$. One has to pay attention to the abuse of notation in using the same minus sign for the λ-ring and for scalars, though these operations are quite different. That is, ϑ_{-1} maps p_n to $2p_n$ if n is odd, and to 0 otherwise. Thus, $\vartheta_{-1}(f(X)) = f((1-q)X)_{q=-1}$ is not the same as $f((1+1)X) = f(2X)$.

In [17], a consistent definition of $\vartheta_q(F) = F((1-q)A)$ has been introduced as follows. One first defines the complete symmetric functions $S_n((1-q)A)$ via their generating series [17, Def. 5.1]

$$\sigma_t((1-q)A) := \sum_{n\geq 0} t^n S_n((1-q)A) = \sigma_{-qt}(A)^{-1}\sigma_t(A)\,, \tag{29}$$

and then ϑ_q is defined as the ring homomorphism such that

$$\vartheta_q(S_n) = S_n((1-q)A). \tag{30}$$

It can then be shown [17, Thm. 4.17] that

$$F((1-q)A) = F(A) * \sigma_1((1-q)A)\,. \tag{31}$$

For generic q, ϑ_q is an automorphism, and its inverse is the $1/(1-q)$-transform

$$\sigma_t\left(\frac{A}{1-q}\right) = \overleftarrow{\prod_{n\geq 0}} \sigma_{tq^n}(A)\,. \tag{32}$$

Computing the image of S_n, one arrives at

$$K_n(q) = (q)_n S_n\left(\frac{A}{1-q}\right) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I(A)\,. \tag{33}$$

Hence, Klyachko's element $n\kappa_n = K(\omega)$ is the specialization of this expression at $q=\omega$. This is puzzling: the commutative image of (33) is a Hall-Littlewood function ($\tilde{Q}'_{(1^n)}$, precisely), and the specialization of such functions at roots of unity are known. In this case, one gets the power sum p_n.

7. Hopf algebras enter the scene

At this point, we can see that the commutative images of our three Lie idempotents are the same: $\frac{1}{n}p_n$. The symmetric functions $\frac{1}{n}p_n$ have two significant properties: (i) they are idempotent for the internal product, and (ii) they are primitive elements for the coproduct of *Sym*.

Identifying $f\otimes g$ with $f(X)g(Y)$, the coproduct of *Sym* can be defined by $\Delta f = f(X+Y)$, which gives for complete functions

$$\Delta h_n = \sum_{i=0}^{n} h_i \otimes h_{n-i}\,, \tag{34}$$

and similarly, **Sym** can be endowed with a bialgebra structure, defined by

$$\Delta S_n = \sum_{i=0}^{n} S_i \otimes S_{n-i} \tag{35}$$

(with the convention $S_0 = 1$).

One may therefore suspect that our Lie idempotents might be primitive elements of **Sym**. That this is true can be verified directly for each of them, but we have the following much stronger result [17]. Let us say that an element e of an algebra is *quasi-idempotent* if $e^2 = c\cdot e$ for some (possibly 0) scalar c.

Theorem 7.1. *Let $F = \alpha(\pi)$ be an element of $\mathbf{Sym}_n$, where $\pi \in \Sigma_n$.*

(i) *The following assertions are equivalent:*

1. *π is a Lie quasi-idempotent;*
2. *F is a primitive element for Δ;*
3. *F belongs to the Lie algebra $L(\Psi)$ generated by the Ψ_n.*

(ii) *Moreover, π is a Lie idempotent iff $F - \frac{1}{n}\Psi_n$ is in the Lie ideal $[L(\Psi)\,,\, L(\Psi)]$.*

Thus, Lie idempotents are essentially the same thing as "noncommutative power sums" (up to a factor n), and we shall from now on identify both notions: a Lie idempotent in $\mathbf{Sym}_n$ is a primitive element whose commutative image is p_n/n.

8. A one parameter family of Lie idempotents

Theorem 7.1 suggests a recipe for constructing new examples. Start from a known family, e.g., Dynkin elements, and take its image by a bialgebra automorphism, e.g., ϑ_q^{-1}. The result is then automatically a sequence of Lie idempotents. In the case under consideration, we get

$$\varphi_n(q) = \frac{1-q^n}{n}\Psi_n\left(\frac{A}{1-q}\right) = \frac{1}{n}\sum_{|I|=n} \frac{(-1)^{\ell(I)-1}}{\begin{bmatrix} n-1 \\ \ell(I)-1 \end{bmatrix}_q}\, q^{\mathrm{maj}\,(I)-\binom{\ell(I)}{2}}\ R_I(A)\,. \tag{36}$$

The obtention of the closed form in the r.h.s. requires a fair amount of calculation, but the fact that $\varphi_n(q)$ is a Lie idempotent is automatic. This being granted, it is not difficult to show that

$$\varphi_n(0) = \frac{1}{n}\,\Psi_n\ ,\ \ \varphi_n(\omega) = \frac{1}{n}K_n(\omega)\ ,\ \ \varphi_n(1) = \frac{1}{n}\,\Phi_n\ . \tag{37}$$

Other Hopf automorphisms, like the noncommutative analogs of the transformation

$$f(X) \longrightarrow f\left(\frac{1-t}{1-q}X\right) \tag{38}$$

used in the theory of Macdonald polynomials, lead to other families of Lie idempotents. It is not always possible, however, to obtain such a clean closed form for them.

9. The iterated q-bracketing and its diagonalization

There is another one-parameter family of Lie idempotents, for which no closed expression is known, but which is of fundamental importance.

The reproducing kernel of ϑ_q, $S_n((1-q)A)$ is easily seen to be the image under α of the iterated q-bracketing operator

$$S_n((1-q)A) = (1-q)\alpha\left([[\cdots[1,2]_q,3]_q,\ldots,n]_q\right)\ , \tag{39}$$

a natural q-analog of Dynkin's idempotent. For generic q, this is not an idempotent at all, but an automorphism.

The most important property of ϑ_q is its diagonalization [17, Thm. 5.14]: there is a unique family of Lie idempotents $\pi_n(q)$ with the property

$$\vartheta_q(\pi_n(q)) = (1-q^n)\pi_n(q)\,. \tag{40}$$

Moreover, ϑ_q is semi-simple, and its eigenvalues in $\mathbf{Sym}_n$, the nth homogeneous component of $\mathbf{Sym}$, are $p_\lambda(1-q) = \prod_i (1-q^{\lambda_i})$ where λ runs over the partitions of n. The projectors on the corresponding eigenspaces are the maps $F \mapsto F * \pi^I(q)$ [17, Sec. 3.4].

Here are the first values of $\pi_n(q)$:

$$\pi_1(q) = \Psi_1\ ,\quad \pi_2(q) = \frac{\Psi_2}{2}\ ,\quad \pi_3(q) = \frac{\Psi_3}{3} + \frac{1}{6}\,\frac{1-q}{(1+2q)}\,[\Psi_2,\Psi_1]\ ,$$

$$\pi_4(q) = \frac{\Psi_4}{4} + \frac{1}{12}\,\frac{(1-q)(2q+1)}{(1+q+2q^2)}\,[\Psi_3,\Psi_1] + \frac{1}{24}\,\frac{(1-q)^2}{(1+q+2q^2)}\,[[\Psi_2,\Psi_1],\Psi_1]\ ,$$

$$\begin{aligned}\pi_5(q) = \frac{\Psi_5}{5} &+ \frac{1}{20}\,\frac{(1-q)(3q^2+2q+1)}{(2q^3+q^2+q+1)}\,[\Psi_4,\Psi_1] + \frac{1}{30}\,\frac{(1-q)(q+2)}{(2q^2+2q+1)}\,[\Psi_3,\Psi_2]\\
&+\frac{1}{60}\,\frac{(1-q)^2(4q^3+7q^2+7q+2)}{(2q^2+q+2)(2q^3+q^2+q+1)}\,[[\Psi_3,\Psi_1],\Psi_1]\\
&-\frac{1}{120}\,\frac{(1-q)^2(4q^2+9q+7)}{(q^2+3q+1)(2q^2+1+2q)}\,[[\Psi_1,\Psi_2],\Psi_2]\\
&+\frac{1}{120}\,\frac{(1-q)^3(2q^5+2q^4+q^3+5q^2+9q+6)}{(2q^3-q^2+q+3)(2q^2+q+2)(2q^3+q^2+q+1)}\,[[[\Psi_2,\Psi_1],\Psi_1],\Psi_1]\ .\end{aligned}$$

The idempotents $\pi_n(q)$ have interesting specializations. The easiest one is $q=1$:

$$\pi_n(1) = \frac{\Psi_n}{n}\ . \tag{41}$$

This has the strange consequence that, for *any* Lie idempotent $F_n \in \mathbf{Sym}_n$,

$$\lim_{q\to 1}\ \frac{F_n((1-q)A)}{1-q^n} = \frac{\Psi_n}{n}\ . \tag{42}$$

Next, we have, for ω a primitive n-th root of unity,

$$\pi_n(\omega) = \kappa_n\,. \tag{43}$$

Again, a curious consequence is that

$$\lim_{q\to\omega}\ (1-q^n)\,F_n(\frac{A}{1-q}) = \kappa_n\,, \tag{44}$$

for any Lie idempotent $F_n \in \mathbf{Sym}_n$.

To describe the next specialization, we need to introduce a new family of noncommutative power sums. The *noncommutative power sums* of the *third kind* Z_n are defined by

$$\sigma_t(A) = \exp(Z_1\,t)\,\exp(\frac{Z_2}{2}\,t^2)\,\ldots\,\exp(\frac{Z_n}{n}\,t^n)\,\ldots \tag{45}$$

The Fer-Zassenhauss formula (*cf.* [31]) shows that every Z_n is a Lie element. It is also clear that the commutative image of Z_n is p_n.

The first values of Z_n are

$$Z_1 = \Psi_1\ ,\ Z_2 = \Psi_2\ ,\ Z_3 = \Psi_3 + \frac{1}{2}[\Psi_2, \Psi_1]\ ,$$

$$Z_4 = \Psi_4 + \frac{1}{3}[\Psi_3, \Psi_1] + \frac{1}{6}[[\Psi_2, \Psi_1], \Psi_1]\ ,$$

$$Z_5 = \Psi_5 + \frac{1}{4}[\Psi_4, \Psi_1] + \frac{1}{3}[\Psi_3, \Psi_2] + \frac{1}{12}[[\Psi_3, \Psi_1], \Psi_1]$$
$$-\frac{7}{24}[\Psi_2, [\Psi_2, \Psi_1]] + \frac{1}{24}[[[\Psi_2, \Psi_1], \Psi_1], \Psi_1]\ .$$

Then, we have [17, 8]

$$\pi_n(0) = \frac{Z_n}{n}\ . \tag{46}$$

10. Decompositions of the descent algebras

Let $F = (F_n)$ be a sequence of Lie quasi-idempotents with non-zero normalization factors, F_n belonging to $\mathbf{Sym}_n$. We define as usual

$$F^I = F_{i_1}\, F_{i_2}\ \ldots\ F_{i_r}$$

for a composition $I = (i_1, i_2, \ldots, i_r)$. The family (F^I) is then a basis of $\mathbf{Sym}$. It follows that

$$S_n = \sum_{|I|=n} c_I\, F^I$$

for some scalar coefficients (c_I). Then we define

$$E_\lambda(F) = \sum_{I\downarrow=\lambda} c_I\, F^I$$

where $I \downarrow$ is the nondecreasing reordering of I and λ a partition of n. Hence,

$$S_n = \sum_{\lambda\vdash n} E_\lambda(F)\ .$$

For $F_n = \Phi_n$, the $E_\lambda(\Phi)$ are the Garsia-Reutenauer idempotents [13] (a complete set of orthogonal idempotents of the descent algebra). More generally, we have:

Theorem 10.1. *The family $(E_\lambda(F))_{\lambda\vdash n}$ is a complete family of orthogonal idempotents of* $\mathbf{Sym}_n$.

With $F_n = \Psi_n$, one finds

$$E_{(3)}(\Psi) = \frac{1}{3}\Psi_3 = \frac{1}{3}[R_3 - R_{12} + R_{111}]\ ,$$

$$E_{(21)}(\Psi) = \frac{1}{6}\Psi^{21} + \frac{1}{3}\Psi^{12} = \frac{1}{6}[3R_3 - R_{21} + R_{12} - 3R_{111}]\ ,$$

$$E_{(111)}(\Psi) = \frac{1}{6}\Psi^{111} = \frac{1}{6}[R_3 + R_{21} + R_{12} + R_{111}]\ .$$

and for $n = 4$:

$$E_4(\Psi) = \frac{1}{4}[R_4 - R_{13} + R_{112} - R_{1111}]$$

$$E_{(31)}(\Psi) = \frac{1}{12}[4R_4 + R_{31} - 3R_{22} + 3R_{211} + 2R_{13} - R_{121} - 2R_{112} + 4R_{1111}]\ ,$$

$$E_{(22)} = \frac{1}{8}\left[R_4 - R_{31} + R_{22} - R_{211} - R_{13} + R_{121} - R_{112} + R_{1111}\right] \ ,$$

$$E_{(211)}(\Psi) = \frac{1}{12}\left[3R_4 + R_{22} - 2R_{211} + 2R_{13} - R_{121} - 3R_{1111}\right] \ ,$$

$$E_{(1111)}(\Psi) = \frac{1}{24}\left[R_4 + R_{31} + R_{22} + R_{211} + R_{13} + R_{121} + R_{112} + R_{1111}\right] \ .$$

One can define analogs of the Eulerian idempotents associated to any series (F_n) of Lie idempotents, by the formula

$$E_n^{[k]}(F) = \sum_{|\lambda|=n,\, \ell(\lambda)=k} E_\lambda(F) \ . \tag{47}$$

In all cases, these idempotents generate an n-dimensional commutative subalgebra of Σ_n. The original Eulerian idempotents are obtained for $F_n = \Phi_n$.

11. Decompositions of the tensor algebra

Any complete set $(e_\lambda)_{\lambda\vdash n}$ of orthogonal idempotents of the descent algebra Σ_n determines a decomposition

$$T^n(V) = \bigoplus_{\lambda\vdash n} V_\lambda \tag{48}$$

where $V_\lambda = T^n(V)e_\lambda$.

In the case where the e_λ are the Garsia-Reutenauer idempotents $e_\lambda = \alpha^{-1}(E_\lambda(\Phi))$, the characters L_λ of the $GL(V)$-modules V_λ are known. Indeed, if $\lambda = (1^{m_1}2^{m_2}\dots n^{m_n})$, then V_λ is isomorphic to a product of symmetric powers

$$S^{m_1}(L^1(V)) \otimes S^{m_2}(L^2(V)) \otimes \cdots \otimes S^{m_n}(L^n(V))$$

of homogeneous components $L^i(V)$ of the free Lie algebra $L(V)$ over V. Thus, as a symmetric function, L_λ is given by a product of plethysms

$$L_\lambda = h_{m_1}[\ell_1]h_{m_2}[\ell_2]\cdots h_{m_n}[\ell_n] \tag{49}$$

where $\ell_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{n/d}$ is the character of $GL(V)$ in $L^n(V)$.

The character of V_λ is always the same, whenever the e_λ are defined by the above construction. In fact, this character depends only on the commutative image of the noncommutative symmetric function $E_\lambda = \alpha(e_\lambda)$.

Theorem 11.1. [17] *Let e'_λ and e''_λ be two idempotents of Σ_n such that the associated noncommutative symmetric functions $E'_\lambda = \alpha(e_\lambda)$ and $E''_\lambda = \alpha(e''_\lambda)$ both have as commutative image the normalized product of power sums p_λ/z_λ. Set $V'_\lambda = T^n(V)e'_\lambda$ and $V''_\lambda = T^n(V)e''_\lambda$. Then, the endomorphism ϕ of $T^n(V)$ defined by*

$$\phi(v_1\otimes\cdots\otimes v_n) = (v_1\otimes\cdots\otimes v_n)(id - e'_\lambda - e''_\lambda)$$

induces an isomorphism of $GL(V)$-modules between V'_λ and V''_λ.

Hence, the $GL(V)$-modules $V^{\otimes n}e_\lambda$ constructed from the complete sets of orthogonal idempotents associated to any sequence of Lie idempotents are all isomorphic. It is possible to describe them more precisely:

Proposition 11.2. *Let $\pi = (\pi_n)_{n\geq 1}$ be a sequence of Lie idempotents, with $\pi_n \in$ $\mathbf{Sym}_n$, let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of n and let*

$$E_\lambda(\pi) = \sum_{I\downarrow=\lambda} c_I \, \pi^I$$

be the idempotent associated with λ as above. The image $T^n(V)e_\lambda$ of this idempotent is spanned by the polynomials

$$\sum_{\sigma\in\mathfrak{S}_r} c_{\lambda\cdot\sigma} \, P_{l_{\sigma(1)}} \, P_{l_{\sigma(2)}} \, \dots \, P_{l_{\sigma(r)}}$$

where P_{l_k} is a homogeneous Lie polynomial of degree l_k.

The proposition is a special case of the following lemma:

Lemma 11.3. *With the same hypotheses, let*

$$E = \sum_{I\downarrow=\lambda} c_I \, \pi^I$$

be an arbitrary linear combination of all π^I indexed by a permutation of λ. The image of this element is then spanned by the polynomials

$$\sum_{\sigma\in\mathfrak{S}_r} c_{\lambda\cdot\sigma} \, P_{l_{\sigma(1)}} \, P_{l_{\sigma(2)}} \, \dots \, P_{l_{\sigma(r)}}$$

where as above, P_{l_k} is a homogeneous Lie polynomial of degree l_k.

For each partition $\lambda = (\lambda_1, \dots, \lambda_r)$, there exists then a "probability distribution" c_λ on $\mathfrak{S}_r$ with the following properties:

- $c_\lambda(\sigma) = c_\lambda(\tau)$ whenever the compositions $\lambda\cdot\sigma$ and $\lambda\cdot\tau$ are equal.
- The image of the projector $E_\lambda(\pi)$ is the module generated by all c_λ-symmetrized products

  $$(P_1, \dots, P_r)_{c_\lambda} = \sum_{\sigma\in\mathfrak{S}_r} c_\lambda(\sigma) \; P_{\sigma(1)} \, \dots \, P_{\sigma(r)}$$

  where each P_i is a homogeneous Lie polynomial of degree λ_i.

The family $\Phi = (\Phi_n)_{n\geq 1}$ of Solomon idempotents is characterized by the fact that the associated probability distributions are uniform.

The distributions associated with the Dynkin idempotents are given by:

$$c_\lambda(\sigma) = \frac{\lambda_1 \dots \lambda_r}{\lambda_{\sigma(1)} \, (\lambda_{\sigma(1)} + \lambda_{\sigma(2)}) \, \dots \, (\lambda_{\sigma(1)} + \dots + \lambda_{\sigma(r)})} \tag{50}$$

More examples can be found in [6].

12. General deformations

Let X be a totally ordered commutative alphabet (which may be "virtual"), and let π_n be a Lie quasi-idempotent in $\mathbf{Sym}_n$. Then, it is easy to show [6] that

$$S_n(X\,A) * \pi_n(A) = p_n(X)\ \pi_n(A)\,. \tag{51}$$

On the other hand, $\pi_n(XA) = \pi_n(A) * S_n(XA)$ is not in general proportional to $\pi_n(A)$. If π_n is a Lie idempotent in $\mathbf{Sym}_n$, then, $\pi_n(XA)$ is another one. Actually, one can show the following:

Proposition 12.1. *For any two sequences (Π_n) and π_n of Lie idempotents, there exists a virtual alphabet X (i.e., a specialization of quasi-symmetric functions) such that*

$$\Pi_n(A) = \frac{\pi_n(X\,A)}{p_n(X)}\,.$$

One can again show that the operator $S_n(X\,A)$ is diagonalizable with $p_\lambda(X)$ as eigenvalues (where λ is a partition of n) and that the corresponding eigenspaces can still be constructed by the above method, based on the unique family of Lie idempotents $\pi_{X,n}$ such that

$$\pi_{X,n}(X\,A) = p_\lambda(X)\,\pi_{X,n}(A)\ . \tag{52}$$

here are the first $\pi_{X,n}$. Their coefficients are rational quasi-symmetric functions of X.

$$\pi_{X,1} = \Psi_1\ ,\quad \pi_{X,2} = \frac{\Psi_2}{2}\ ,$$

$$\pi_{X,3} = \frac{\Psi_3}{3} + \frac{\frac{1}{6}\,M_{12} - \frac{1}{3}\,M_{21}}{M_{12}+M_{21}}\,[\Psi_2,\Psi_1]\ .$$

For $n=3$, the spectral projectors are

$$E_3 = \pi_{X,3}\ ,\ E_{21} = \frac{M_{21}\,\pi_X^{21} + M_{12}\,\pi_X^{12}}{M_{12}+M_{21}}\ ,\ E_{111} = \frac{\Psi^{111}}{6}\ .$$

References

[1] F. Bergeron, N. Bergeron and A.M. Garsia, *Idempotents for the free Lie algebra and q-enumeration, in* Invariant theory and tableaux, D. Stanton *ed.*, IMA Volumes in Mathematics and its Applications, Vol. 19, Springer, 1988.

[2] I. Bialynicki-Birula, B. Mielnik and J. Plebański, *Explicit solution of the continuous Baker-Campbell-Hausdorff problem*, Annals of Physics **51** (1969), 187-200.

[3] D. Blessenohl and H. Laue, *Algebraic combinatorics related to the free Lie algebra*, Actes du 29-ième Séminaire Lotharingien de Combinatoire, A. Kerber Ed., Publ. IRMA, Strasbourg, 1993, 1-21.

[4] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. 2 et 3, Hermann, 1972.

[5] G. Duchamp, F. Hivert, and J.-Y. Thibon, *Noncommutative symmetric functions VI: free quasi-symmetric functions and related algebras*, Internat. J. Alg. Comput. **12** (2002), 671–717.

[6] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, *Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras*, Discrete Mathematics and Theoretical Computer Science **1** (1997), 159–216.

[7] G. Duchamp, D. Krob, B. Leclerc and J.-Y. Thibon, *Déformations de projecteurs de Lie*, C.R. Acad. Sci. Paris **319** (1994), 909-914.

[8] G. Duchamp, D. Krob and E. A. Vassilieva, *Zassenhaus Lie idempotents, q-bracketing and a new exponential/logarithm correspondence*, J. Algebraic Combin. 12 (2000), no. 3, 251–277.
[9] E.B. Dynkin, *Calculation of the coefficients in the Campbell-Baker-Hausdorff formula*, Dokl. Akad. Nauk. SSSR (N.S.) **57** (1947), 323-326 (in Russian).
[10] H.O. Foulkes, *Characters of symmetric groups induced by char acters of cyclic subgroups*, in *Combinatorics* (Proc. Conf. Comb. Math. Inst. Oxford 1972), Inst. Math. Appl., Southend-on-Sea, 1972, 141-154.
[11] K. O. Friederichs, *Mathematical aspects of the quantum theory of fields*, Interscience, New-York, 1951.
[12] A.M. Garsia, *Combinatorics of the free Lie algebra and the symmetric group*, *in* Analysis, et cetera ..., Jürgen Moser Festschrift, Academic press, New York, (1990), 309-82.
[13] A.M. Garsia and C. Reutenauer, *A decomposition of Solomon's descent algebra*, Advances in Math. **77** (1989), 189-262.
[14] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. **112** (1995), 218–348.
[15] A.A. Klyachko, *Lie elements in the tensor algebra*, Siberian Math. J. **15** (1974), 1296-1304.
[16] D.E. Knuth, The art of computer programming, vol.3: Sorting and searching, Addison-Wesley, 1973.
[17] D. Krob, B. Leclerc and J.-Y. Thibon, *Noncommutative symmetric functions II: Transformations of alphabets*, Internat. J. Alg. Comput. **7** (1997), 181–264.
[18] B. Leclerc, T. Scharf and J.-Y. Thibon, *Noncommutative cyclic characters of symmetric groups*, J. Comb. Theory A **75** (1996), 55–69.
[19] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
[20] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Comm. Pure Appl. Math. **VII** (1954), 649-673.
[21] B. Mielnik and J. Plebański, *Combinatorial approach to Baker-Campbell-Hausdorff exponents*, Ann. Inst. Henri Poincaré, Section A, vol. **XII** (1970), 215-254.
[22] F. Patras, *L'algèbre des descentes d'une bigèbre graduée*, J. Algebra **170** (1994), 547–566.
[23] R. Ree, *Generalized Lie elements*, Canad. J. Math. **12** (1960), 493–502.
[24] C. Reutenauer, *Theorem of Poincaré-Birkhoff-Witt, logarithm and representations of the symmetric group whose order are the Stirling numbers*, *in* Combinatoire énumérative, Proceedings, Montréal 1985 (G. Labelle and P. Leroux Eds.), Lecture Notes in Math., **1234**, Springer, (1986), 267-284.
[25] C. Reutenauer, *Free Lie algebras*, Oxford, 1993.
[26] L. Solomon, *On the Poincaré-Birkhoff-Witt theorem*, J. Comb. Theory **4** (1968), 363-375.
[27] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–268.
[28] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (electronic), http://www.research.att.com/~njas/sequences/
[29] W. Specht, *Die linearen Beziehungen zwischen höheren Kommutatoren*, Math. Zeit. **51** (1948), 367-376.
[30] F. Wever, *Über Invarianten in Lieschen Ringen*, Math. Annalen **120** (1949), 563-580.
[31] R.M. Wilcox, *Exponential operators and parameter differentiation in Quantum Physics*, J. Math. Phys. **8** (1967), 962-982.

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