Tree expanded series in some combinatorial Hopf algebras

Jean-Yves Thibon

Université Paris-Est/Marne-la-Vallée

(joint with F. Hivert and J.-C. Novelli)

Tree expanded series

Occur in several contexts

- Statistical physics, quantum field theory
- Numerical analysis
- Renormalization, resummation of divergent series ...

but none of these will be discussed here.

Back to basics

Rather, we shall start with a trivial example, and apply one of Feynman's maxims: "each time you learn a new trick, try it on your favorite problems"

Binary tree expansions arise from the formal solutions of functional equations

$$x = a + B(x, x)$$

where $B(x, y)$ is a bilinear map.

Tree expansion for $x = a + B(x, x)$

By iterated substitution

$$x = a + B(a, a) + B(B(a, a), a) + B(a, B(a, a)) + \cdots$$

$$= a + \quad \begin{array}{ccc} & B & \\ a & & a \end{array} \quad + \quad \begin{array}{ccc} & B & \\ B & & a \\ a & & a \end{array} \quad + \quad \begin{array}{ccc} & B & \\ a & & B \\ & a & a \end{array} \quad + \cdots$$

$$x = \sum \quad B_T(a)$$

T: Complete Binary Tree

A simple example

The function $x(t) = \frac{1}{1-t}$ is the unique solution of

$$\frac{dx}{dt} = x^2 \ , \quad x(0) = 1$$

This is equivalent to the fixed point problem

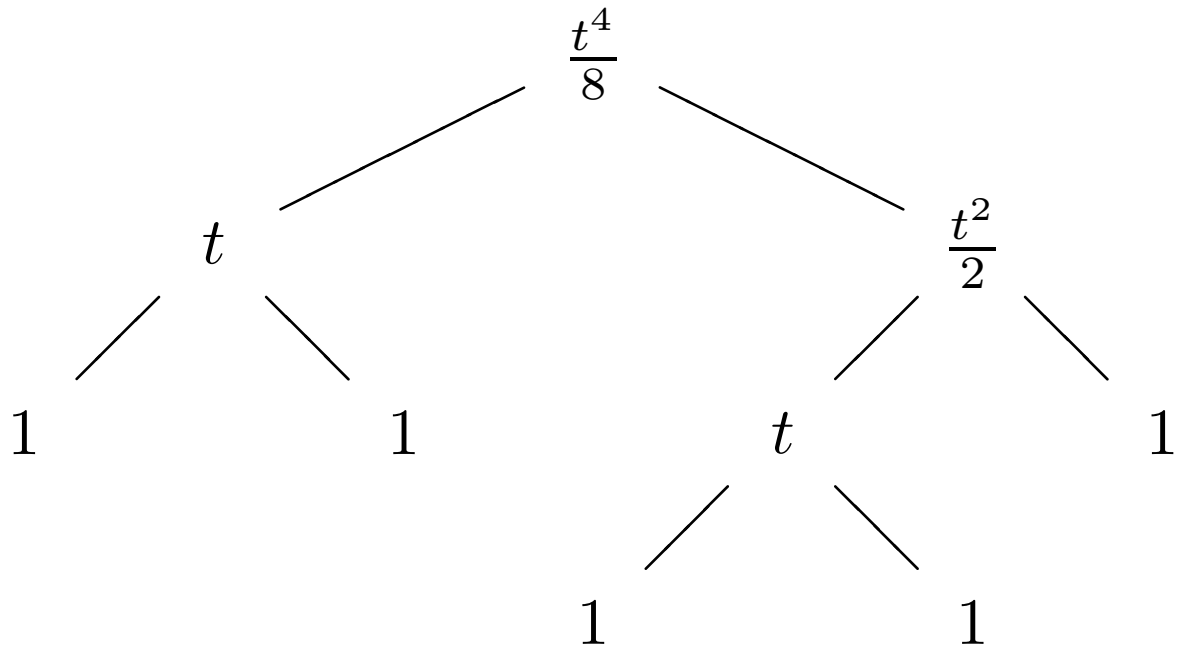
$$x = 1 + \int_0^t x^2(s)ds = 1 + B(x, x)$$

where

$$B(x, y) := \int_0^t x(s)y(s)ds$$

Example of a term

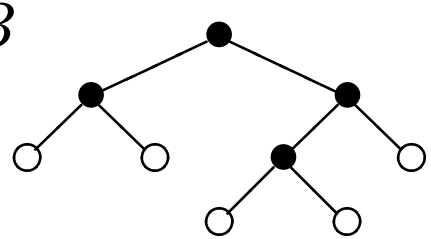
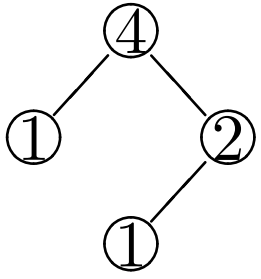
$$B(x,y) := \int_0^t x(s)y(s)ds$$



Hook-length formula

$$B_T(1) = t^{\#(T')} \prod_{o \in T'} \frac{1}{HL(o)}$$

- T' : incomplete tree associated to T;
- $HL(o)$: size of the subtree rooted at o.

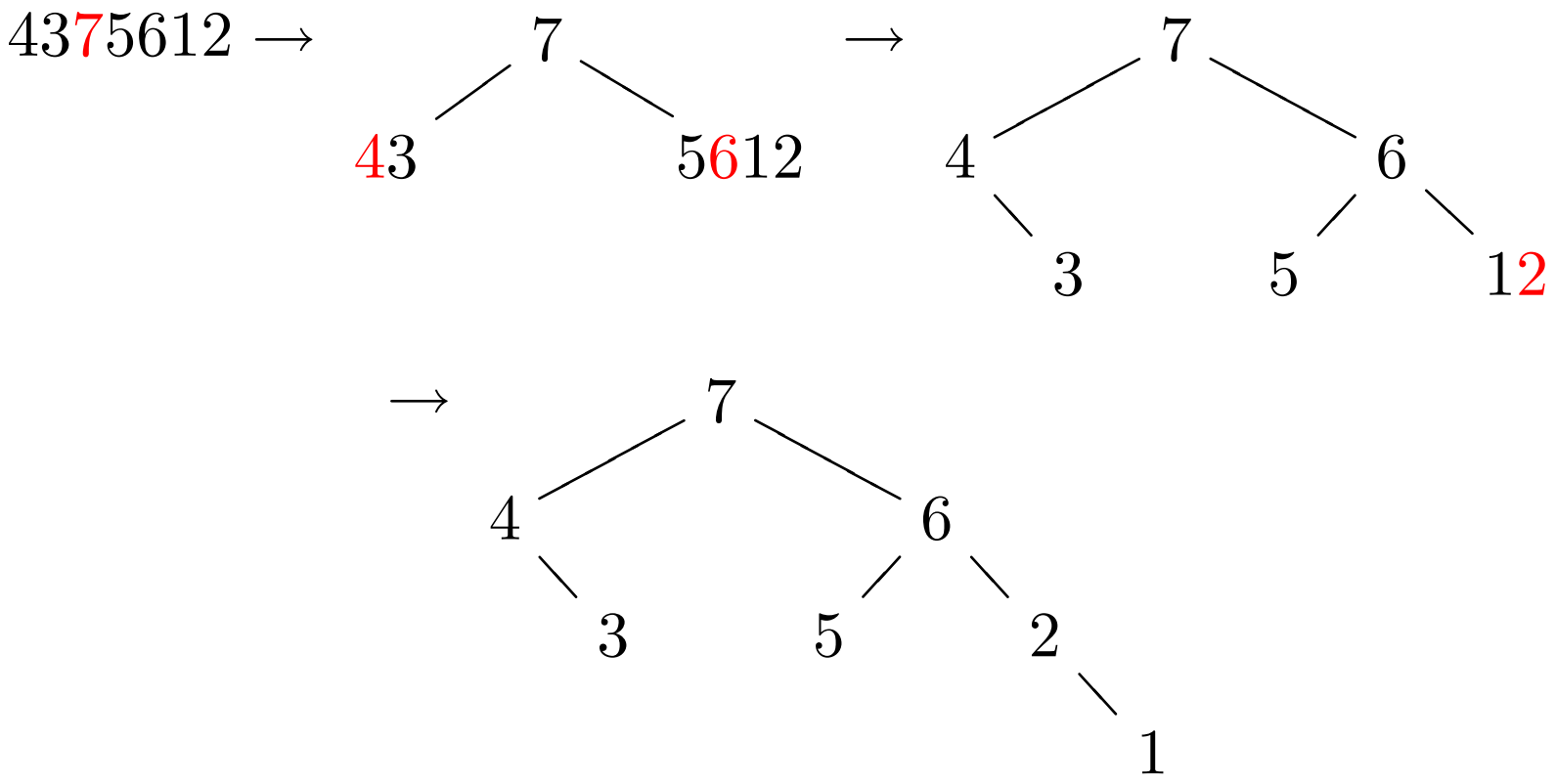
B  $(1) \quad = \quad t^4 \prod$  $= \quad \frac{t^4}{8}$

Combinatorial meaning ?

- The solution is $x(t) = \sum_{n \geq 0} t^n$
- The coefficient 1 of t^n is obtained as a sum of reciprocal integers $1/d_T$ (a probability distribution on trees)
- the denominator d_T divides $n!$
- multiplying by $n!$ gives a decomposition $n! = \sum_T e_T$
- The positive integers $e_T = n!/d_T$ should count classes of permutations

Actually, their interpretation is well-known.

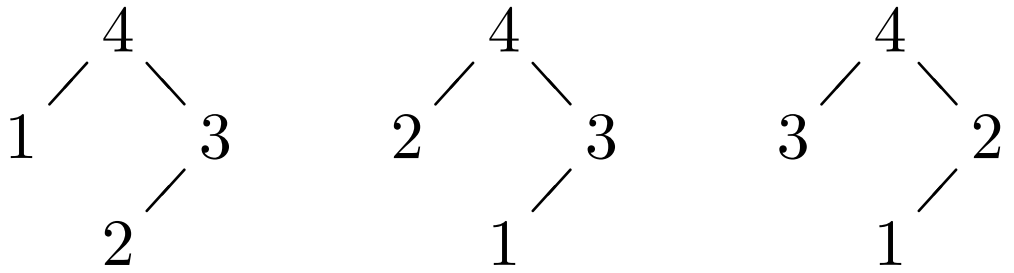
Decreasing tree of a permutation



The hook-length formula for decreasing trees

[Knuth - AOCP 3] The number of permutations whose decreasing tree has shape T is

$$n! \prod_{o \in T'} \frac{1}{HL(o)}$$



How to explain this ?

The new trick: Combinatorial Hopf Algebras

- based on permutations (**FQSym**, *aka* Malvenuto-Reutenauer)
- based on binary trees (**PBT**, Loday-Ronco), subalgebra of **FQSym**

There is a basis $\mathbf{G}_\sigma$ ($\sigma \in \mathfrak{S}_n$) of **FQSym** such that

$$\mathbf{G}_1^n = \sum_{\sigma \in \mathfrak{S}_n} \mathbf{G}_\sigma$$

and

$$\phi : \ \mathbf{G}_\sigma \longmapsto \frac{t^n}{n!}$$

is an algebra morphism, so that

$$x(t) = \frac{1}{1-t} = \phi\left((1-\mathbf{G}_1)^{-1}\right)$$

Moreover, the natural basis of **PBT** is

$$\mathbf{P}_T = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma$$

where $\mathcal{T}(\sigma)$ denotes the shape of the decreasing tree of σ.

What we have to do is to find a functional equation for $\mathbf{X} = (1-\mathbf{G}_1)^{-1}$ in **FQSym**, such that $B_T(1) = \mathbf{P}_T$.

Standardization of a Word

$$\begin{array}{ccc} \text{word of length } n & \longmapsto & \text{permutation of } \mathfrak{S}_n \\ w = l_1 l_2 \dots l_n & \longmapsto & \sigma = \mathrm{std}(w) \end{array}$$

for all $i < j$ set $\sigma(i) > \sigma(j)$ iff $a_i > a_j$.

Example: $\mathrm{std}(abcadbcaa) = 157296834$

a	b	c	a	d	b	c	a	a
a_1	b_5	c_7	a_2	d_9	b_6	c_8	a_3	a_4
1	5	7	2	9	6	8	3	4

Free Quasi-Symmetric Functions

Subspace of the free associative algebra $K\langle A\rangle$ spanned by

$$\mathbf{G}_\sigma(A) := \sum_{\mathrm{std}(w)=\sigma} w\,.$$

It is a subalgebra, with product rule for $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma = u\cdot v \\ \mathrm{Std}(u)=\alpha,\ \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma\,.$$

$$\mathbf{G}_{21}\mathbf{G}_{213} = \mathbf{G}_{54213}+\mathbf{G}_{53214}+\mathbf{G}_{43215}+\mathbf{G}_{52314}+\mathbf{G}_{42315}+\mathbf{G}_{32415} + \mathbf{G}_{51324} + \mathbf{G}_{41325} + \mathbf{G}_{31425} + \mathbf{G}_{21435}$$

The canonical character of FQSym

There are $\binom{m+n}{n}$ terms in the product $\mathbf{G}_\alpha \mathbf{G}_\beta$ for $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$, so that the map

$$\phi : \ \mathbf{G}_\sigma \longmapsto \frac{t^n}{n!} \quad (\sigma \in \mathfrak{S}_n)$$

is an algebra morphism. Thus, we have to find a derivation ∂ of **FQSym**, lifting $\frac{d}{dt}$, and such that $\partial \mathbf{X} = \mathbf{X}^2$.

A derivation in FQSym

$\partial =$ "remove n" operator : $\quad \partial \mathbf{G}_{316425} := \mathbf{G}_{31425}$

$$\phi(\partial F) = \frac{d}{dt}\phi(F)$$

The map ∂ is a derivation of **FQSym**:

$$\partial(AB) = A\partial(B) + \partial(A)B$$

It is the adjoint of the linear map $F \mapsto F \cdot \mathbf{F}_1$.

- $(\mathbf{F}_\sigma)$ dual basis of $(\mathbf{G}_\sigma)$: $\quad \langle \mathbf{F}_\sigma, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau}$.
- $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$

Dendriform structure

Splitting the product

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \mathbf{G}_\alpha \prec \mathbf{G}_\beta + \mathbf{G}_\alpha \succ \mathbf{G}_\beta$$

$$\mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma = u \cdot v, \max(u) > \max(v) \\ \mathrm{Std}(u) = \alpha,\ \mathrm{Std}(v) = \beta}} \mathbf{G}_\gamma \qquad \mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma = u \cdot v, \max(u) < \max(v) \\ \mathrm{Std}(u) = \alpha,\ \mathrm{Std}(v) = \beta}} \mathbf{G}_\gamma$$

Dendriform derivation

$$\partial(\mathbf{G}_\alpha \prec \mathbf{G}_\beta) = \partial \mathbf{G}_\alpha \prec \mathbf{G}_\beta, \qquad \partial(\mathbf{G}_\alpha \succ \mathbf{G}_\beta) = \mathbf{G}_\alpha \succ \partial \mathbf{G}_\beta$$

Differential equation in FQSym

$\mathbf{X} = (1 - \mathbf{G}_1)^{-1}$ satisfies

$$\partial \mathbf{X} = \mathbf{X}^2, \quad \mathbf{X}_0 = 1$$

Note that $\mathbf{X} = \sum_\sigma \mathbf{G}_\sigma = \sum_{w \in A^*} w\,.$

Can we lift to **FQSym** the bilinear map B so that the fixed point equation $\mathbf{X} = 1 + B(\mathbf{X}, \mathbf{X})$ hold ?

The bilinear map

$$\alpha \in \mathfrak{S}_m,\ \beta \in \mathfrak{S}_n,$$

$$B(\mathbf{G}_\alpha, \mathbf{G}_\beta) := \sum_{\substack{\gamma = u \ (m+n+1)\ v \\ \mathrm{Std}(u)=\alpha,\ \mathrm{Std}(v)=\beta}} \mathbf{G}_\gamma$$

$$B(\mathbf{G}_{21}, \mathbf{G}_{213}) = \mathbf{G}_{546213} + \mathbf{G}_{536214} + \mathbf{G}_{436215} + \mathbf{G}_{526314} + \mathbf{G}_{426315} +$$
$$\mathbf{G}_{326415} + \mathbf{G}_{516324} + \mathbf{G}_{416325} + \mathbf{G}_{316425} + \mathbf{G}_{216435}$$

By definition, $\partial B(\mathbf{G}_\alpha, \mathbf{G}_\beta) = \mathbf{G}_\alpha \mathbf{G}_\beta$, so that

$$\mathbf{X} = 1 + B(\mathbf{X}, \mathbf{X}).$$

Theorem

The binary tree solution of $\mathbf{X} = 1 + B(\mathbf{X}, \mathbf{X})$ is

$$B_T(1) = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma \tag{1}$$

$B_T(1) = \mathbf{P}_T$, the natural basis of Loday-Ronco algebra.

Corollary The number of permutations whose decreasing tree has shape T is $n! \prod_{o \in T} \frac{1}{HL(o)}$.

In fact, this is trivial. So, why bother with such a sophisticated argument ?

Specialisations

$\mathbb{A}$: any totally ordered alphabet.

$$\mathbf{G}_\sigma(\mathbb{A}) = \sum_{w \in \mathbb{A}^*,\ \mathrm{std}(w)=\sigma} w\,.$$

$\mathbb{X}$: commutative (totally ordered) alphabet

$$\mathbf{F}_\sigma(\mathbb{X}) = F_{\mathrm{Des}(\sigma^{-1})}(\mathbb{X})$$

F_I : Gessel's fundamental quasi-symmetric functions

A *virtual alphabet* is a way of denoting a Hopf algebra morphism $\varphi :$ **FQSym** $\to H$: one sets $\varphi(F) = F(\mathbb{A})$.

The alphabet $1/(1-q)$

$$\mathbb{X} = \frac{1}{1-q} := \{\ldots < q^n < q^{n-1} < \ldots < q < 1\}$$

Then if $\sigma \in \mathfrak{S}_n$,

$$\mathbf{G}_\sigma\left(\frac{1}{1-q}\right) = \frac{q^{\mathrm{iMaj}(\sigma)}}{(q)_n}$$

- $\mathrm{iMaj}(\sigma) = \mathrm{Maj}(\sigma^{-1})$
- $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

q-analog of the fundamental character

The map ϕ_q defined by

$$\phi_q(\mathbf{G}_\sigma) := \frac{q^{\mathrm{iMaj}(\sigma)} t^n}{[n]_q!} = (t(1-q))^n \ \mathbf{G}_\sigma \left(\frac{1}{1-q} \right)$$

is a homomorphism of algebras.

q-Specialization of the fixed point equation

$$x = 1 + B_q(x, x)$$

The bilinear map is now a q-integral:

If $f(t) = \phi_q(F)$ and $g(t) = \phi_q(G)$, then

$$B_q(f, g) := \phi_q(B(F, G)) = \int_0^t f(s) g(qs)\, d_q s\,.$$

- $\int_0^t s^n\, d_q s := \frac{t^{n+1}}{1+q+\cdots+q^n}$.

The q-hook length formula of Björner-Wachs (1989)

$$\sum_{\mathcal{T}(\sigma)=T} q^{\mathrm{iMaj}(\sigma)} = [n]_q! \prod_{o \in T} \frac{q^{\delta_o}}{[HL(o)]_q},$$

where δ_o is the number of nodes in the right subtree of o

A similar argument shows that the same formula holds for the generating series by inversion number, using free quantum quasi-symmetric functions $\mathbf{FQSym}_q$.

New results

The bilinear map is $B(U,V) = U \prec \mathbf{G}_1 \succ V$.

This remark allows us to use more complicated specializations (characters).

For example, the virtual alphabet $\mathbb{X}_{q,t} = \frac{|1-t}{1-q|}$ can be defined by

$$\mathbf{F}_\sigma(\mathbb{X}_{q,t}) = \frac{1}{(q)_n} \sum_{\epsilon \in \{\pm 1\}^n} (-t)^{|\epsilon|} q^{\mathrm{Maj}(\sigma,\epsilon)}$$

where (σ, ϵ) is a signed permutation, and $|\epsilon|$ the number of minus signs.

Then, $\mathbf{P}_T(\mathbb{X}_{q,t})$ is the generating function by major index and number of signs of signed permutations whose underlying unsigned permutations have a decreasing tree of shape T.

On can prove an analog of the "hook-content formula" for it, best stated in a recursive form:

Let T be a binary tree and T_1 (resp. T_2) be its left (resp. right) subtree. Let T_2' be the left subtree of T_2.

$$\mathbf{P}_T(\mathbb{X}_{q,t}) = \frac{(q^{\#T_2} - q^{\#T_2'}t)(1-q^{\#T_1})}{(1-q^{\#T_2'}t)(1-q^n)}\mathbf{P}_{T_1}(\mathbb{X}_{q,t})\mathbf{P}_{T_2}(\mathbb{X}_{q,t})$$

Another direction is to use more general trees.

Word quasi-symmetric functions

WQSym : quasi-symmetric functions in noncommuting variables

Packing of a word :

b	d	e	f	b	f	d
1	2	3	4	1	4	2

$$\mathbf{M}_u = \sum_{\mathrm{Pack}(w)=u} w$$

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{\substack{v,w;\ u=vw \\ Pack(v)=u',\mathrm{Pack}(w)=u''}} \mathbf{M}_u$$

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1121} + \mathbf{M}_{1132} + \mathbf{M}_{2221} + \mathbf{M}_{2231} + \mathbf{M}_{3321}.$$

A finite difference operator

Erase all occurences of the maximal letter:

$$(5211354)' = 21134$$

Linear map δ, $\quad \delta \mathbf{M}_u := (\mathbf{M}_u)' = \mathbf{M}_{u'}$.

$$\delta \mathbf{M}_u(A) = \mathbf{M}_u(A \hat{+} 1) - \mathbf{M}_u(A)$$

$A \hat{+} 1$ is the ordered sum of A and $\{1\}$

$$\delta(FG) = (\delta F)G + (\delta F)(\delta G) + F(\delta G)$$

The multilinear maps

Packed words $u_1, \ldots, u_k$

$$\mathbf{F}_k(\mathbf{M}_{u_1}, \ldots, \mathbf{M}_{u_k}) = \sum \mathbf{M}_w$$

where the sum runs over packed words w such that

$$w = w_1 m w_2 m \cdots m w_k \,,$$

$\mathrm{Pack}(w_i) = u_i \,,\ m = \max(w_1, \ldots, w_k) + 1.$

For example,

$\mathbf{F}_2(\mathbf{M}_{11}, \mathbf{M}_{21}) = \mathbf{M}_{11321} + \mathbf{M}_{11432} + \mathbf{M}_{22321} + \mathbf{M}_{22431} + \mathbf{M}_{33421}.$

Difference equation and fixed point equation

$$\mathbf{X} := (1 - q\mathbf{M}_1)^{-1} = \sum_u q^{|u|}\mathbf{M}_u = \sum_w q^{|w|} w\,.$$

Finite difference equation

$$\delta\mathbf{X} = \sum_{n\geq 2} q^{n-1}\mathbf{X}^n\,.$$

Fixed point equation

$$\mathbf{X} = 1 + \sum_{n\geq 2} F_n(\mathbf{X}, \mathbf{X}, \ldots, \mathbf{X})$$

Specialization

$$\begin{array}{rccc} \psi: & \mathbf{WQSym} & \longrightarrow & K[[\alpha]] \\ & \mathbf{M}_u & \longmapsto & \binom{\alpha}{\max(u)} \end{array}$$ is a morphism of algebras. It maps δ to the usual finite difference operator

$$\psi(\delta F) = \Delta\psi(F) = \psi(F)(\alpha+1) - \psi(F)(\alpha)$$

Fixed point equation using the discrete integral:

$$F_n(x_1, \ldots, x_n) = \sum_{i=0}^{\alpha-1} x_1(i)x_2(i)\cdots x_n(i)$$

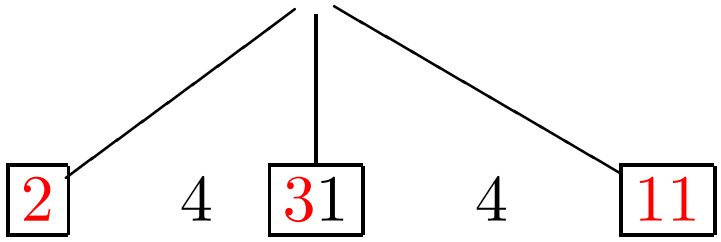
Plane tree of a word

$$\mathfrak{T}(2431411)$$

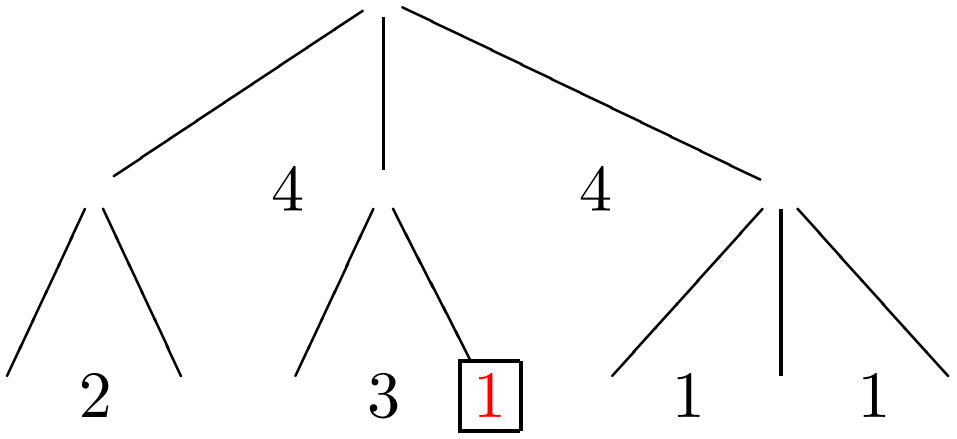
Plane tree of a word

2431411

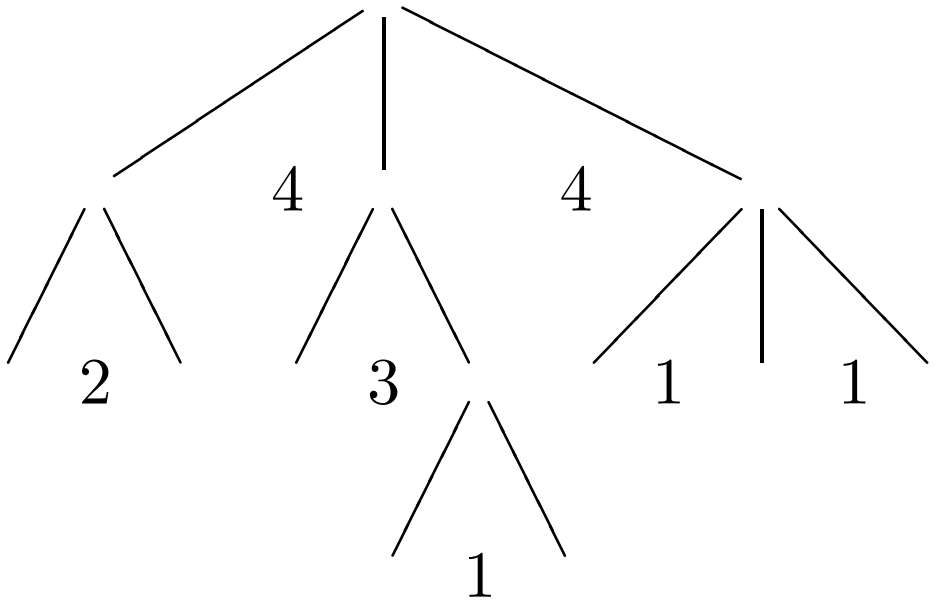
Plane tree of a word



Plane tree of a word



Plane tree of a word



"Hook-length" type formula for plane trees

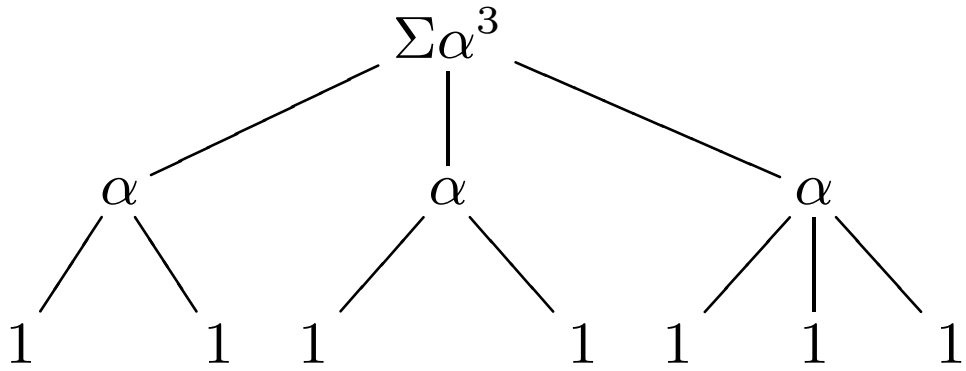
The coefficients c_k of the expansion of the term $F_T(1)$ in the plane tree solution

$$F_T(1) = \sum_k c_k \binom{\alpha}{k}$$

are the number of packed words u with maximal letter k such that $\mathfrak{T}(u) = T$.

Example

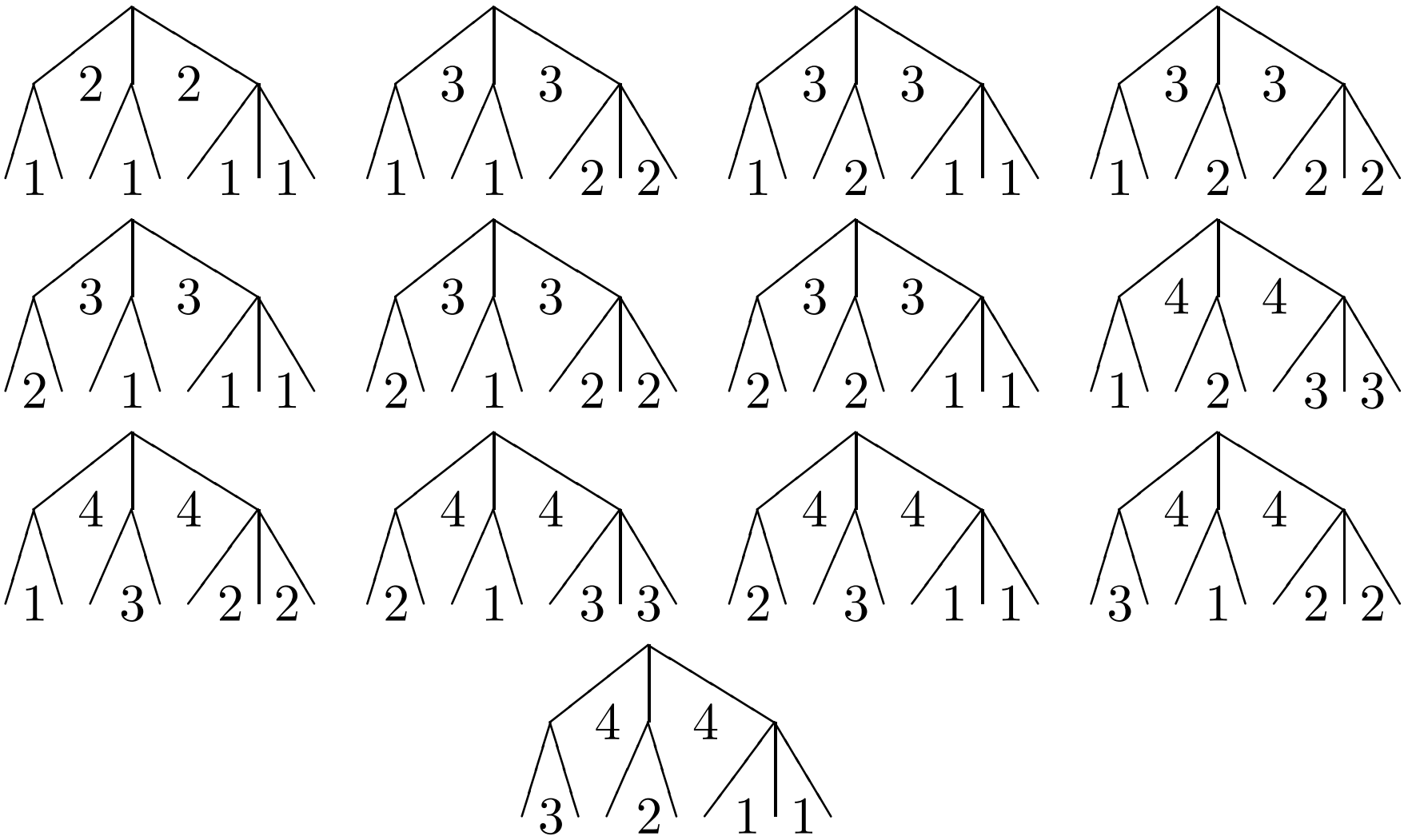
$\mathbf{F}_3(\mathbf{F}_2(1,1),\mathbf{F}_2(1,1),\mathbf{F}_3(1,1,1)) =$



gives

$$\sum_{i=0}^{\alpha-1} i^3 = 6\binom{\alpha}{4} + 6\binom{\alpha}{3} + \binom{\alpha}{2}$$

so that there are $6+6+1=13$ packed words whose plane trees have this shape.


2 2
1 1 1 1
3 3
1 1 2 2
3 3
1 2 1 1
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1 2 2 2
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2 1 1 1
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2 1 2 2
3 3
2 2 1 1
4 4
1 2 3 3
4 4
1 3 2 2
4 4
2 1 3 3
4 4
2 3 1 1
4 4
3 1 2 2
4 4
3 2 1 1

(q,t)-analog

This result can be interpreted as an evaluation of

$$\lim_{q\to 1} \mathbf{M}_T \left(\frac{|1-q^\alpha}{1-q|} \right)$$

where

$$\mathbf{M}_T = \sum_{\mathfrak{T}(u)=T} \mathbf{M}_u$$

and again, a (q,t)-analog can be obtained from the tridendriform structure of **WQSym**.

Let $\mathrm{int}(T)$ denote the list of internal nodes of T, root not included.

Let T be a plane tree with n nodes. Then

$$\mathbf{M}_T(\mathbb{X}_{q,t}) = \frac{1-t}{1-q^n} \prod_{i \in \mathrm{int}(T)} \frac{q^{s(i)-1} - t^{a(i)-1}}{1-q^{s(i)-1}},$$

where $a(i)$ is the arity of i and $s(i)$ the number of internal nodes of T below i.

Conclusion

- Other combinatorial Hopf algebras ? (e.g., can one explain the hook-content formula for Schur functions in a similar way ?)
- Other specializations ?
- How to connect this to others applications of tree expanded series
- This involves only the multiplicative structure. What about functional composition ?