Combinatorial Hopf algebras and inverse problems in representation theory

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Symmetric functions and representation theory

Frobenius: characters of $\mathfrak{S}_n$ can be encoded by homogeneous symmetric functions of degree n

$$ch(M) := \sum_{\lambda \vdash n} \chi_M(\lambda) \frac{p_\lambda}{z_\lambda}\,, \qquad \chi_M(\lambda) = \langle ch(M), p_\lambda \rangle\ .$$

Induction and restriction correspond to the natural product and coproduct on Sym

$$ch(M) \cdot ch(N) = ch(M \widehat{\otimes} N) = ch(M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}).$$

Other finite groups

- Spin symmetric groups (Schur, Q-functions)
- Wreath products $H \wr \mathfrak{S}_n$ (Specht, $Sym^{\otimes r}$)
- Finite linear groups (Green, Hall-Littlewood functions)

The resulting Hopf algebras (commutative, positive, self-dual) have been classified by Zelevinsky

Inductive towers of algebras

This game can be played with non semi-simple algebras as well:

- Sequence of finite dimensional associative algebras

$$A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_n \hookrightarrow \ldots ;$$

- embeddings $\rho_{m,n} \ : \ A_m \otimes A_n \hookrightarrow A_{m+n}$;
- associativity conditions:

$$\rho_{i,j+k} \circ (\mathrm{Id}_i \otimes \rho_{j,k}) = \rho_{i+j,k} \circ (\rho_{i,j} \otimes \mathrm{Id}_k) \, .$$

Grothendieck ring of a tower

$G_0(A_n)$ free abelian group over $\{V_\lambda\}$ (simple modules)
$K_0(A_n)$ free abelian group over $\{P_\lambda\}$ (indecomposable projective modules)

$$0 \to M \to L \to N \to 0 \;\Leftrightarrow\; [L] = [M] + [N]$$

Grothendieck rings:

$\mathcal{G} := \bigoplus_{n\geq 0} G_0(A_n)$, $\mathcal{K} := \bigoplus_{n\geq 0} K_0(A_n)$.

Product/coproduct defined by induction/restriction:

$$[M] \cdot [N] = [M \widehat{\otimes} N] = [M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}].$$

The inverse problem of representation theory

There are many "combinatorial" Hopf algebras resembling symmetric functions

- graded and connected
- based on combinatorial objects
- canonical basis with nonnegative integers as structure constants
- product and coproduct given by combinatorial algorithms
- morphisms to some kind of symmetric functions

Oldest example: Compositions: Quasisymmetric functions $QSym$ or its dual, Noncommutative Symmetric functions **Sym**

Other possibilities: Odd compositions (Peak algebras), Colored compositions, binary trees, rooted trees, parking functions, set partitions, permutations, colored permutations, integer matrices ...

Can one find a tower of algebra admitting a given combinatorial Hopf algebra as Grothendieck ring ?

It seems that the answer has to do with *deformations* ...

Example: Hecke Algebra $H_n(q)$

Generated by $(T_i)_{i=1..n-1}$ with relations

$$\begin{aligned}
(T_i - q)(T_i + 1) &= 0 && \text{for } 1 \leq i \leq n-1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leq i \leq n-2, \\
T_i T_j &= T_j T_i && \text{for } |i-j| \geq 2.
\end{aligned}$$

- Dimension $n!$
- Base $T_\sigma := T_{\sigma_1} T_{\sigma_2} \ldots T_{\sigma_k}$ where $\sigma_1 \sigma_2 \ldots \sigma_3$ is a reduced word for σ.

Special values of q

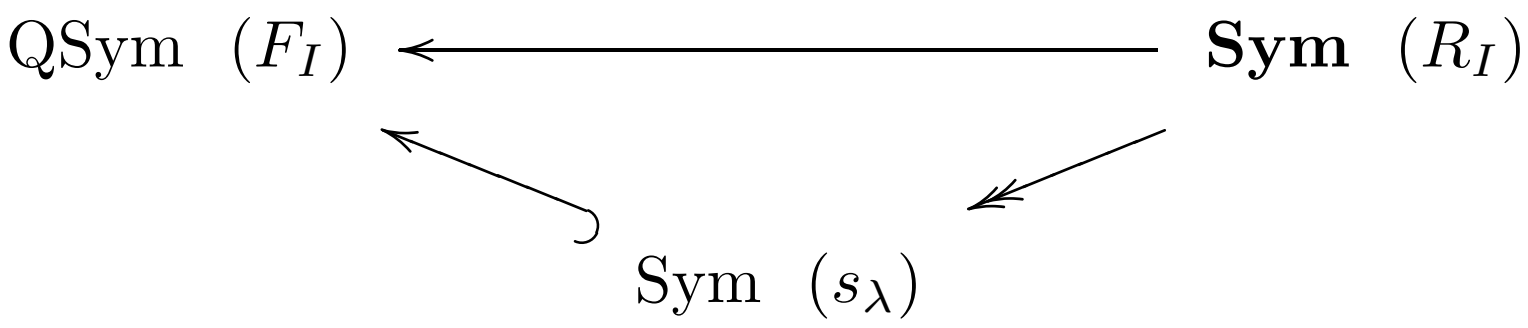
- $q = 1$: gives back $\mathbb{C}\mathfrak{S}_n$
- "generic" q: $\mathbb{C}\mathfrak{S}_n$ again (non trivial)
- q a primitive k-th root of unity: $\mathcal{G} = Sym/(p_{mk})$, $\mathcal{K} = \mathbb{Z}[p_r | r \not\equiv 0[k]]$
- $q = 0$ is what we are looking for ...

Cartan-Brauer triangle of $H_n(0)$

The degenerate Hecke Algebra: $A_n := H_n(0)$.

$$\mathcal{G} \simeq \mathrm{QSym}\,, \qquad [S_I] = F_I = L_I\,.$$

$$\mathcal{K} \simeq \mathbf{Sym}\,, \qquad [P_I] = R_I = F_I^*\,.$$



Comments

- This case was easy enough: representations were known [Norton 1975]
- Grothendieck rings worked out for the NCSF project [Duchamp-Krob-Leclerc-T. 1995]
- Natural filtration led to quantum quasi-symmetric functions $QSym_q$ [Ung-T. 1996]
- $QSym_q$ led to **FQSym** [Duchamp-Hivert-T. 2002]
- duality with quantum groups: hypoplactic monoid [Krob-T. 1997]

- the hypoplactic monoid led to the sylvester monoid [Hivert-Novelli-T. 2002] and to the Schröder algebra [Novelli-T. 2003]
- **FQSym** led to **MQSym** [Duchamp-Hivert-T. 2002] and to parking functions [Novelli-T. 2003]

It would be interesting to find other examples!

Recently solved

- (Peak, Peak*) [Stembridge 1997] Subalgebra of the descent algebra, basis (Θ_P, Π_P) indexed by peak compositions, solved by [N. Bergeron-Hivert.T. 2004]
- ($\mathrm{QMR}^{(r)}$, $\mathbf{MR}^{(r)}$), [Mantaci-Reutenauer 1995] Descent algebra for wreath product, basis $(F_{(I,c)}, R_{(I,c)})$ indexed by colored compositions, solved by [Hivert-Novelli-T. 2004]

Peaks and 0-Hecke-Clifford Algebra

Hecke-Clifford superalgebra [Olshanski 1992]

$$
\begin{aligned}
t_i^2 &= (q - q^{-1})t_i + 1 && \text{for } 1 \le i \le n-1, \\
t_i t_j &= t_j t_i && \text{for } |i-j| > 1, \qquad (1) \\
t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} && \text{for } 1 \le i \le n-2.
\end{aligned}
$$

$$
\begin{aligned}
t_i c_j &= c_j t_i && \text{for } i \ne j, j+1, \\
t_i c_i &= c_{i+1} t_i && \text{for } 1 \le i \le n-1, \\
& && \qquad (2) \\
(t_i + q - q^{-1}) c_{i+1} &= c_i (t_i + q - q^{-1}) && \text{for } 1 \le i \le n-1.
\end{aligned}
$$

The degenerate version

Set $t_i = q^{-1}T_i$ and take the limit $q \to 0$

The 0-Hecke-Clifford superalgebra $HCl_n(0)$:

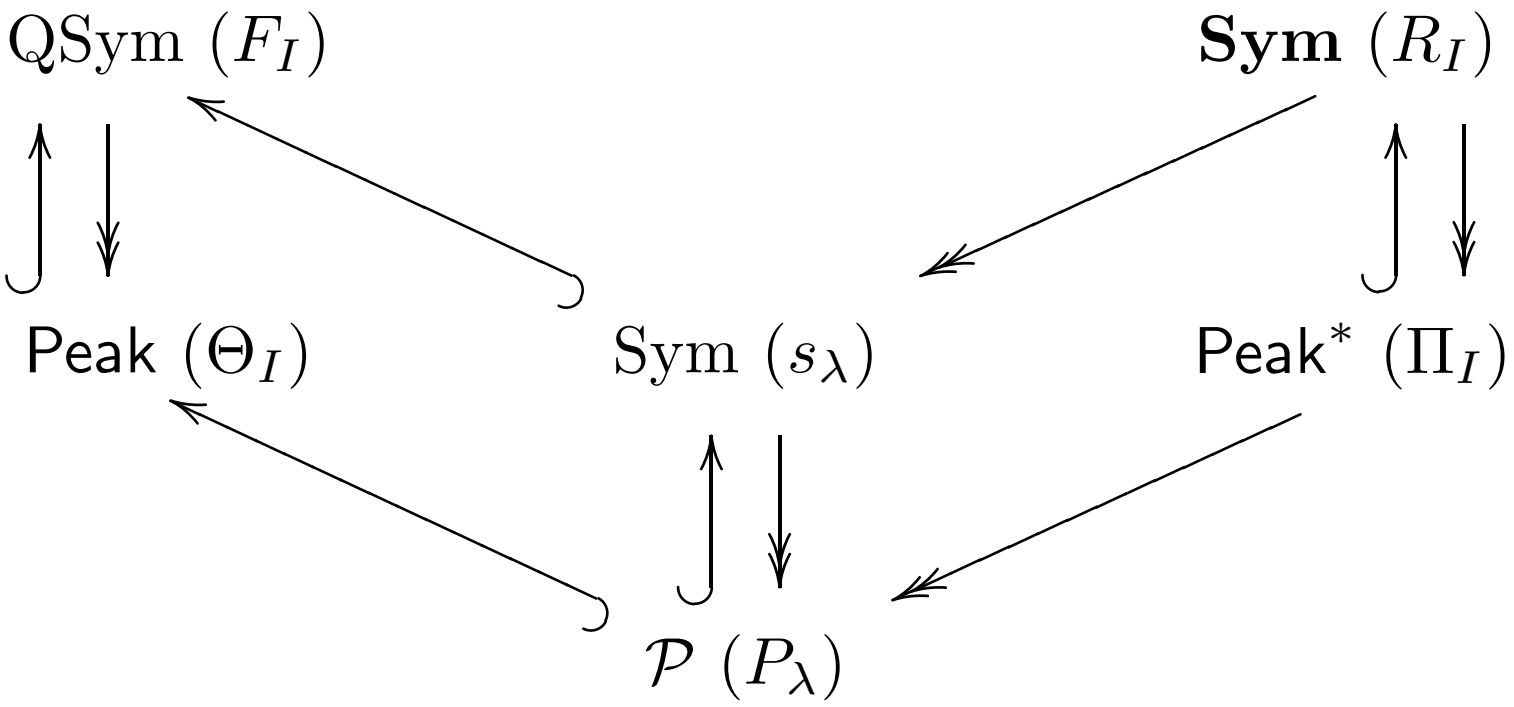
generated by the 0-Hecke algebra and the Clifford algebra, with the cross-relations

$$\begin{aligned} T_i c_j &= c_j T_i && \text{for } i \neq j, j+1, \\ T_i c_i &= c_{i+1} T_i && \text{for } 1 \leq i \leq n-1, \\ (T_i + 1)c_{i+1} &= c_i(T_i + 1) && \text{for } 1 \leq i \leq n-1. \end{aligned} \tag{3}$$

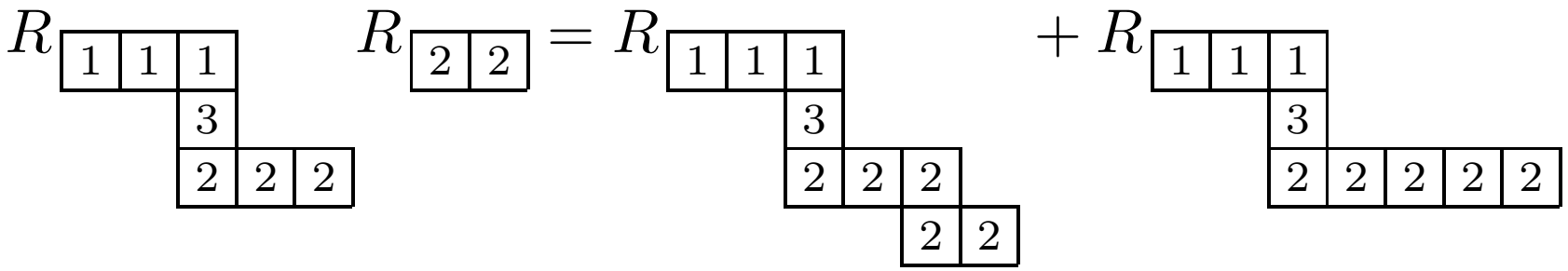
Sketch of the theory

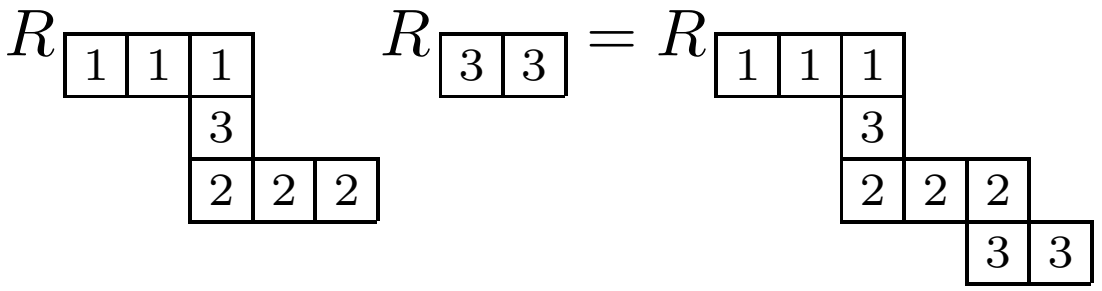
- take the quasi-symmetric charateristic of restrictions to $H_n(0)$
- induce simple modules S_I of $H_n(0)$
- the induced (super)-modules are isomorphic whenever compositions have the same peak sets
- all simple modules occur as composition factors of such induced modules
- their characteristics are Stembridge's Θ-functions
- the ring of projective modules is obtained by duality

The complete diagram



Mantaci-Reutenauer : Colored Ribbons





Colored permutations and wreath products

Find a degenerate Hecke algebra associated with the wreath product $Z_l \wr \mathfrak{S}_n$:

Should be somthing like a 0-Ariki-Koike cyclotomic Hecke algebra.

Problem: find the correct presentation where one can put $q = 0$ and get the desired algebra (computer exploration: the usual presentation does not work!).

Shoji's presentation

$$(T_i - q)(T_i + 1) = 0$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i$$

$$(\xi_j - u_1) \cdots (\xi_j - u_r) = 0$$

$$\xi_i \xi_j = \xi_j \xi_i$$

$$T_i \xi_i = \xi_{i+1} T_i - (q-1) \sum_{a<b} (u_b - u_a) P_a(\xi_i) P_b(\xi_{i+1})$$

$$T_i(\xi_{i+1} + \xi_i) = (\xi_{i+1} + \xi_i) T_i$$

$$T_i \xi_j = \xi_j T_i \quad (j \neq i+1, i)$$

Lagrange interpolator basis

$C := 1, 2 \ldots, r$ and $u_1, \ldots, u_r$ distinct complex numbers.

$$P(X) := \prod_l (X - u_l) \quad \text{and} \quad P_k(X) := \prod_{l \neq k} \frac{X - u_l}{u_k - u_l}.$$

Clearly, $P_k(X)$ is a basis of $\mathbb{C}[X]/P$, so that
$\mathbb{C}[\xi_1, \ldots, \xi_n]/(P(\xi_i))_i$ has for basis

$$L_{\mathbf{c}} := P_{c_1}(\xi_1) \cdots P_{c_n}(\xi_n) \text{ where } \mathbf{c} = (c_1, \ldots, c_n) \in C^n.$$

The 0-Ariki-Koike-Shoji algebras $\mathcal{H}_{n,r}(0)$

0-Hecke algebra + commuting variables $(\xi_i)_{i=1..n}$
satisfying the relations $P(\xi_i) = 0$

Basis $(L_{\mathbf{c}} T_\sigma)$ where $\mathbf{c}$ is a word on C and $\sigma \in \mathfrak{S}_n$.

$$T_i L_{\mathbf{c}} = L_{\mathbf{c}\sigma_i} T_i + \begin{cases} -L_{\mathbf{c}} & \text{if} \quad c_i < c_{i+1}, \\ 0 & \text{if} \quad c_i = c_{i+1}, \\ L_{\mathbf{c}\sigma_i} & \text{if} \quad c_i > c_{i+1}. \end{cases}$$

Main result

Grothendieck ring of the tower of $\mathcal{H}_{n,r}(0)$:

$$\mathcal{G}^{(r)} := \bigoplus_{n \geq 0} G_0(\mathcal{H}_{n,r}(0))$$

Characteristic map ch :

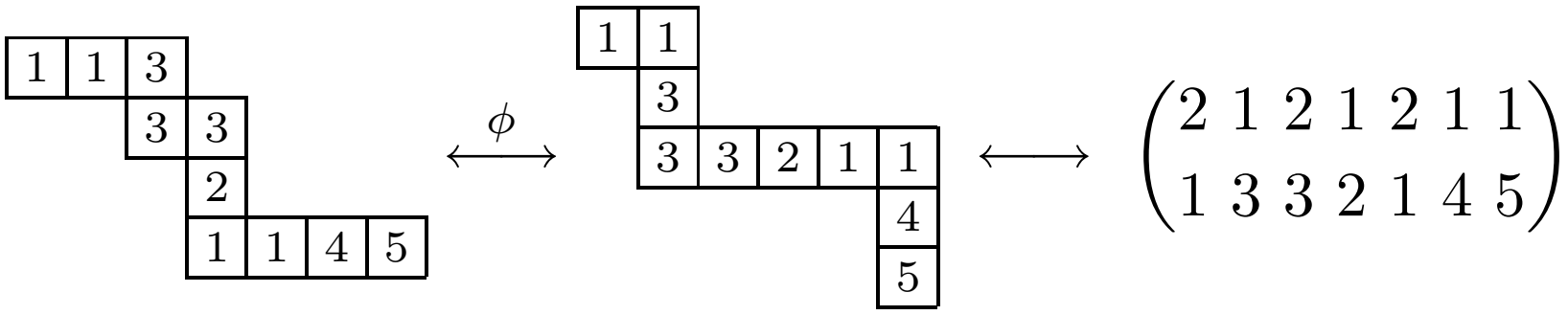
$$\begin{array}{rl} \mathcal{G}^{(r)} & \to \mathrm{QMR}^{(r)} \\ [S_{[I,\mathbf{c}]}] & \mapsto F_{[I,\mathbf{c}]}. \end{array}$$

ch is an isomorphism of Hopf algebras.

Cyclotomic Ribbons

Cyclorobbon: filling of a ribbon with colors weakly *increasing* in rows and weakly *decreasing* in columns.

Anticycloribbon: filling of a ribbon with colors weakly *decreasing* in rows and weakly *increasing* in columns.



Simple modules

Simple modules of the 0-Hecke Algebra $S_I = \mathbb{C}\eta_I$

$$T_j \eta_I = \begin{cases} -\eta_I & \text{if } j \in \mathrm{Des}(I), \\ 0 & \text{otherwise.} \end{cases}$$

Simple modules of $\mathcal{H}_{n,r}(0)$

$$\eta_{[I,\mathbf{c}]} := L_{\mathbf{c}}\ \eta_I = \eta_{I'} L_{\mathbf{c}'}$$

where $[I', \mathbf{c}']$ are the shape and the mirror image of the reading of the anticycloribbon $\phi([I, \mathbf{c}])$.

Simple modules

Let $[I, \mathbf{c}]$ be a cyclo-ribbon. Then

$$S_{[I,\mathbf{c}]} := \mathcal{H}_{n,r}(0)\eta_{[I,\mathbf{c}]} = \mathbb{C}\eta_{[I,\mathbf{c}]}$$

form a full set of pairwise non isomorphic simple modules of $\mathcal{H}_{n,r}(0)$. They are all of dimension 1.

$$\xi_i \eta_{[I,\mathbf{c}]} = c_i \eta_{[I,\mathbf{c}]}$$

$$T_j \eta_{[I,\mathbf{c}]} = \begin{cases} -\eta_{[I,\mathbf{c}]} & \text{if } j \in \mathrm{Des}(\Phi(I)), \\ 0 & \text{otherwise.} \end{cases}$$

Projective modules

$$\mathcal{K} = \bigoplus_{n \geq 0} K_0(\mathcal{H}_{n,r}(0)) \simeq \mathbf{MR}^{(r)}$$

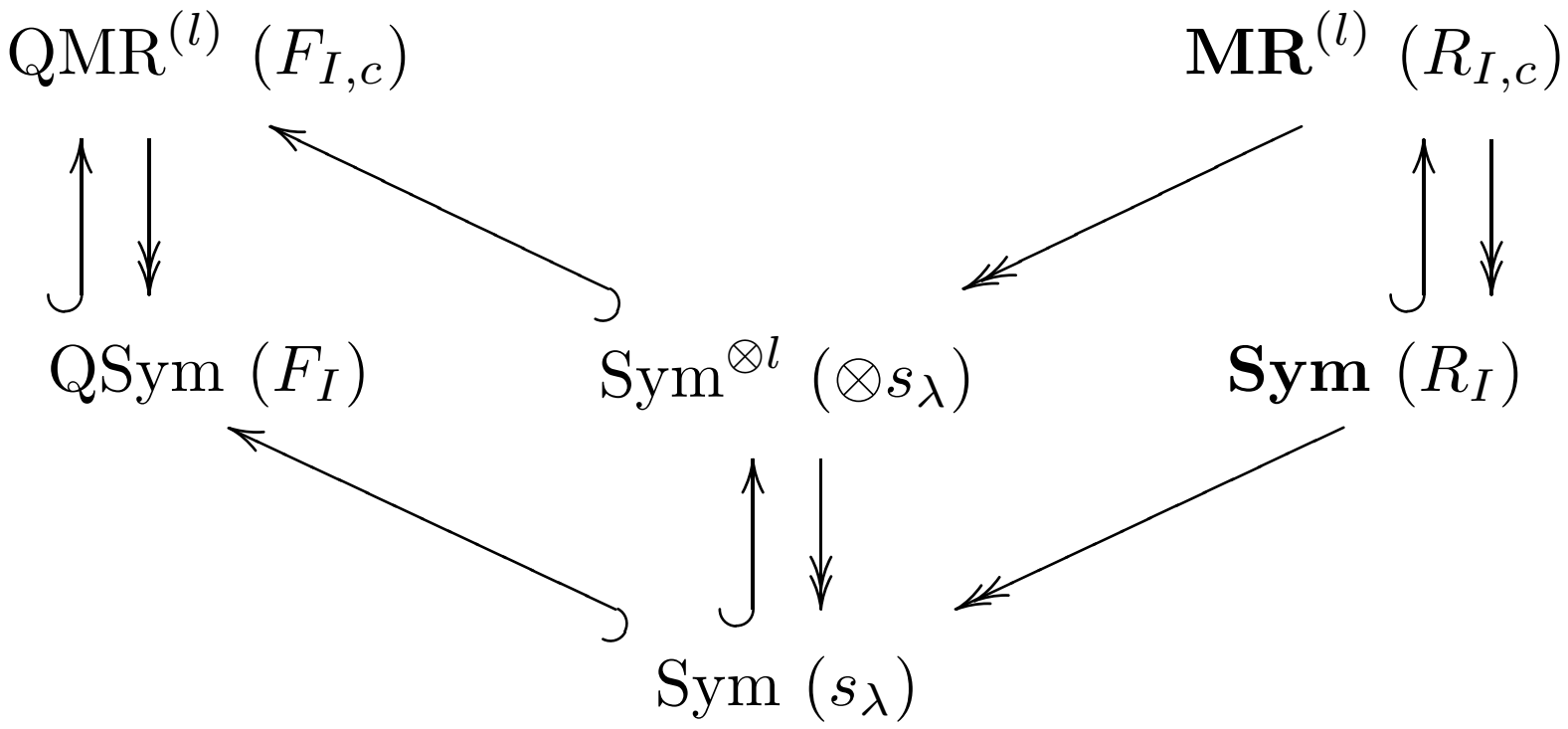
is the Mantaci-Reutenauer algebra

The classes of the indecomposable projective modules $P_{(I,u)}$ are the colored ribbons, which satisfy the very simple multiplication rule:

$$R_{(I,u)} R_{(J,v)} = R_{(I,u)\cdot(J,v)} + \begin{cases} R_{(I,v)\triangleright(J,v)} & \text{if } u_p = v_1, \\ 0 & \text{if } u_p \neq v_1. \end{cases} \tag{4}$$

As a consequence, all $P_{(I,u)}$ can be factored into induction products of monochromatic modules, which are given by $H_n(0)$

Morphisms


QMR$^{(l)}$ $(F_{I,c})$
$\mathbf{MR}^{(l)}$ $(R_{I,c})$
QSym (F_I)
Sym$^{\otimes l}$ $(\otimes s_\lambda)$
Sym (R_I)
Sym (s_λ)

Perspectives

Most of the known combinatorial Hopf algebras come in several flavors, which can probably be cast into continuous families

For example, the Loday-Ronco algebra of plane binary trees has a cocommutative version, whose dual can be realized by polynomials which refine quasi-symmetric functions [Novelli-T. 2003]

One can reasonably conjecture that at least this one is a Grothendieck ring

If this is true, the other algebras of trees should admit

similar interpretations

The relevant tower of algebras are probably specializations of affine or toroidal Hecke algebras

The difficult point is to find the good presentation: two presentations of the same algebra can have non isomorphic degeneracies