Noncommutative Symmetric Bessel Functions

Jean-Yves Thibon

Université Paris-Est/Marne-la-Vallée

(joint with J.-C. Novelli)

An exercise on 0-Hecke algebras

- Symmetric functions encode representations of symmetric groups
- The tensor product of representations gives rise to the *internal product*
- Quasi-symmetric functions encode representations of 0-Hecke algebras
- Again, in this case, a tensor product can be defined
- So what ?

Symmetric functions and representation theory

Characters of $\mathfrak{S}_n \longleftrightarrow$ symmetric functions of degree n

$$\mathrm{ch}(M) := \sum_{\lambda \vdash n} \chi_M(\lambda) \frac{p_\lambda}{z_\lambda}, \qquad \chi_M(\lambda) = \langle ch(M), p_\lambda \rangle .$$

Induction corresponds to the natural product on Sym

$$\mathrm{ch}(M) \cdot \mathrm{ch}(N) = \mathrm{ch}(M \widehat{\otimes} N) = \mathrm{ch}(M \otimes N \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}).$$

Tensor (Kronecker) product corresponds to the internal product $*$

$$\mathrm{ch}(M) * \mathrm{ch}(N) = \mathrm{ch}(M \otimes_{\mathbb{C}\mathfrak{S}_n} N)$$

Duality

The ordinary (polynomial product) of Sym is dual to the "sum" coproduct

$$\Delta(f) = f(X+Y) \quad (\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n)$$

The internal product is dual to the "product" coproduct

$$\delta(f) = f(XY) \quad (\delta(p_n) = p_n \otimes p_n)$$

with the usual encoding $f(X)f(Y) \equiv f \otimes g$

The 0-Hecke Algebra $H_n(0)$

Generated by $(T_i)_{i=1..n-1}$ with relations

$$\begin{aligned} T_i^2 &= -T_i && \text{for } 1 \leq i \leq n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i && \text{for } |i-j| \geq 2. \end{aligned}$$

- Dimension $n!$
- Basis $T_\sigma := T_{\sigma_1} T_{\sigma_2} \dots T_{\sigma_k}$ (reduced word)
- 2^{n-1} simple modules S_I, all 1-dimensional, parametrized by compositions I of n

Quasisymmetric characteristic

$\mathsf{S}_I = \mathbb{C}\eta_I$, where $T_i\eta_I = -\eta_I$ if $i \in \mathrm{Des}(I)$, $= 0$ otherwise.

If we set

$$\mathrm{ch}(\mathsf{S}_I) = F_I \ \in \ QSym$$

(Gessel's fundamental basis of quasi-symmetric functions), then, again

$$\mathrm{ch}(M) \cdot \mathrm{ch}(N) = \mathrm{ch}(M \widehat{\otimes} N) = \mathrm{ch}(M \otimes N \uparrow_{H_m(0) \otimes H_n(0)}^{H_{m+n}(0)})$$

and so on. This can be summarized in terms of Grothendieck rings as follows.

$\boxed{\textbf{Cartan-Brauer triangle of } H_n(0)}$

Grothendieck rings of the tower $A_n := H_n(0)$.

$$\mathcal{G} \simeq QSym\,, \qquad [S_I] = F_I = L_I\,.$$

$$\mathcal{K} \simeq \mathbf{Sym}\,, \qquad [P_I] = R_I = F_I^*\,.$$

$$\begin{tikzcd}
QSym\ \ (F_I) & & \mathbf{Sym}\ \ (R_I) \arrow[ll] \arrow[dl, two heads] \\
& Sym\ \ (s_\lambda) \arrow[ul, hook] &
\end{tikzcd}$$

Tensor products

No natural tensor products for $H_n(q)$-modules, but ...

... $H_n(0)$ is the algebra of a monoid (in more than one way): $H_n(0) = \mathbb{C}[\Pi_n]$, where Π_n is generated by idempotents $\pi_1, \ldots, \pi_{n-1}$ ($\pi_i = -T_i$) satisfying the braid relations.

[Other choice: the idempotents $\bar{\pi}_i = 1 - \pi_i$]

This yields a canonical coproduct on $H_n(0)$

$$\delta_\wedge \pi = \pi \otimes \pi \text{ for } \pi \in \Pi_n \tag{1}$$

so that tensor products of $H_n(0)$-modules can be defined.

$\boxed{\textbf{Internal product on } Qsym}$

For H, K, I compositions of n,

$$\mathsf{S}_H \otimes \mathsf{S}_K = \mathsf{S}_I \quad \text{where } \mathrm{Des}(I) = \mathrm{Des}(H) \cap \mathrm{Des}(K)\,.$$

This induces an internal product $\wedge$ on $QSym_n$,

$$F_H \wedge F_K = F_I$$

where $I = H \wedge K$, that is, $\mathrm{Des}(I) = \mathrm{Des}(H) \cap \mathrm{Des}(K)$.

Duality again

This looks deceptively trivial.

In fact, the interesting object is the dual coproduct on **Sym**:

$$\gamma_\wedge R_I = \sum_{\mathrm{Des}(I)=\mathrm{Des}(H)\cap\mathrm{Des}(K)} R_H \otimes R_K \,.$$

Theorem

The coproduct $\gamma_\wedge$ is a morphism for the ordinary (outer) product of noncommutative symmetric functions:

$$\gamma_\wedge(FG) = \gamma_\wedge(F)\gamma_\wedge(G)$$

It is completely determined by the images of the elementary functions, $\gamma_\wedge \Lambda_n = \Lambda_n \otimes \Lambda_n$, whence the inversion formula

$$\left(\sum_{n\geq 0}(-1)^n \Lambda_n \otimes \Lambda_n\right)^{-1} = \sum_{\mathrm{Des}(H)\cap\mathrm{Des}(K)=\emptyset} R_H \otimes R_K\,.$$

Connection with Bessel functions

Under the exponential specialization

$$S_n \to S_n(t\mathbb{E}) = \frac{t^n}{n!} \;\Rightarrow\; R_I(t\mathbb{E}) = d_I \frac{t^n}{n!}$$

where d_I is the number of permutations with descent composition I, the previous equation becomes

$$\frac{1}{J_0(2t)} = \sum_{n\geq 0} a_n \frac{t^{2n}}{(n!)^2}\ .$$

where a_n is the number of pairs of permutations $(\sigma,\tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $\mathrm{Des}(\sigma) \subseteq \mathrm{Des}(\tau)$, a classical result of Carlitz, Scoville and Vaughan.

Noncommutative Bessel functions

A and B two mutually commuting alphabets. Recall that $\sigma_z = \sum S_n z^n$ and $\lambda_z = \sum \Lambda_n z^n = 1/\sigma_{-z}$. The noncommutative Bessel functions $\mathbf{J}_n(A, B)$ are defined by the generating series

$$\sum_{n \in \mathbb{Z}} z^n \mathbf{J}_n(A, B) = \lambda_{-1/z}(A) \sigma_z(B)$$

that is,

$$\mathbf{J}_n(A, B) = \sum_{m \geq 0} (-1)^m \Lambda_{m-n}(A) S_m(B) \, .$$

For $A = B = t\mathbb{E}$, this is the usual Bessel function $J_n(2t)$.

A couple of tricks

- The right residual operator $\overleftarrow{\partial}$ (for J_1/J_0-type formulas)
- θ-alphabets (Carlitz-Koszul duality, encoding of various objects by biwords)

$\overleftarrow{\partial}$ and J_1/J_0

Linear operator on **Sym** (acting on the right)

$$S^{(i_1,\ldots,i_r)} \overleftarrow{\partial} = S^{(i_1,\ldots,i_r-1)} .$$

It satisfies

$$(FG) \overleftarrow{\partial} = F \cdot (G \overleftarrow{\partial}) + (F \overleftarrow{\partial}) \cdot G_0 ,$$

where G_0 denotes the constant term of G, and

$$R_I \overleftarrow{\partial} = \begin{cases} R_{i_1,\ldots,i_r-1} & \text{if } i_r > 1 , \\ 0 & \text{if } i_r = 1 \end{cases}$$

In particular, if $G_0 = 0$,

$$(1-G)^{-1} \overleftarrow{\partial} = (1-G)^{-1}(G \overleftarrow{\partial})$$

Applying this with $\overleftarrow{\partial} = \overleftarrow{\partial}_B$ acting only on $\mathbf{Sym}(B)$ to

$$\mathbf{J}_0(A,B)^{-1} = (1 - \sum_{n\geq 1} (-1)^{n-1} \Lambda_n(B) S_n(A))^{-1} = \sum_I S^I(A) R_I(B)$$

we obtain

$$\mathbf{J}_0(A,B)^{-1} \mathbf{J}_{-1}(A,B) = \sum_I S^I(A)(R_I \overleftarrow{\partial} (B))$$

Example

Specializing $A = B = t\mathbb{E}$, we get another result of Carlitz et al: The coefficient c_n in

$$\frac{J_1(2t)}{J_0(2t)} = \sum_{n\geq 1} c_n \frac{t^{2n-1}}{(n-1)!n!}$$

is equal to the number of pairs of permutations $(\alpha, \beta) \in \mathfrak{S}_n \times \mathfrak{S}_n$ such that $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(\beta)$ and $\beta(n) = n$.

One can take more complicated specializations. To interpret these, we need θ-alphabets.

θ-Alphabets

$\theta \subseteq A \times A$ any binary relation.

$\overline{\theta}$ complement of θ in $A \times A$. Let

$$X = X(A;\theta) = \{w = a_1 \cdots a_n \in A^* | a_1 \theta a_2 \theta \ldots \theta a_n\}\,,$$

$$Y = Y(A;\theta) = X(A;\overline{\theta})\,,$$

where $a\theta b$ means $(a,b) \in \theta$.

The empty word 1 and the letters belong to both X and Y.

The θ-specialization $\mathbf{Sym}(A;\theta)$ is defined by

$$\Lambda_n(A;\theta) = \sum_{w\in X\cap A^n} w\,. \tag{2}$$

Carlitz-Koszul duality for alphabets:

The complete symmetric functions $S_n(A;\theta)$ are given by

$$S_n(A;\theta) = \Lambda_n(A;\overline{\theta})\,.$$

More generally, if $\theta\mathrm{Adj}(w) = \{i|a_i\theta a_{i+1}\}$ (the θ-adjacency set of $w = a_1a_2\cdots a_n$), and if $C_\theta(w)$ is the associated composition of n,

$$R_I(A;\theta) = \sum_{C_\theta(w)=I} w\,.$$

Double Eulerian polynomials (Fédou-Rawlings)

The noncommutative Bessel function $\mathbf{J}_0(A,B)$ is the generating series of θ elementary symmetric functions, for the product alphabet $\mathbb{J} = (A \times B, \theta)$, with

$$(a,b)\theta(a',b') \;\Leftrightarrow\; a > a' \text{ and } b \le b'\,.$$

We denote words over $A \times B$ by biwords

$$w = [u,v] = \begin{bmatrix} u \\ \\ v \end{bmatrix} \;\; u \in A^n\,,\;\; v \in B^n\,.$$

Observing that

$$\theta\mathrm{Adj}([u,v]) = \mathrm{Des}(u) \cap \overline{\mathrm{Des}(v)} = \mathrm{Des}(u)\backslash\mathrm{Des}(v)\,,$$

we can now write

$$\sum_{w=(u,v)\in(A\times B)^*} t^{\theta\mathrm{adj}(w)} z^{l(w)} w = \frac{1-t}{\mathbf{J}_0((1-t)z;A,B)-t}$$

$$= \sum_K z^{|K|} t^{l(K)-1} R_K(A,B;\theta)$$

where from now on we shall use the notation

$$\mathbf{J}_0(x;A,B) = \lambda_{-x}(\mathbb{J}) = \lambda_{-x}(A,B;\theta)\,.$$

Summing over all specializations $A_i = [i+1]_q$ and $B_j = [j+1]_p$, one finds [Fédou-Rawlings]:

$$\sum_{i,j\geq 0} x^i y^j \frac{1-t}{\mathbf{J}_0((1-t)z; A_i, B_j) - t}$$
$$= \sum_{n\geq 0} \frac{z^n}{(x;q)_{n+1}(y;p)_{n+1}} \times$$
$$\times \sum_{\alpha,\beta\in\mathfrak{S}_n} t^{\mathrm{desris}(\alpha,\beta)} x^{\mathrm{des}(\alpha^{-1})} y^{\mathrm{des}(\beta^{-1})} q^{\mathrm{coimaj}(\alpha)} p^{\mathrm{coimaj}(\beta)},$$

where $\mathrm{desris}(\alpha,\beta) = |\mathrm{Des}(\alpha) \backslash \mathrm{Des}(\beta)|$.

Heaps of segments and polyominos

Bessel functions and their multiparameter analogs play a crucial role in the enumerative theory of polyominos (Bousquet-Mélou, Viennot, Delest, Fédou ...).

Combinatorial proofs of enumerative results can be achieved by means of Viennot's theory of heaps of segments.

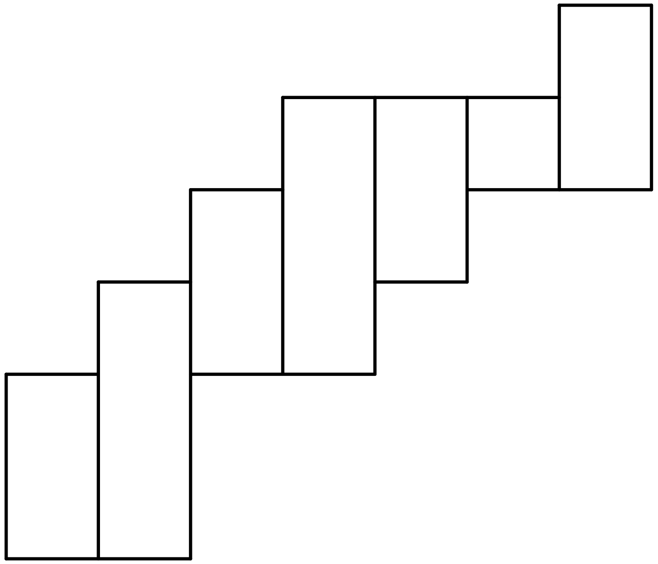
This can also be conveniently formulated in terms of θ-noncommutative symmetric functions.

Example: staircase polyominos

A *parallelogram* (or *staircase*) *polyomino* P, (a connected skew Young diagram) can be encoded as a biword

$$w = a_{i_1 j_1} \cdots a_{i_n j_n} = \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

where j_k is the height of the kth column C_k, and i_k is the number of common rows between C_k and C_{k+1} (with a conventional value $i_n = 1$ for the last column).



is encoded by the biword

$$\begin{bmatrix} 2122111 \\ 2323212 \end{bmatrix}$$

The biwords corresponding to polyominos are the words over the alphabet

$$A = \{a_{ij} | i \leq j\}$$

satisfying the θ-adjacency conditions

$$a_{i_k j_k} \, \theta \, a_{i_{k+1} j_{k+1}} \iff i_k \leq j_{k+1}$$

and the ending condition

$$i_n = 1 \, .$$

The generating series (by length) of such biwords is obtained from

$$\lambda_t(A,\theta) = \left(1 - \sum_{n\geq 1}(-1)^{n-1}t^n \sum_{i_k > j_{k+1}} \begin{bmatrix} i_1 \cdots i_n \\ j_1 \cdots j_n \end{bmatrix}\right)^{-1}$$

by applying the operator

$$D = \sum_{j\geq 1} \overleftarrow{\partial}_{a_{1j}}\ a_{ij}\ ,$$

which yields

$$\left(1-\sum_{n\geq 1}(-1)^{n-1}t^{n}\sum_{i_k>j_{k+1}}\begin{bmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{bmatrix}\right)^{-1}\times$$

$$\times\sum_{n\geq 1}(-1)^{n-1}t^{n}\sum_{i_k>j_{k+1};i_n=1}\begin{bmatrix} i_1\cdots i_n \\ j_1\cdots j_n \end{bmatrix}.$$

Again, it acquires the structure J_1/J_0 once A is specialized to

$$a_{ij}=xy^{j-i}q^{j},$$

giving the generating series by width, height and area (a result of Bousquet-Mélou).