Flag Varieties and the Yang-Baxter Equation

Alain Lascoux[*], Bernard Leclerc[†] and Jean-Yves Thibon[‡]

April 9, 1996

Abstract

We investigate certain bases of Hecke algebras defined by means of the Yang-Baxter equation, which we call Yang-Baxter bases. These bases are essentially self-adjoint with respect to a canonical bilinear form. In the case of the degenerate Hecke algebra, we identify the coefficients in the expansion of the Yang-Baxter basis on the usual basis of the algebra with specializations of double Schubert polynomials. We also describe the expansions associated to other specializations of the generic Hecke algebra.

1 Introduction

Yang's original motivation for introducing the Yang-Baxter equation [29] was the n-body problem on a circle with Hamiltonian

$$H(\mathbf{x}) = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j) \ ,$$

where δ is the Dirac distribution. The problem was to solve the Schrödinger equation

$$H(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

with periodic boundary conditions.

Using a variant of the Bethe Ansatz, Yang looked for solutions of the form

$$\psi(\mathbf{x}) = \sum_{\tau \in \mathfrak{S}(n)} \theta(\mathbf{x}^\tau) \sum_{\mu} A_\mu(\tau) e^{\mathrm{i}\mathbf{u}^\mu \cdot \mathbf{x}^\tau}$$

where $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{C}^n$ is the vector of spectral parameters, $\mathbf{u}^\mu = (u_{\mu(1)}, \ldots, u_{\mu(n)})$, and θ is the characteristic function of the domain $x_1 < x_2 < \ldots < x_n$. The unknown coefficients $A_\mu(\tau)$ form an $n! \times n!$ matrix, and it is convenient to regard each A_μ as a

[*]Institut Blaise pascal, LITP, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 05, France

[†]Université de Caen, Département de Mathématiques, Esplanade de la Paix, BP 5186, 14032 Caen Cedex, France

[‡]Institut Gaspard Monge, Université de Marne-la-Vallée, 2, rue de la Butte-Verte, 93166 Noisy-le-Grand Cedex, France

function on the symmetric group, or equivalently as an element of its group algebra. Then, denoting by σ_j the elementary transposition $(j, j+1)$, the induction $\nu \to \mu = \nu\sigma_j$, implies the recursion

$$A_\mu = A_\nu Y_j(u_{\nu(j)}, u_{\nu(j+1)}) \ , \tag{1}$$

where $Y_j(u,v) = \dfrac{\mathrm{i}(u-v)\sigma_j + c}{\mathrm{i}(u-v) - c}$ satisfies the Yang-Baxter equation

$$Y_i(u,v)Y_{i+1}(u,w)Y_i(v,w) = Y_{i+1}(v,w)Y_i(u,w)Y_{i+1}(u,v) \ . \tag{2}$$

Thus, Yang's problem was to expand the operator A_μ on the basis of permutations. He obtained this expansion by means of the recurrence relation (1), and as far as we know, no closed expression for the coefficients $A_\mu(\tau)$ is known. One aim of this note is to investigate this question. We give a partial answer to the original problem, and a complete one to some variants that we explain below.

Getting rid of the normalization constants, we replace Yang's operators by

$$Y_i(u,v) = 1 - (u-v)\sigma_i \ .$$

More general solutions of (2) can be obtained by replacing the elementary transpositions σ_i by the generators T_i of a Hecke algebra, satisfying the braid relations and a quadratic relation

$$(T_i - q_1)(T_i - q_2) = 0 \ .$$

For generic values of q_1, q_2 this solution is

$$Y_j = 1 + \frac{v/u - 1}{q_1 + q_2} T_j \ ,$$

while for instance, in the degenerate case $T_i^2 = 0$ one can take $Y_i = 1 - (u-v)T_i$.

To each of these solutions and to each system of spectral parameters, one can associate a basis $\{Y_\mu(\mathbf{u}) \mid \mu \in \mathfrak{S}_n\}$ of the corresponding Hecke algebra, called the Yang-Baxter basis.

From this data, we define a bilinear form on the Hecke algebra. A fundamental property of the Yang-Baxter basis is that it is essentially self-adjoint with respect to this form (*i.e.* its adjoint basis is obtained by permuting the indices and rescaling). We also give closed formulas for the coefficients of the Yang-Baxter basis on the usual basis $\{T_\mu\}$ in the two degenerate cases $T_i^2 = 0$ and $T_i^2 = -T_i$. The solution is obtained in terms of specializations of double Schubert and Grothendieck polynomials, which were originally defined as canonical bases of the cohomology and Grothendieck rings of flag manifolds. The first case can be interpreted as giving the terms of lowest degree in Yang's coefficients. The existence of a connection between Schubert polynomials and the Yang-Baxter equation was first noticed by Fomin and Kirilllov [10]. All that we present here is for the root system A_n. Generalizations of Schubert polynomials for types B_n, C_n, D_n are given in [27, 14] (see also [26]). The case B_n is studied in detail in [12] and [2].

This paper is organized as follows. In sections 2 and 3 we recall the definitions of Schubert and Grothendieck polynomials, and their geometric interpretations. Then we introduce the Yang-Baxter basis (Section 4) and the bilinear form on the Hecke algebra A with respect to which this basis is well behaved (Section 5). Finally, we present the expansions of the YB basis on the standard basis in terms of Schubert and Grothendieck polynomials (Section 6).

2 Flag varieties and Hecke algebras

Hecke algebras arise naturally in the theory of flag manifolds, as algebras of operators acting on the cohomology [1, 7] or on the Grothendieck ring [7, 24].

Recall that a flag of vector spaces is an increasing sequence of vector spaces $V_1 \subset V_2 \subset \cdots \subset \mathbf{C}^n$. A flag is said to be complete when it is of the type

$$V_1 \subset V_2 \subset \cdots \subset V_n = \mathbf{C}^n \ ,$$

with $\dim(V_i) = i$, $i = 1, \ldots, n$.

The set of complete flags in $\mathbf{C}^n$ is an algebraic projective variety $\mathcal{F}_n = \mathcal{F}(\mathbf{C}^n)$. This variety is equipped with tautological line bundles $L_1, L_2, \ldots, L_n$ which are defined as follows: for each $1 \le i \le n$, the collection $W_i := \{(F, V_i),\ V_i \in F, F \in \mathcal{F}(\mathbf{C}^n)\}$, is a vector bundle on $\mathcal{F}(\mathbf{C}^n)$. Then, writing $E^\vee$ for the dual of a vector bundle E, one has

$$L_i := (W_i / W_{i-1})^\vee \ .$$

The cohomology and Grothendieck ring (of classes of vector bundles) of the flag variety are both quotients of the ring of polynomials in n variables. More precisely, writing $1+\gamma_i$ for the Chern class of L_i, and ξ_i for the class of L_i in the Grothendieck ring, $1 \le i \le n$, the cohomology ring is

$$\mathbf{C}[\gamma_1, \ldots, \gamma_n] = \mathbf{C}[x_1, \ldots, x_n]/\mathcal{J}(0)$$

and the Grothendieck ring

$$\mathbf{C}[\xi_1, \ldots, \xi_n] = \mathbf{C}[x_1, \ldots, x_n]/\mathcal{J}(1) \ ,$$

where $\mathcal{J}(0)$ is the ideal generated by the $f(x_1, \ldots, x_n) - f(0, \ldots, 0)$, f symmetrical, and $\mathcal{J}(1)$ is the ideal generated by the $f(x_1, \ldots, x_n) - f(1, \ldots, 1)$.

Both rings are graded modules of rank $n!$ over the ring of symmetric polynomials, with natural bases parametrized by the symmetric group $\mathfrak{S}(n)$. The Poincaré polynomial of both spaces is

$$(1+q)\,(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \ .$$

The equivariant Grothendieck ring $K_{GL(n)}(\mathcal{F}_n)$ is the ring of Laurent polynomials $\mathbf{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where x_i is now the equivariant class of L_i. For the purpose of introducing bases depending on parameters, we shall work in the extended ring $R := \mathbf{C}[y_1, \ldots, y_n; x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Again, R is a free module of rank $n!$ over $\mathbf{Sym}(X)[t]$, where $t = (x_1 \cdots x_n)^{-1}$ and $\mathbf{Sym}(X)$ is the ring of symmetric polynomials in the x_i's with coefficients in $\mathbf{C}[y_1, \ldots, y_n]$.

The flag variety can be constructed as an iterated sequence of projective bundles (see [17]). This implies the existence of natural linear operators acting on the ring of polynomials:

- permutations, generated by the elementary transpositions

$$\sigma_i \ : \quad f \longmapsto \sigma_i f = f(\ldots, x_{i+1}, x_i, \ldots)$$

- divided differences
$$\partial_i f = \frac{f - \sigma_i f}{x_i - x_{i+1}}$$
used in [1] and [7] to construct a basis of the cohomology,

- $s_i = \sigma_i + \partial_i$ (see [19]),

- isobaric divided differences $\pi_i(f) = \partial_i(x_i f)$, which appear in Demazure's character formula [8],

- $\bar\pi_i = \pi_i - 1$ (*cf.* [24]).

- $T_i = -(q_1+q_2)\bar\pi_i + q_2\sigma_i$ (*cf.* [9]).

Each of these families of operators satisfies the braid relations

$$D_i D_{i+1} D_i = D_i D_{i+1} D_i \ , \tag{3}$$

$$D_i D_j = D_j D_i \quad |i-j| > 1 \ . \tag{4}$$

This ensures the existence of operators D_μ, $\mu \in \mathfrak{S}(n)$, which are obtained by taking, for a given permutation μ, an arbitrary reduced decomposition $\mu = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$, and putting $D_\mu = D_{i_1}D_{i_2}\cdots D_{i_r}$

The simple operators D_i are of the Hecke type, *i.e.* satisfy a quadratic relation:

$$\sigma_i^2 = 1, \quad \partial_i^2 = 0, \quad s_i^2 = 1, \quad \pi_i^2 = \pi_i, \quad \bar\pi_i^2 = -\bar\pi_i, \quad (T_i - q_1)(T_i - q_2) = 0 \ . \tag{5}$$

We shall write $\mathcal{H}^\sigma$, $\mathcal{H}^\partial$, $\mathcal{H}^s$, $\mathcal{H}^\pi$, $\mathcal{H}^{\bar\pi}$ and $\mathcal{H}^T$ for the different algebras generated by the corresponding simple operators. The algebra generated by the T_i's together with the variables x_i (considered as operators $f \mapsto x_i f$) is the affine Hecke algebra [4, 5]. The other algebras are specializations of this one. In particular, the semidirect product $\mathcal{H}^s \times \mathbf{C}[x_1, \ldots, x_n]$ is the degenerate affine Hecke algebra [3, 4]. Note that we distinguish between $\mathcal{H}^\sigma$ and $\mathcal{H}^s$ which have a different action on R.

3 Schubert polynomials and Grothendieck polynomials

The ring R admits two distinguished bases as a free module over the ring $R^{\mathfrak{S}(X)}$ of symmetric polynomials in X, the *double Schubert Polynomials* X_μ and the *double Grothendieck Polynomials* G_μ, $\mu \in \mathfrak{S}(n)$, which are defined as follows. For $\mu = \omega := (n, \ldots, 2, 1)$,

$$X_\omega := \prod_{i+j \le n} (x_i - y_j) \quad , \quad G_\omega := \prod_{i+j\le n} (1 - y_j/x_i)$$

and otherwise

$$X_\mu := \partial_{\mu^{-1}\omega} X_\omega \tag{6}$$

$$G_\mu := \pi_{\mu^{-1}\omega} G_\omega \ . \tag{7}$$

The specialization $X_\mu(y_1 = 0, \ldots, y_n = 0)$ is a representative of a Schubert cycle in the cohomology [22, 25]. The specialization $G_\mu(y_1 = 1, \ldots, y_n = 1)$ is a representative of the class of the structure sheaf of a Schubert variety in the Grothendieck ring [16].

Double Schubert polynomials also have an interpretation as universal polynomials for degeneracy loci (*cf.* [15]). The simplest case corresponds to a pair of vector bundles E, F, and a map $f : E \longrightarrow F$. The r-th degeneracy locus of this map is the set of points where the corank of f is $\geq r$.

Under suitable genericity conditions, the class of this degeneracy locus is a polynomial in the Chern classes of E and F, which can be identified with some Schubert polynomial in the Chern roots of E and F. General Schubert polynomials correspond to a pair of flags of vector bundles, a map f between them, and corank conditions on f.

4 Yang-Baxter bases of Hecke algebras

We first define the elementary Yang-Baxter operators Y_i for the various realizations of Hecke algebras considered in Section 2. The Yang-Baxter basis will then be constituted by the operators Y_μ ($\mu \in \mathfrak{S}_n$), defined inductively by

$$Y_\mu(\mathbf{u}) = Y_\nu(\mathbf{u}) Y_j(u_{\nu(j)}, u_{\nu(j+1)})$$

for $\mu = \nu\sigma_j$, $\ell(\mu) > \ell(\nu)$.

The elementary operators are as follows:

$$\begin{aligned}
Y_j^\sigma(u, v) &:= 1 - (u - v)\sigma_j \ \in \mathcal{H}^\sigma \\
Y_j^\partial(u, v) &:= 1 - (u - v)\partial_j \ \in \mathcal{H}^\partial \ , \\
Y_j^{\bar\pi}(u, v) &:= 1 + (1 - v/u)\bar\pi_j \ \in \mathcal{H}^{\bar\pi}_, \\
Y_j^T(u, v; q_1, q_2) &:= 1 + \frac{v/u - 1}{q_1 + q_2} T_j \ \in \mathcal{H}^T \ .
\end{aligned}$$

One can recover Y_j^∂, $Y_j^{\bar\pi}$ and Y_j^σ from the last case Y_j^T by various specializations. Thus, setting

$$\tilde{Y}_j^T(u, v; q_1, q_2) = Y_j^T\left(e^{(q_1+q_2)u},\ e^{(q_1,+q_2)v)}; q_1, q_2\right) \ ,$$

one has

$$\begin{aligned}
Y_j^\sigma(u, v) &= \lim_{q_1+q_2 \to 0} \tilde{Y}_j^T(u, v; q_1, q_2) \ , \\
Y_j^{\bar\pi}(u, v) &= \tilde{Y}_j^T(u, v; -1, 0) \ .
\end{aligned}$$

The operator Y_j^∂ can also be obtained from Y_j^T by combining a similar specialization with an appropriate homographic substitution on the variables x_i.

In summary, given an arbitrary choice of the parameters $\mathbf{u} = (u_1, \ldots, u_n)$, for all the above Hecke algebras there exists a linear basis $\{Y_\mu(\mathbf{u}),\ \mu \in \mathfrak{S}(n)\}$, that we call the *Yang-Baxter basis.* The problem that we shall examine is to express Yang-Baxter basis in the standard bases $\{\mu\}$, $\{\partial_\mu\}$, $\{\bar\pi_\mu\}$ or $\{T_\mu\}$.

We shall deduce all relations from the following recursion (in the case of the generic Hecke algebra)

$$Y_{\mu\sigma_j}^T(\mathbf{u}) = Y_\mu^T(\mathbf{u}) \cdot (1 + \frac{u - 1}{q_1 + q_2} T_j) \ , \tag{8}$$

when $\ell(\mu\sigma_j) > \ell(\mu)$, with $u = u_{\mu(j+1)}/u_{\mu(j)}$.

We note that, in the case $\ell(\mu\sigma_j) < \ell(\mu)$, one has, with the same u,

$$Y^T_{\mu\sigma_j}(\mathbf{u}) \cdot (1 + \frac{1/u - 1}{q_1 + q_2} T_j) = Y^T_\mu(\mathbf{u}) \cdot (1 - \frac{2 - u - 1/u}{(q_1 + q_2)^2} q_1 q_2) \ . \tag{9}$$

More generally, the Hecke relation implies, for all j, u, v,

$$(1 + \frac{u - 1}{q_1 + q_2} T_j) \cdot (1 + \frac{v - 1}{q_1 + q_2} T_j) = 1 - \frac{(u-1)(v-1)q_1 q_2}{(q_1 + q_2)^2} + \frac{uv - 1}{q_1 + q_2} T_j \ . \tag{10}$$

This product is thus a constant iff $u = 1/v$.

Cherednik [6] has shown that one can recover orthogonal idempotents in the group algebra of the symmetric group by some specialization of the Yang-Baxter basis. He uses them to describe bases of representations of the symmetric group corresponding to skew partitions.

5 Orthogonality properties of the Yang-Baxter basis

We shall only treat the case of the generic Hecke algebra, and recover the other ones by specialization.

Define the anti-isomorphism φ on $\mathcal{H}[\mathbf{u}]$ by

$$\varphi(T_\mu) = T_{\mu^{-1}}, \qquad \varphi(u_i) = u_{n-i+1} \tag{11}$$

and a bilinear form $< , >$, given $h_1, h_2 \in \mathcal{H}$ by

$$< h_1 \, , \, h_2 > := h_1 \cdot \varphi(h_2)|_{T_\omega} \ , \tag{12}$$

where $h\,|_{T_\omega}$ denotes the coefficient of T_ω in the expansion of $h \in \mathcal{H}$ in the basis T_μ.

Theorem 5.1 *Let* $\mathbf{u}$ *be an arbitrary system of spectral parameters, and*

$$\Delta^T(\mathbf{u}) := \prod_{1 \le i < j \le n} \frac{u_j/u_i - 1}{q_1 + q_2}$$

Then the Yang-Baxter basis $\{Y^T_\mu\}$, $\mu \in \mathfrak{S}(n)$, *is adjoint to* $\{Y^T_{\omega\mu} \Delta^T(\mathbf{u}^{\omega\mu})^{-1}\}$, i.e. *one has*

$$< Y^T_\mu \, , \, Y^T_\nu > = \Delta^T(\mathbf{u}^{\omega\mu}) \delta_{\nu,\omega\mu}$$

Proof — Let μ and j be such that $\ell(\mu\sigma_j) > \ell(\mu)$. Then

$$Y^T_\mu \cdot (1 + \frac{u - 1}{q_1 + q_2} T_j) \cdot \varphi(Y^T_\nu) = Y^T_\mu \cdot \varphi(Y^T_\nu \cdot (1 + \frac{1/u - 1}{q_1 + q_2} T_j))$$

Now, by induction on the length of μ, we can suppose that we know all the pairings $< Y^T_\mu \, , \, Y^T_\eta >$. Since for any constant c, $Y^T_\nu(1 + cT_j)$ is a linear combination of T_ν and $T_{\nu\sigma_j}$, the non-zero pairings can occur only for $\mu = \nu$ or $\mu = \nu\sigma_j$. The conclusion follows from (10). □

Corollary 5.2 *Given an arbitrary set of parameters* $\mathbf{u}$*, any element* h *of the Hecke algebra* $\mathcal{H}^T$ *can be expressed as*

$$h = \sum_\mu \frac{1}{\Delta^T(\mathbf{u}^{\omega\mu})} < h\,,\, Y^T_{\omega\mu}(\mathbf{u}) > Y^T_\mu(\mathbf{u})\ .$$

Example 5.3 We take $n = 3$, and we write for short $(ji)T_k$ instead of $(1 + \frac{u_j/u_i - 1}{q_1+q_2} T_k)$ and Y_μ instead of Y^T_μ. Then, the Yang-Baxter basis is

$$\begin{array}{ccc} & Y_{123} = 1 & \\ Y_{213} = (21)T_1 & & (32)T_2 = Y_{132} \\ Y_{231} = (21)T_1(31)T_2 & & (32)T_2(31)T_1 = Y_{312} \\ (21)T_1(31)T_2(32)T_1 & = Y_{321} = & (32)T_2(31)T_1(21)T_2 \end{array}$$

and its image under φ is

$$\begin{array}{ccc} & 1 & \\ (23)T_1 & & (12)T_2 \\ (13)T_2(23)T_1 & & (13)T_1(12)T_2 \\ (23)T_1(13)T_2(12)T_1 & = & (12)T_2(13)T_1(23)T_2 \end{array}$$

Conversely, the expressions of the elements T_μ in the Yang-Baxter basis are, writing Δ for $\Delta^T(\mathbf{u})$:

$\Delta\, T_{123} = \Delta Y_{123}$; $\Delta\, T_{213} = \frac{(u_3/u_2-1)(u_3/u_1-1)}{(q_1+q_2)^2}(Y_{213} - Y_{123})$

$\Delta\, T_{132} = \frac{(u_2/u_1-1)(u_3/u_1-1)}{(q_1+q_2)^2}(Y_{132} - Y_{123})$

$\Delta\, T_{231} = \frac{u_3/u_2-1}{q_1+q_2}(Y_{231} - Y_{213}) - \frac{u_3/u_1-1}{q_1+q_2}(Y_{132} - Y_{123})$

$\Delta\, T_{312} = \frac{u_2/u_1-1}{q_1+q_2}(Y_{312} - Y_{132}) - \frac{u_3/u_1-1}{q_1+q_2}(Y_{213} - Y_{123})$

$\Delta\, T_{321} = Y_{321} - Y_{231} - Y_{312} + Y_{213} + Y_{132} + (1 - \frac{1+u_3/u_1-u_3/u_2-u_2/u_1}{(1+q_1/q_2)(1+q_2/q_1)})Y_{123}$

6 Explicit coefficients

An easy way to get the sequence of parameters used in the factorized expression of a Yang-Baxter element is provided by a planar representation of a permutation due to Rothe (1800), that we shall call its Rothe-Riguet diagram [25].

Represent a permutation μ, in the Cartesian plane $\mathbf{N} \times \mathbf{N}$, by the set of points of coordinates $(i, \mu(i))$. The diagram of μ is the set of points (denoted by boxes $\square$) $\{(i, \mu(j))\,:\, i < j, \mu(i) > \mu(j)\}$. This diagram is a convenient way to represent the inversions of a permutation. The number of boxes in the diagram of μ is exactly the length $\ell(\mu)$ of μ.

Each box $(i, \mu(j))$ is then filled with the factor

$$1 + \frac{u_{\mu(j)}/u_{\mu(i)} - 1}{q_1 + q_2} T_k$$

where $k - i$ is the number of boxes in the same column strictly below the given box. Thus in each column, the indices of the T_k are increasing by 1 upwards, starting from the index of the column.

Example 6.1 Let $\mu = 31542$. Still writing $(ji)T_k$ for $(u_j/u_i-1)/(q_1+q_2)\,T_k$, and denoting the points $(i,\ \mu(i))$ by a $\star$, the diagram of μ is

$$\begin{array}{cccccc}
5 & \cdot & \star & \cdot & \cdot & \cdot \\
4 & \cdot & (54)T_4 & \cdot & \star & \cdot \\
3 & \star & \cdot & \cdot & \cdot & \cdot \\
2 & (32)T_2 & (52)T_3 & \cdot & (42)T_4 & \star \\
1 & (31)T_1 & (51)T_2 & \star & \cdot & \cdot \\
 & & & & & \\
\mu\ = & 3 & 5 & 1 & 4 & 2
\end{array}$$

The following proposition, obtained by induction on the length $\ell(\mu)$ tells us that

$$Y_\mu^T = (54)T_4\,(32)T_2\,(52)T_3\,(42)T_4\,(31)T_1\,(51)T_2$$

Proposition 6.2 *The reading of the Rothe diagram of a permutation μ (in the occidental way, from left to right and top to bottom), as the product of the factors contained in the boxes, gives an expression of the Yang-Baxter element $Y_\mu^T(\mathbf{u})$.* $\square$

Let us now consider the cases of degenerate Hecke algebras. The following formulas could in principle be derived from Theorem 5.1, but it will be more convenient to provide direct proofs in each case. We first treat the case of the simplest Hecke relation $D_i^2 = 0$, *i.e.* the algebra $\mathcal{H}^\partial$.

Proposition 6.3 *The entries of the transition matrix $Y_\mu^\partial(\mathbf{u}) \to \partial_\nu$ of $\mathcal{H}^\partial$ are specializations of double Schubert polynomials. More precisely,*

$$Y_\mu^\partial(\mathbf{u}) = \sum_\nu X_\nu(\mathbf{u}^\mu, \mathbf{u})\,\partial_\nu\ . \tag{13}$$

Proof — We use the induction $\mu \to \mu\sigma_k$ with $\ell(\mu\sigma_k) > \ell(\mu)$. Assume that (13) holds for μ, *i.e.* that

$$Y_\mu^\partial = \sum_{\nu:\ell(\nu\sigma_k)=\ell(\nu)+1} [X_\nu(\mathbf{u}^\mu, \mathbf{u})\partial_\nu + X_{\nu\sigma_k}(\mathbf{u}^\mu, \mathbf{u})\partial_{\nu\sigma_k}]$$

Multipliying by the factor $(1-(u_i-u_j)\partial_k)$, where $i=\mu(k)$, $j=\mu(k+1)$, one obtains the coefficients of ∂_ν and $\partial_{\nu\sigma_k}$ in $Y_{\mu\sigma_k}$. The one of $\partial_{\nu\sigma_k}$ is equal to $X_{\nu\sigma_k}(\mathbf{u}^\mu,\mathbf{u}) - (u_i - u_j)X_\nu(\mathbf{u}^\mu,\mathbf{u})$. Taking into account the relation

$$(u_i-u_j)X_\nu(\mathbf{u}^\mu,\mathbf{u}) = X_{\nu\sigma_k}(\mathbf{u}^\mu,\mathbf{u}) - X_{\nu\sigma_k}(\mathbf{u}^{\mu\sigma_k},\mathbf{u})$$

which follows directly from the definition (6), we see that this coefficient is indeed given by (13). $\square$

There are several related formulas involving Schubert polynomials and the nilCoxeter algebra. The first one is a recursion on Schubert polynomials ([21], Prop. 1.7 and 1.12, see also [25], (7.2) p. 95) in $\mathcal{H}^\partial$ (called *algèbre des différences divisées* in [21, 22]). This recursion is directly equivalent to the generating function obtained in [13, 10] by a different method involving the Yang-Baxter equation. The coefficients $X_\nu(\mathbf{u}^\mu,\mathbf{u})$ also appear in the expansion of permutations in the divided differences algebra $\mathcal{H}^\partial$. According to [22, 23], one has

$$\mu = \sum_\nu X_\nu(\mathbf{x}^\mu, \mathbf{x})\partial_\nu \ . \qquad (14)$$

This follows from the Newton interpolation formula, since for any polynomial f,

$$\sum_\nu X_\nu(\mathbf{x}^\mu, \mathbf{y})\partial_\nu f(\mathbf{y}) = f(\mathbf{x}^\mu) \ .$$

Note however that in (14) the divided differences ∂_ν act on the variables x_i. To recover permutations from the Yang-Baxter operators, one just has to compose them with the specialization homomorphism $\xi : u_i \mapsto x_i$. In other terms, if one defines normal ordering in the divided differences algebra by the condition that for a monomial M in the x_i and the ∂_j, $: M :$ is the same expression where all the x_i have been moved to the left of the difference operators, without alterating the relative order of the ∂_j, on has

Corollary 6.4 *As an operator on polynomials, the permutation μ is equal to the normal ordering of the Yang-Baxter element $Y^\partial_\mu(\mathbf{x})$*

$$\mu = \, : Y^\partial_\mu(\mathbf{x}) : \, = \ \xi \circ Y^\partial_\mu(\mathbf{u}) \ .$$

For example,

$$Y^\partial_{321}(\mathbf{u}) = (1-(u_2-u_3)\partial_1)(1-(u_1-u_3)\partial_2)(1-(u_1-u_2)\partial_1)$$

$$= 1 + \cdots + (u_3-u_2)(u_3-u_1)(u_2-u_1)\partial_{321}$$

and, writing $\sigma_i = 1-(x_i-x_{i+1})\partial_i$,

$$(321) = \sigma_1\sigma_2\sigma_1 = (1-(x_1-x_2)\partial_1)(1-(x_2-x_3)\partial_2)(1-(x_1-x_2)\partial_1)$$

$$= 1 + \cdots + (x_2-x_1)(x_3-x_1)(x_3-x_2)\partial_{321} = \, : Y^\partial_{321}(\mathbf{x}) :$$

the reduction to a normally ordered expression involving repetitive use of the Leibniz formula $\partial_i(PQ) = (\partial_i P)Q + \sigma_i(P)\partial_i Q$.

One could think of obtaining the expansion (13) by taking the specialization of the generating function of double Schubert polynomials. For example, for $\mu = (1324)$, the generating function of Fomin and Kirillov specializes to

$$(1+(x_2-x_1)\partial_3)(1+(x_3-x_1)\partial_2)(1+(x_1-x_1)\partial_1)(1+(x_3-x_2)\partial_3)(1+(x_1-x_2)\partial_2)(1+(x_1-x_3)\partial_3)$$

and it is not immediate that this expression is equal to the Yang-Baxter element

$$Y^\partial_{1324} = 1-(x_2-x_3)\partial_2 \ .$$

In other words, formula (13) implies

Corollary 6.5 *Let $\mathcal{F}(\mathbf{x}, \mathbf{y})$ be the Fomin-Kirillov generating function for double Schubert polynomials (see [10]). Then,*

$$\mathcal{F}(\mathbf{u}^\mu, \mathbf{u}) = Y^\partial_\mu(\mathbf{u}) \ .$$

Next, we turn back to the original case of Yang and Baxter, *i.e.* the case where $\mathcal{H} = \mathcal{H}^\sigma$ is the group algebra of the symmetric group.

Proposition 6.6 *The Yang-Baxter coefficients $A_\nu(\mu)$, which are the coefficients of the expression of the Yang-Baxter basis in the basis of permutations, are non-homogeneous polynomials whose leading terms (i.e. term of lowest degree) are equal to the specializations $X_\nu(\mathbf{u}^\mu, \mathbf{u})$ of double Schubert polynomials.*

Proof — In the case of $\mathcal{H}^\partial$, expanding a Yang-Baxter element $Y_\mu^\partial(\mathbf{u})$, after having chosen a reduced decomposition w of μ, involves only subwords of w which are reduced decompositions (because products of ∂_i which are not reduced decompositions vanish). These subwords give exactly the same contribution in the case of $\mathcal{H}^\sigma$. Extra terms involve at least a square $\cdots(u_i - u_j)\sigma_k(u_{i'} - u_{j'})\sigma_k\cdots$ and thus give a contribution to $A_\nu(\mu)$ which is of degree strictly bigger than $\ell(\nu)$.

For example, for $\mathfrak{S}(3)$, the only coefficient which does not coincide with a Schubert polynomial is $A_{123}(321) = 1 + (u_1 - u_2)(u_2 - u_3)$. □

Another interesting case for geometry is the algebra $\mathcal{H}^{\bar\pi}$

Proposition 6.7 *The entries of the transition matrix $Y_\mu^{\bar\pi}(\mathbf{u}) \to \bar\pi_\nu$ of $\mathcal{H}^{\bar\pi}$ are specializations of double Grothendieck polynomials. More precisely,*

$$Y_\mu^{\bar\pi}(\mathbf{u}) = \sum_\nu G_\nu(\mathbf{u}^\mu, \mathbf{u})\, \partial_\nu \ .$$

The proof is the same as in the case of the $Y_\mu^\partial(\mathbf{u})$, the relations $\partial_i^2 = 0$ being replaced by $\pi_i\bar\pi_i = 0$. Remark that the expansion of a product of k factors $1 + (u-1)\bar\pi_i$ involves 2^k terms, in contrast with the case of $1 + (u-1)\partial_i$ where only those terms which are reduced decompositions give a non zero contribution.

Fomin and Kirillov [11] have given a generating function for Grothendieck polynomials in the algebra $\mathcal{H}^{\bar\pi}$, related to the recursion of [21] (Th. 2.5) and to formula (4.11) of [23]. One has the same type of specialization as for Schubert polynomials:

Corollary 6.8 *Let $\mathcal{G}$ be the Fomin-Kirillov generating function of double Grothendieck polynomials. Then,*

$$Y_\mu^{\bar\pi}(u_1^{-1}, \ldots, u_n^{-1}) = \mathcal{G}(\mathbf{u}^\mu, \mathbf{u}) \ .$$

One can recover Schubert polynomials from Grothendieck polynomials. Indeed, the change of variables $x_i \to 1/(1-a_i)$, $y_j \to 1/(1-b_j)$ transforms the maximal Grothendieck polynomial $G_\omega(X,Y)$ into $\prod_{i,j:i+j\le n}(a_i - b_j)/(1-b_j)$, i.e. into the Schubert polynomial $X_\omega(A,B)$, up to a factor independent of the a_i. On another hand, the operator π_i becomes $f \longrightarrow \partial_i((1 + a_{i+1}) \cdot f)$, where the ∂_i are now the operators corresponding to the variables $\{a_i\}$. Thus, the term of smallest degree (in the a_i) of the polynomial $G_\mu(x_i, y_j)$, when expressed in terms of the a_i, b_j, is the Schubert polynomial $X_\mu(a_i, b_j)$.

7 Appendix

7.1 Action on the ring of polynomials

We have already mentioned that the different Hecke algebras naturally occur as algebras of operators on the ring $\mathbf{C}[x_1, \ldots, x_n]$. This action is in fact related to the χ_y-characteristic of Hirzebruch [17] (see [9]). However, in the preceding sections we did not make use of the action on the variables x_i. Let us mention only two applications [9, 19].

In the case where the spectral parameters are chosen to be $u_i = q^{i-1}$, with $q_1 = q$, $q_2 = -1$, the Yang-Baxter element Y_μ, for μ the maximal element of a Young subgroup $\mathfrak{S}(I) = \mathfrak{S}_{i_1} \times \mathfrak{S}_{i_2} \times \cdots \times \mathfrak{S}_{i_r}$ factorizes into

$$Y_\mu(f) = \Delta_1 \Delta_2 \cdots \Delta_r \, \partial_\mu(f)$$

where Δ_k is the q-Vandermonde associated to the k-th factor of the Young subgroup, *i.e.*, setting $m_k = i_1 + \cdots + i_k$

$$\Delta_k = \prod_{m_k+1 \leq i < j \leq m_{k+1}} (q x_j - x_i) \ .$$

In [9] is described how to use these special elements to obtain bases of the irreducible representations of the Hecke algebra, as well as q-idempotents generalizing the Young idempotents.

The case of $\mathcal{H}^s$, where the parameters are now $u_i = i$, is given in [19]. The images of the monomial $x_1^{n-1} x_2^{n-2} \cdots x_1^0$ under Yang-Baxter elements Y_μ^s constitute a linear basis of the cohomology of the flag manifold, which can be regarded as a deformation of the Schubert basis.

Yang-Baxter elements corresponding to the maximal elements of a Young subgroup still satisfy a factorization property. For example, if ω is the permutation $(n, n-1, \ldots, 1)$, one has

$$Y_\omega^s(f) = \prod_{1 \leq i < j \leq n} (1 + x_j - x_i) \, \partial_\omega(f) \ .$$

The polynomial $\prod_{1 \leq i < j \leq n}(1 + x_j - x_i)$ can be interpreted as the total Chern class of the flag variety.

7.2 Examples

For $n = 4$, the double Schubert polynomials are:

$X_{4321} = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_2)(x_3 - y_1)$

$X_{3421} = X_{4321}/(x_1 - y_3)$; $X_{4231} = X_{4321}/(x_2 - y_2)$; $X_{4312} = X_{4321}/(x_3 - y_1)$

$X_{2431} = (x_1 - y_1)(x_2 - y_1)(x_3 - y_1)(x_1 + x_2 - y_2 - y_3)$; $X_{3241} = X_{4321}/(x_1 - y_3)(x_2 - y_2)$; $X_{3412} = X_{4321}/(x_1 - y_3)(x_3 - y_1)$; $X_{4132} = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 + x_3 - y_1 - y_2)$; $X_{4213} = X_{4321}/(x_2 - y_2)(x_3 - y_1)$

$X_{2341} = (x_1 - y_1)(x_2 - y_1)(x_3 - y_1)$; $X_{1432} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_2^2 x_3 - (x_1^2 + x_1 x_2 + x_2^2)(y_1 + y_2) - (x_1 x_2 + x_1 x_3 + x_2 x_3)(y_1 + y_2 + y_3) + (x_1 + x_2)(y_1^2 + y_1 y_2 + y_2^2) + (x_1 + x_2 + x_3)(y_1 y_2 + y_1 y_3 + y_2 y_3) - (y_1^2 y_2 + y_1^2 y_3 + y_1 y_2^2 + y_1 y_2 y_3 + y_2^2 y_3)$; $X_{2413} = (x_1 - y_1)(x_2 - y_1)(x_1 + x_2 - y_2 - y_3)$; $X_{3142} = (x_1 - y_1)(x_1 - y_2)(x_2 + x_3 - y_1 - y_2)$; $X_{3214} = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1)$; $X_{4123} = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)$.

$X_{1342} = x_1x_2 + x_1x_3 + x_2x_3 - (y_1+y_2)(x_1+x_2+x_3) + y_1^2 + y_1y_2 + y_2^2$; $X_{1423} = x_1^2 + x_1x_2 + x_2^2 - (x_1+x_2)(y_1+y_2+y_3) + y_1y_2 + y_1y_3 + y_2y_3$; $X_{2143} = (x_1-y_1)(x_1+x_2+x_3-y_1-y_2-y_3)$; $X_{2314} = (x_1-y_1)(x_2-y_1)$; $X_{3124} = (x_1-y_1)(x_1-y_2)$.

$X_{1243} = x_1+x_2+x_3-y_1-y_2-y_3$; $X_{1324} = x_1+x_2-y_1-y_2$; $X_{2134} = x_1-y_1$.

$X_{1234} = 1$

The Grothendieck polynomials for $n=3$ are

$$\begin{tikzcd}[column sep=small]
 & G_{321} = (1-\frac{y_1}{x_1})(1-\frac{y_1}{x_2})(1-\frac{y_2}{x_1}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_{231} = (1-\frac{y_1}{x_1})(1-\frac{y_1}{x_2}) \arrow[d, "\pi_2"'] & & (1-\frac{y_1}{x_1})(1-\frac{y_2}{x_1}) = G_{312} \arrow[d, "\pi_1"] \\
G_{213} = (1-\frac{y_1}{x_1}) \arrow[dr, "\pi_1"'] & & (1-\frac{y_1y_2}{x_1x_2}) = G_{132} \arrow[dl, "\pi_2"] \\
 & G_{123} = 1 &
\end{tikzcd}$$

Let us also illustrate the relations between the Yang-Baxter elements associated with the various Hecke algebras.

Take $\mu = (35142)$ as in example 6.1.

The coefficient of $T_2 := T_{13245}$ in Y^T_{35142} is the same as in the product

$$\left(1+\frac{x_3x_2^{-1}-1}{q_1+q_2}T_2\right)\left(1+\frac{x_5x_1^{-1}-1}{q_1+q_2}T_2\right)\left(1+\frac{x_5x_4^{-1}-1}{q_1+q_2}T_4\right)\left(1+\frac{x_4x_2^{-1}-1}{q_1+q_2}T_4\right)$$

that is,

$$\frac{(x_3x_5-x_1x_2)(x_2x_4-q_1q_2(q_1+q_2)^{-2}(x_4-x_2)(x_5-x_4))}{(q_1+q_2)x_1x_2^2x_4} .$$

The coefficient of $\bar\pi_2 := \bar\pi_{13245}$ in $Y^{\bar\pi}_{35142}$ is the same as in the product

$$(1+(1-\frac{x_3}{x_2})\bar\pi_2)(1+(1-\frac{x_5}{x_1})\bar\pi_2)$$

that is,

$$\left(1-\frac{x_3}{x_2}\right)+\left(1-\frac{x_5}{x_1}\right)-\left(1-\frac{x_3}{x_2}\right)\left(1-\frac{x_5}{x_1}\right)$$
$$= 1-\frac{x_3x_5}{x_1x_2} .$$

It is the specialization of the preceding coefficient for $q_1=-1, q_2=0$.

The coefficient of $\partial_2 := \partial_{13245}$ in Y^∂_{35142} is the same as in

$$(1+(x_2-x_3)\partial_2)(1+(x_1-x_5)\partial_2) ,$$

that is,

$$x_1+x_2-x_3-x_5 .$$

The coefficient of $\sigma_2 = (13245)$ in Y^{σ}_{35142} is the same as in the product

$$(1 + (x_3 - x_2)\sigma_2)(1 + (x_5 - x_1)\sigma_2)(1 + (x_5 - x_4)\sigma_4)(1 + (x_4 - x_2)\sigma_4)$$

that is,

$$(x_1 + x_2 - x_3 - x_5)(1 + (x_5 - x_4)(x_4 - x_2)) ,$$

whose term of lowest degree corresponds to the preceding case.

References

[1] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and the cohomology of the spaces G/P*, Russian Math. Surveys **28** (1973), 1-26.

[2] S. C. Biley and M. Haiman, *Schubert polynomials for the classical groups*, preprint, 1993.

[3] I.V. Cherednik, *Special bases of irreducible representations of a degenerate affine Hecke algebra*, Funct. Anal. Appl. **20** (1986), 76-78.

[4] I.V. Cherednik, *On R-matrix quantization of formal loop groups*, *in* Group theoretical methods in physics, Proc. 3rd Semin. Yurmala (USSR), 1985, vol. 2, 161-180 (1986).

[5] I.V. Cherednik, *Quantum groups as hidden symmetries of classic representation theory*, *in* Differential geometric methods in theoretical physics (A.I. Solomon ed.), World Scientific, Singapore, 1989, 47-54.

[6] I.V. Cherednik, *A new interpretation of Gelfand-Tzetlin bases*, Duke. Math. J. **54** (1987), 563-577.

[7] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math **21** (1973), 287-301.

[8] M. Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. **98** (1974), 163-172.

[9] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf and J.-Y. Thibon, *Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras*, Publ. RIMS Kyoto **31** (1995), 179-201.

[10] S. Fomin and A.N. Kirillov, *The Yang-Baxter equation, symmetric functions and Schubert polynomials*, Proc. 5th International Conference 'Formal Power Series and Algebraic Combinatorics', Firenze, 1993 (to appear in Discrete Math.).

[11] S. Fomin and A.N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proceedings of FPSAC'94, DIMACS, Rutgers University, 183-189.

[12] S. Fomin and A.N. Kirillov, *Combinatorial B_n-analogues of Schubert polynomials*, preprint.

[13] S. Fomin and R.P. Stanley, *Schubert polynomials and the nilCoxeter algebra*, Adv. in Math. **103** (1994), 196-207.

[14] W. Fulton, *Schubert varieties in flag bundles for the Classical Groups*, preprint, University of Chicago, 1994.

[15] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. **65** (1991), 381-420.

[16] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), 711-729.

[17] F. Hirzebruch, *Topologocal methods in algebraic geometry*, 3rd ed., Springer, 1978.

[18] A. Lascoux, *Classes de Chern des variétés de drapeaux*, C.R. Acad. Sci. Paris **295** (1982), 393-398.

[19] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Twisted action of the symmetric group on the cohomology of a flag manifold*, Banach Center Publ. (to appear).

[20] A. Lascoux and M.P. Schützenberger, *Polynômes de Schubert*, C.R. Acad. Sci. Paris **294** (1982), 447-450.

[21] A. Lascoux and M.P. Schützenberger, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C.R. Acad. Sci. Paris **295** (1982), 629-633.

[22] A. Lascoux and M.P. Schützenberger, *Symmetrization operators on polynomial rings*, Funct. Anal. Appl. **21** (1987), 77-78.

[23] A. Lascoux and M.P. Schützenberger, *Décompositions dans l'algèbre des différences divisées*, Discrete Math. **99**(1992), 165-179.

[24] G. Lusztig, *Equivariant K-theory and representations of Hecke Algebras*, Proc. Amer. Math. Soc. **94** (1985), 337-342.

[25] I.G. Macdonald, *Notes on Schubert polynomials*, Publ. LACIM **6**, UQAM, Montréal, 1991.

[26] P. Pragacz, *Symmetric polynomials and divided differences in formulas of intersection theory*, to appear in "Parameter Spaces", Banach Center Publications 36 (1996)

[27] P. Pragacz and J. Ratajski, *Formulas for Lagrangian and orthogonal degeneracy loci: the $\tilde{Q}$-polynomials approach*, Max-Planck-Institut fur Mathematik Preprint (1994); to appear in Compositio Math..

[28] S. Veigneau, *SP, a Maple package for Schubert polynomials*, Université de Marne-la-Vallée, 1994.

[29] C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312-1315.