Some generalizations of quasi-symmetric functions and noncommutative symmetric functions

Gérard Duchamp[1], Florent Hivert[2], and Jean-Yves Thibon[2]

[1] Laboratoire d'Informatique Fondamentale et
Appliquée de Rouen – Faculté des Sciences – Université de Rouen
76821 Mont-Saint-Aignan CEDEX FRANCE
Gerard.Duchamp@liafa.jussieu.fr

[2] Institut Gaspard Monge – Université de Marne-la-Vallée
5, bd Descartes – Champs sur Marne
77454 Marne-la-Vallée CEDEX 2 FRANCE
Florent.Hivert@univ-mlv.fr, jyt@univ-mlv.fr

Abstract. In this paper, we investigate various kinds of generalisations of symmetric functions. The classical algebra Sym of symmetric functions is embedded in $QSym$, the algebra of quasi-symmetric functions, and is also a quotient of the algebra $\mathbf{Sym}$ of noncommutative symmetric functions. A q-analogue $QSym_q$ of the algebra $QSym$ provides a kind of unification of both generalizations $QSym$ and $\mathbf{Sym}$ of Sym. In this paper we introduce some further genarlizations whose natural bases are various combinatorial objects like standard tableaux, permutations or integer matrices.

1 Introduction

The classical algebra of symmetric functions, denoted here by Sym, can be embedded in the algebra $QSym$ of quasi-symmetric functions [4], and is also a quotient (the abelianization) of the algebra $\mathbf{Sym}$ of noncommutative symmetric functions [3].

Thus, we have, for the homogeneous components of degree n of these algebras, a commutative diagram

$$\begin{tikzcd} Sym_n \arrow[rr, "d"] & & QSym_n \\ & \mathbf{Sym}_n \arrow[ul, "e"] \arrow[ur, "c"'] & \end{tikzcd}$$

which has a neat interpretation in representation theory: it is the Cartan-Brauer triangle of $H_n(0)$, the Hecke algebra of type A_{n-1} at $v = 0$ [2, 1, 7]. This means that $QSym_n$ is to be interpreted as $G_0(H_n(0))$, the Grothendieck group of the category of finitely generated $H_n(0)$-modules, $\mathbf{Sym}_n$ as $K_0(H_n(0))$, the Grothendieck group of finitely generated projective $H_n(0)$-modules, and Sym_n as the Grothendieck group $R(H_n(v)) = G_0(H_n(v)) = K_0(H_n(v))$ of the semi-simple algebra $H_n(v)$, for generic v. The inclusion map $d: \ Sym_n \to QSym_n$ corresponds to the decomposition map. Indeed, the simple $H_n(v)$ modules $V_\lambda(v)$ correspond to the Schur functions s_λ, with $\lambda \vdash n$, and the coefficients of the expansion

$$s_\lambda = \sum_{|I|=n} d_{\lambda I} F_I \tag{1}$$

are the multiplicities of the simple $H_n(0)$ modules $\mathbf{D}_I$ (parametrized by compositions I of n) as composition factors of the specialized module $V_\lambda(0)$.

This interpretation leads to a q-analogue of the algebra of quasi-symmetric functions: the quantum quasi-symmetric functions defined in [12]. Indeed, $H_n(0)$-modules are endowed with a natural filtration, which can be taken into account in the description of the composition factors of the induced modules

$$\mathbf{D}_I \widehat{\otimes} \mathbf{D}_J = \mathbf{D}_I \otimes \mathbf{D}_J \uparrow_{H_n(0)\otimes H_m(0)}^{H_{m+n}(0)} .$$

The multiplicity c_{IJ}^K of $\mathbf{D}_K$ as a composition factor of this module is equal to the coefficient of F_K in the product $F_I F_J$. The rule to evaluate this product is as follows: take any permutation u of $1, \ldots, n$ with descent composition $C(u) = I$ and any permutation v of $n+1, \ldots, n+m$ such that $C(v) = J$. Then the shuffle of the two words u and v is a sum of permutations of $\{1, \ldots, n+m\}$

$$u \sqcup\!\sqcup v = \sum_{w \in \mathfrak{S}_{m+n}} c_w w \tag{2}$$

and the product is given by

$$F_I F_J = \sum_{w \in \mathfrak{S}_{m+n}} c_w F_{C(w)} . \tag{3}$$

There exists a q-analogue of the shuffle product, which is defined by

$$\text{if } u = au' \text{ and } v = bv' \text{ with } a, b \in A, \quad u \sqcup\!\sqcup_q v = a(u' \sqcup\!\sqcup_q v) + q^{|u|} b(u \sqcup\!\sqcup_q v') \tag{4}$$

where $|u|$ is the length of u. It can be shown that this operation is associative, and that when q is not a root of unity, the q-shuffle algebra is isomorphic to the concatenation algebra, which corresponds to the case $q = 0$.

The induced representation $\mathbf{D}_I \widehat{\otimes} \mathbf{D}_J$ is generated by a single vector u. There is a filtration of this module whose k-th slice M_k is spanned by the elements $T_\sigma u$ for permutations σ of length k. Now, if one computes the product $F_I F_J$ by using the q-shuffle instead of the ordinary one in formula (3), the coefficient of $q^k F_H$ in the result is the multiplicity of $\mathbf{D}_H$ at level k of the filtration. The algebra $QSym_q$ of quantum quasi-symmetric functions is defined accordingly as the algebra with generators F_I and multiplication rule

$$F_I F_J = \sum_w < w | u \sqcup\!\sqcup_q v > F_{C(w)} \tag{5}$$

for permutations u and v as above, $< w | u \sqcup\!\sqcup_q v >$ being the coefficient of w in $u \sqcup\!\sqcup_q v$.

All the usual bases of $QSym$, in particular (M_I), are defined in $QSym_q$ by the same expressions in terms of the F_I as in the classical case.

For generic values of q, $QSym_q$ is freely generated by the one-part quasi-ribbons F_n, or as well by the power-sums M_n, or any sequence corresponding to a free set of generators of the algebra of symmetric functions in the classical case. This means that if we define for a composition $I = (i_1, \ldots, i_r)$

$$F^I = F_{i_1} F_{i_2} \cdots F_{i_r} \quad \text{and} \quad M^I = M_{i_1} M_{i_2} \cdots M_{i_r} \ \in \ QSym_q \tag{6}$$

then the F^I (resp. the M^I) form a basis of $QSym_q$. This is clearly not true for $q = 1$, as in this case these elements are symmetric functions.

Thus, for generic q, $QSym_q$ is isomorphic to the algebra of noncommutative symmetric functions. Actually, it can be obtained by specializing the formal variables of the polynomial realization $\mathbf{Sym}(A)$ of $\mathbf{Sym}$ (see [3], Sec. 7.3) to the generators of the (infinite dimensional) quantum affine space $\mathbb{C}_q[X] = \mathbb{C}_q[x_1, x_2, \ldots]$, the associative algebra generated by an infinite sequence of elements x_i subject to the q-commutation relations

$$\text{for } j > i, \quad x_j x_i = q x_i x_j \, . \tag{7}$$

More precisely, let $\mathbf{Sym}(X)$ be the subalgebra of $\mathbb{C}_q[X]$ generated by the specialization $a_i \to x_i$ of the noncommutative symmetric functions. Then, $\mathbf{Sym}(X)$ is isomorphic to $QSym_q$, the correspondence being given by

$$M_I \longleftrightarrow \overline{M}_I = \sum_{j_1 < \cdots < j_r} x_{j_1}^{i_1} \cdots x_{j_r}^{i_r} \, . \tag{8}$$

That is, if one defines

$$\overline{F}_I = \sum_{J \succeq I} \overline{M}_J \ , \tag{9}$$

one has for u a permutation of $1, \ldots, n$ and v a permutation of $n+1, \ldots, n+m$

$$\overline{F}_{C(u)} \overline{F}_{C(v)} = \sum_w < w | u \sqcup\!\sqcup_q v > \overline{F}_{C(w)} \tag{10}$$

where $< w | u \sqcup\!\sqcup_q v >$ is the coefficient of the word w in the q-shuffle $u \sqcup\!\sqcup_q v$.

Thus, $QSym_q$ provides a kind of unification of both generalizations $QSym$ and $\mathbf{Sym}$ of Sym. In this article, we introduce some further generalizations of symmetric functions, whose natural bases are labelled by various combinatorial objects (permutations, standard tableaux or integer matrices), which culminates with a self-dual bialgebra, whose bases are labelled by double cosets of symmetric groups modulo parabolic subgroups, and which accomodates as subalgebras or as quotients all the other examples.

2 Free quasi-symmetric functions

Our first generalization is obtained by lifting the multiplication rule (3) to the free associative algebra, where it becomes multiplicity free. One arrives in this way to an algebra with basis labelled by all permutations, which turns out to be isomorphic to the dual of the convolution algebra $\mathbb{Z}S$ introduced by Malvenuto and Reutenauer in [9], Sec. 3.

Definition 1. *The* free quasi-ribbon $\mathbf{F}_\sigma$ *labelled by a permutation* $\sigma \in \mathfrak{S}_n$ *is the noncommutative polynomial*

$$\mathbf{F}_\sigma = \sum_{\mathrm{std}\,(w)=\sigma^{-1}} w \quad \in \mathbb{Z}\langle A\rangle$$

where $\mathrm{std}\,(w)$ *denotes the standardized of the word* w*.*

The hypoplactic version of the Robinson-Schensted correspondence [7] shows that the commutative image of $\mathbf{F}_\sigma$ is the quasi-symmetric function F_I, where $I = C(\sigma)$. Indeed, the standard ribbon playing the role of the insertion tableau is equal to $\mathrm{std}\,(w)^{-1}$, so that $\mathbf{F}_\sigma$ contains exactly one representative of each hypoplactic class of shape I.

For a word $w = x_1x_2\cdots x_n$ in the letters $1, 2, \ldots$ and an integer p, denote by $w[p]$ the shifted word $(x_1+p)(x_2+p)\cdots(x_n+p)$.

Proposition 1. *Let* $\alpha \in \mathfrak{S}_n$ *and* $\beta \in \mathfrak{S}_m$*. Then,*

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\sigma \in \alpha \,\sqcup\!\sqcup\, \beta[n]} \mathbf{F}_\sigma$$

Therefore, the free quasi-ribbons span a $\mathbb{Z}$*-subalgebra of the free associative algebra.*

Definition 2. *The subalgebra of* $\mathbb{C}\langle A\rangle$

$$\mathbf{FQSym} = \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}\mathbf{F}_\sigma$$

is called the algebra of free quasi-symmetric functions.

An interesting property of this algebra is that it contains a subalgebra with a distinguished basis labelled by standard Young tableaux, which maps to ordinary Schur functions under abelianization, and to which the Littlewood-Richardson rule can be lifted to a multiplicity free formula.

Definition 3. *Let* t *be a standard tableau of shape* λ*. The* free Schur function *labelled by* t *is*

$$\mathbf{S}_t = \sum_{P(\sigma)=t} \mathbf{F}_\sigma \,.$$

As pointed out in [8], Schützenberger's version of the Littlewood-Richardson rule is equivalent to the following statement, which shows that the free Schur functions span a subalgebra of **FQSym**. We will call it the algebra of *free symmetric functions* and denote it by **FSym**. It can be shown that it provides a realization of the algebra of tableaux introduced by Poirier and Reutenauer [11].

Proposition 2 (LRS rule). *Let t', t'' be standard tableaux, and let n be the number of cells of t'. Then,*

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_{t \in \mathrm{Sh}\,(t',t'')} \mathbf{S}_t$$

where $\mathrm{Sh}\,(t',t'')$ *is the set of standard tableaux in the shuffle of t' (regarded as a word via its row reading) with the plactic class of $t''[n]$.*

Note that the algebra of noncommutative symmetric functions $\mathbf{Sym}(A)$ is a subalgebra of **FSym**, since

$$R_I(A) = \sum_{\mathrm{Rec}\,(t)=\mathrm{Des}\,(I)} \mathbf{S}_t \,. \tag{11}$$

The natural homomorphism $a_i \mapsto x_i$ from $\mathbb{C}\langle A\rangle$ to the algebra of quantum polynomials $\mathbb{C}_q[X]$ maps $\mathbf{F}_\sigma$ to the quantum quasi-symmetric function $q^{\ell(\sigma)}F_{C(\sigma)}$. Therefore, $QSym_q$ is a quotient of **FQSym**.

To establish the connection with the results of [9], one first introduces a bialgebra structure as follows:

Proposition 3. **FQSym** *is a bialgebra for the comultiplication*

$$\Gamma \mathbf{F}_\sigma = \sum_{u\cdot v=\sigma} \mathbf{F}_{\mathrm{std}\,(u)} \otimes \mathbf{F}_{\mathrm{std}\,(v)} \,.$$

where $u \cdot v$ denotes the concatenation of u and v.

Let A' and A'' be two mutually commuting ordered alphabets. Identifying $F \otimes G$ with $F(A')G(A'')$, one can check that $\Gamma(F)$ can be identified with $F(A' \oplus A'')$, where $\oplus$ denotes the ordered sum.

Next, define a scalar product on **FQSym** by

$$\langle \mathbf{F}_\alpha \,,\, \mathbf{F}_\beta \rangle = \delta(\alpha\beta) = \begin{cases} 0 & \alpha\beta \neq \mathrm{id} \\ 1 & \alpha\beta = \mathrm{id} \end{cases} \tag{12}$$

and let $\mathbf{G}_\sigma = \mathbf{F}_{\sigma^{-1}}$ be the adjoint basis of $\mathbf{F}_\sigma$.

Proposition 4. *The dual of Γ with repect to $\langle\ ,\ \rangle$ is the convolution product on $\bigoplus_{n\geq 0} \mathbb{C}\mathfrak{S}_n$. That is, if one denotes by $\mu(u \otimes v) = uv$ the concatenation product of $\mathbb{C}\langle A\rangle$, and by $f \star g = \mu \circ (f \otimes g) \circ \Delta$ the convolution product of two endomorphisms of $\mathbb{C}\langle A\rangle$, then, the operation $\star$ defined on* **FQSym** *by*

$$F \star G = \sum_\sigma \langle F \otimes G \,, \Gamma \mathbf{F}_\sigma \rangle \mathbf{G}_\sigma$$

is given by

$$\mathbf{G}_\alpha \star \mathbf{G}_\beta = \sum_\sigma m^\sigma_{\alpha,\beta} \mathbf{G}_\sigma$$

where the coefficients are the same as in

$$\alpha \star \beta = \sum_\sigma m^\sigma_{\alpha,\beta} \sigma \, .$$

Another interesting algebra which can be realized as a subalgebra of the dual $\mathbf{FQSym}^*$ is the Hopf algebra of the planar binary trees of Loday and Ronco [5]. This realization is obtained by taking $T_{\mathcal{A}} = \sum \mathbf{G}_\sigma$, where the sum runs over all permutations having the same underlying binary tree $\mathcal{A}$.

A few identities between series of free quasi-symmetric functions (mainly conjectures) can be found in [13].

3 Matrix quasi-symmetric functions

To define our next generalization, we start from a totally ordered set of commutative variables $X = \{x_1 < \cdots < x_n\}$ and consider the ideal $\mathbb{C}[X]^+$ of polynomials without constant term. We denote by $\mathbb{C}\{X\} = T(\mathbb{C}[X]^+)$ its tensor algebra.

A natural basis of $\mathbb{C}\{X\}$ is formed by tensor products of monomials (called *multiwords* in the sequel), which can be represented by nonnegative integer matrices $M = (m_{ij})$, where m_{ij} is the exponent of the variable x_i in the jth factor of the tensor product. Since constant monomials are not allowed, such matrices have no zero column. We say that they are *horizontally packed.* A multiword $\mathbf{m}$ can be conveniently encoded in the following way. Let A be the *support* of $\mathbf{m}$, that is, the set of those variables x_i such that the ith row of M is non zero, and let P be the matrix obtained form M by removing the null rows. We set $\mathbf{m} = A^P$. A matrix such as P, without zero rows or columns, is said to be *packed.*

For example the multiword $\mathbf{m} = a \otimes ab^3e^5 \otimes a^2d$ is encoded by $\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}$. Its support is the set $\{a, b, d, e\}$, and the associated packed matrix is $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}$.

Let $\mathbf{MQSym}(X)$ be the linear subspace of $\mathbb{C}\{X\}$ spanned by the elements

$$\mathbf{MS}_M = \sum_{A \in \mathcal{P}_k(X)} A^M \tag{13}$$

where $\mathcal{P}_k(X)$ is the set of k-element subsets of X, and M runs over packed matrices of height $h(m) < n$.

Proposition 5. $\mathbf{MQSym}$ *is a subalgebra of* $\mathbb{C}\{X\}$*. Actually,*

$$\mathbf{MS}_P \mathbf{MS}_Q = \sum_{R \in \text{Ш}(P,Q)} \mathbf{MS}_R$$

where the augmented shuffle *of P and Q, $\Cup(P,Q)$ is defined as follows: let r be an integer between* $\max(p,q)$ *and $p+q$, where $p=h(P)$ and $q=h(Q)$. Insert null rows in the matrices P and Q so as to form matrices $\tilde{P}$ and $\tilde{Q}$ of height r. Let R be the matrix $(\tilde{P},\tilde{Q})$. The set $\Cup(P,Q)$ is formed by all the matrices without null rows obtained in this way.*

For example :

$$\mathbf{MS}_{\begin{bmatrix}2&1\\1&0\end{bmatrix}}\mathbf{MS}_{[3\,1]} =$$
$$\mathbf{MS}_{\begin{bmatrix}2&1&0&0\\1&0&0&0\\0&0&3&1\end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix}2&1&0&0\\1&0&3&1\end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix}2&1&0&0\\0&0&3&1\\1&0&0&0\end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix}2&1&3&1\\1&0&0&0\end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix}0&0&3&1\\2&1&0&0\\1&0&0&0\end{bmatrix}}$$

Let $Y=\{y_1<\cdots<y_n\}$ be a second totally ordered set of variables, of the same cardinality as X. We identify the tensor product $\mathbf{MQSym}(X)\otimes\mathbf{MQSym}(X)$ with $\mathbf{MQSym}(X\oplus Y)$, where $X\oplus Y$ denotes the ordered sum of X and Y. The natural embedding

$$\Delta\ :\ \mathbf{MQSym}(X)\longrightarrow\mathbf{MQSym}(X\oplus Y)\simeq\mathbf{MQSym}(X)\otimes\mathbf{MQSym}(X) \tag{14}$$

defined by $\Delta(\mathbf{MS}_M(X))=\mathbf{MS}_M(X\oplus Y)$ can be interpreted as a comultiplication.

For example

$$\Delta\left(\mathbf{MS}_{\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}}\right) = 1\oplus\mathbf{MS}_{\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}} + \mathbf{MS}_{[1\,3]}\oplus\mathbf{MS}_{\begin{bmatrix}0&2&1\\0&0&3\\1&0&2\end{bmatrix}}$$
$$+\ \mathbf{MS}_{\begin{bmatrix}1&0&3\\0&2&1\end{bmatrix}}\oplus\mathbf{MS}_{\begin{bmatrix}0&3\\1&2\end{bmatrix}} + \mathbf{MS}_{\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\end{bmatrix}}\oplus\mathbf{MS}_{[1\,2]} + \mathbf{MS}_{\begin{bmatrix}1&0&3\\0&2&1\\0&0&3\\1&0&2\end{bmatrix}}\oplus 1 \tag{15}$$

From now on, unless otherwise stated, we suppose that X is infinite. Let $\mu\ :\ f\otimes g\mapsto fg$ be the multiplication of $\mathbf{MQSym}$ (induced by the multiplication of the tensor algebra), and let e be the restriction to $\mathbf{MQSym}$ of the augmentation of $T(\mathbb{C}[X]^+)$. Introduce a grading by setting $\deg(\mathbf{MS}_M)=\sum m_{ij}$ and denote by $\mathbf{MQSym}_d$ the homogeneous component of degree d.

Proposition 6. *$(\mathbf{MQSym},\mu,1,\Delta,e)$ is a self dual graded bialgebra, the duality pairing being given by $\langle\mathbf{MS}_P,\mathbf{MS}_Q\rangle=\delta P,{}^tQ$.*

The Hilbert series of $\mathbf{MQSym}$ can be expressed directly or in terms of scalar products of ribbon Schur functions. One has

$$\dim(\mathbf{MQSym}_d)=\sum_{l>0,\ h>0}\binom{d+lh-1}{lh-1}\left(\frac{1}{2}\right)^{l+h+2} \tag{16}$$

$$=\sum_{|I|=|J|=d}2^{2d-\ell(I)-\ell(J)}\langle r_I,r_J\rangle \tag{17}$$

which yields

$$\begin{aligned}&\textstyle\sum_{d\geq 0}\dim(\mathbf{MQSym}_d)\, t^d\\ &\quad = 1 + t + 5t^2 + 33t^3 + 281t^4 + 2961t^5 + 37277t^6 + 546193t^7 + \cdots\end{aligned}$$

To a packed matrix P, we can associate two compositions $\mathrm{Row}(P)$ and $\mathrm{Col}(P)$ formed by the row-sums and column-sums of P. Let Ev be the linear map defined by

$$\begin{array}{rl} \mathrm{Ev} & : \mathbf{MQSym} \longrightarrow QSym \\ & \mathbf{MS}_P \quad \longmapsto \mathrm{Ev}(\mathbf{MS}_P) = M_{\mathrm{Row}(P)} \end{array} \tag{18}$$

Proposition 7. Ev *is an epimorphism of bialgebras. Dually, the transposed map*

$$\begin{array}{rl} {}^t\mathrm{Ev} & : \mathbf{Sym} \ \rightarrow \ \mathbf{MQSym} \\ & \mathbf{MS}_P \longmapsto {}^t\mathrm{Ev}(S^I) = \sum_{\mathrm{Col}(P)=I} \mathbf{MS}_P \end{array} \tag{19}$$

is a monomorphism of bialgebras.

Therefore, **MQSym** admits $QSym$ as a quotient and **Sym** as a subalgebra. The basis $\mathbf{MS}_P$ can be regarded as a simultaneous generalization of the dual bases M_I and S^I. Moreover, $\mathbb{C}\langle X\rangle$ is naturally a subalgebra of $\mathbb{C}\{X\}$, words being identified with multiwords with exponent matrix having only one 1 in each columns. It is clear that this embedding maps $\mathbf{FQSym}(X)$ to a subspace of **MQSym**.

We now elucidate the structure of **MQSym** as an algebra. To describe a generating family, we need the following definitions. Let P be a packed matrix of height h. To a composition $K = (k_1, \dots, k_p)$ of h, we associate the matrix $P\rangle K$ defined as follows. Let $R_1, R_2, \dots, R_h$ be the rows of P. The first row of $P\rangle K$ is the sum of the first k_1 rows of P, the second row is the sum of the next k_2 rows of P, and so on. We end therefore with a matrix of height p. For example

$$\left.\begin{bmatrix} 1&2&0&2\\ 0&1&2&1\\ 1&2&0&0\\ 0&3&1&5\\ 1&3&1&0 \end{bmatrix}\right\rangle (3,2) = \begin{bmatrix} 2&5&2&3\\ 1&6&2&5 \end{bmatrix}.$$

Then, we set

$$\phi\mathbf{S}_P = \sum_{|K|=h} \frac{1}{K!}\mathbf{MS}_{P\rangle K} \tag{20}$$

where $K! = k_1!k_2!\cdots k_p!$. The family $\{\phi\mathbf{S}_P\}$, where P runs over packed matrices, is a homogeneous basis of **MQSym**.

Let us say that a packed matrix A is *connected* if it cannot be written in block diagonal form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

where B and C are not necessarily square matrices.

Theorem 1. **MQSym** *is freely generated by the family* $\{\phi \mathbf{S}_A\}$*, where* A *runs over the set of connected packed matrices.*

We conclude by an interpretation of **MQSym** in terms of invariant theory. First, we reformulate the definition of $\mathbb{C}\{X\}$ (with $X = \{x_1, \ldots, x_n\}$) in a slightly more abstract way. Let V be an n-dimensional vector space, and let $T(S^+(V))$ be the tensor algebra of the augmentation ideal of the symmetric algebra $S(V)$. Let ρ be the natural representation of $GL(V)$ in $\mathrm{End}(T(S^+(V)))$.

Theorem 2. *There exists a canonical homomorphism*

$$\phi \ : \ \mathbf{MQSym} \longrightarrow \mathrm{End}(T(S^+(V)))$$

from **MQSym** *to* $\mathrm{End}(T(S^+(V)))$ *regarded as a convolution algebra, such that for all* d*,* $\phi(\mathbf{MQSym}_d)$ *is the commutant* $\mathrm{End}_{GL(V)}(T(S^+(V))_d)$ *of* $\rho(GL(V))$ *in the homogeneous component of degree* d *of* $\mathrm{End}(T(S^+(V)))$*. Moreover,* ϕ *is one-to-one for* $d \leq n$*.*

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