A crystalizable version of $U_q(gl_N)$ *

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Dedicated to the memory of Marcel–Paul Schützenberger

Abstract: We interpret quasi-symmetric functions and noncommutative symmetric functions as characters of a degenerate quantum group obtained by putting $q = 0$ in a variant of $U_q(gl_N)$.

Résumé: Nous interprétons les fonctions quasi-symétriques et les fonctions symétriques non commutatives comme caractères d'un groupe quantique dégénéré obtenu en posant $q = 0$ dans une variante de $U_q(gl_N)$.

1 Introduction

This paper is a sequel to the series [6, 4, 17, 28] devoted to the representation theoretical interpretations of the algebras of quasi-symmetric functions (see [10]) and of noncommutative symmetric functions [8], two different generalizations of the classical theory of symmetric functions which eventually turned out to be dual to each other [23].

The new theories are very similar to the classical one, the main difference being that the rôle of partitions is in both cases played by compositions. Both algebras possess distinguished bases which can be regarded as analogues of Schur functions. In particular, the structure constants in these bases are nonnegative integers, which suggests the existence of representation-theoretical interpretations similar to the relation between symmetric functions, symmetric groups and general linear groups.

The analogue of the relation between Schur functions and representations of the symmetric group was found to be provided by the 0-Hecke algebra $H_n(0)$ of type A [6, 4, 17]. The known q-analogue of the Schur-Weyl duality between generic Hecke algebras $H_n(q)$ and the quantum groups associated with general linear groups [12] suggested the existence of an interpretation of quasi-symmetric functions and noncommutative symmetric functions as characters of some quantum group at $q = 0$.

However one has to be careful since the standard quantum groups are not defined for $q = 0$. Indeed, their presentations are generally symmetric in q and q^{-1}, and

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have no limit for $q \to 0$. The theory of crystal bases [13] which is sometimes loosely referred to as "the theory of quantum groups at $q = 0$" allows only to set $q = 0$ in certain formulas, obtained from the generic case, and involving only polynomials after some normalization. That is to say, the theory of crystal bases does not deal with a degenerate case, but rather describes the combinatorial aspects of the generic case (e.g. the combinatorics of Young tableaux in the case of gl_N). The relation between quantum groups at $q = 0$ and classical combinatorics was first pointed out in [2].

One can associate two kinds of quantum algebras to a classical group G. First a q analogue $F_q(G)$ of the algebra of polynomial functions on the group (*cf.* [7]) whose comodules are in duality with polynomial representations of the group. Second a q-analogue $U_q(\mathfrak{g})$ of the universal enveloping algebra of its Lie algebra [12].

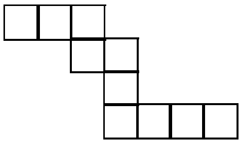
There exists a non-standard version of $F_q(GL_N)$, due to Dipper and Donkin [3] in which the specialization $q = 0$ is possible. This allowed us to intepret quasi-symmetric functions and noncommutative symmetric functions as characters of $F_0(GL_N)$ comodules, an exact analogue of the relation between Schur functions and polynomial representations of GL_N [17]. Moreover this point of view leads to an intepretation of quasi-symmetric functions similar to the plactic interpretation of symmetric functions introduced by Lascoux and Schützenberger in [20].

The Dipper-Donkin version of $F_q(GL_N)$ and the standard one are in fact specializations of a two-parameter quantization discovered by Takeuchi [27], who also gives the corresponding family $U_{\alpha,\beta}(gl_N)$ of quantized enveloping algebras. Jimbo's quantization corresponds to $U_{q,q}(gl_N)$, and the one appropriate to the Dipper-Donkin algebra is $U_{1,q}(gl_N)$.

The specialization related to quasi-symmetric functions and noncommutative symmetric functions is $\mathcal{U}_0(gl_N) = U_{0,1}(gl_N)$. It is this algebra that we study in this paper. We describe the basic properties of $U_{q,1}(gl_N)$, including the quantum Schur-Weyl duality in the generic case, and study the degeneracy at $q = 0$. We can then prove that distinguished bases of quasi-symmetric functions and of noncommutative symmetric functions are actually characters of irreducible and of indecomposable polynomial modules over the degenerate algebra and we give the classification of such modules.

2 Notations and background

A composition of n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers whose sum is equal to n. The length of I is $\ell(I) = r$. A composition can be represented by a skew Young diagram called a *ribbon diagram* of shape I. For example, the ribbon diagram of shape $I = (3, 2, 1, 4)$ is



The *mirror image* of I is $\overline{I} = (i_r, \ldots, i_1)$. The *conjugate* composition $I^\sim$ of I is obtained by reading from right to left the heights of the columns of the ribbon diagram of I. For example, $(3,2,1,4)^\sim = (1,1,1,3,2,1,1)$.

One associates with a composition $I = (i_1, \ldots, i_r)$ of n the subset $D(I)$ of $[1, n-1]$ defined by $D(I) = \{\, i_1, i_1 + i_2, \ldots, i_1 + \ldots + i_{r-1} \,\}$. The *reverse refinement order*, denoted $\preceq$, on the set of all compositions of the same integer n is defined by $I \preceq J$ iff $D(I) \subseteq D(J)$. For example $(3,2,6) \preceq (2,1,2,3,1,2)$. One associates to a permutation σ of $\mathfrak{S}_n$ a composition $C(\sigma)$ of n called the *descent composition* of σ, whose ribbon diagram is called the shape of σ. The descent composition of σ encodes its *descent set* $\mathcal{D}(\sigma) = \{ i \in [1, n-1] \mid \sigma(i) > \sigma(i+1) \}$. That is, $D(C(\sigma)) = \mathcal{D}(\sigma)$. For example, $C(245316) = (3,1,2)$ and $\mathcal{D}(245316) = \{3,4\}$..

The set D_I of all permutations $\sigma \in \mathfrak{S}_n$ having I as descent composition is an interval $[\alpha(I), \omega(I)]$ for the weak order (permutohedron) of $\mathfrak{S}_n$, where $\alpha(I)$ is the permutation obtained by filling the columns of the ribbon diagram of shape I from bottom to top and from left to right with $1, 2, \ldots, n$ and $\omega(I)$ is the permutation obtained by filling the rows of the ribbon diagram of shape I from left to right and from bottom to top with $1, 2, \ldots, n$. For example, $\alpha(2,2,1,1,3) = 132765489$ and $\omega(2,2,1,1,3) = 896754123$.

Let $(A, <)$ be a totally ordered alphabet. A *tableau* of ribbon shape I is a skew Young tableau obtained by filling a ribbon diagram R of shape I by letters of A in such a way that each row of r is nondecreasing from left to right and each column of r is strictly increasing from bottom to top. A word w is said to be of *ribbon shape* I if it can be obtained by reading from top to bottom and from left to right the columns of a tableau of ribbon shape I, which amounts to say that the descent composition of w is I. For example, the ribbon shape of $w = aacbacdcd$ is $(3,1,3,2)$. A *quasi-tableau* of ribbon shape I is an object obtained by filling a ribbon diagram r of shape I by letters of A in such a way that each row of r is nondecreasing from left to right and each column of r is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-tableau of shape I. For example, the word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-tableau. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3,1,3,2)$.

The algebra of *noncommutative symmetric functions* [8] is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \ldots, i_r)$, one sets $S^I = S_{i_1} S_{i_2} \ldots S_{i_r}$. The family (S^I) is a linear basis of $\mathbf{Sym}$. Although it is convenient to define $\mathbf{Sym}$ as an abstract algebra, a useful realisation can be obtained by taking an infinite alphabet $A = \{\, a_1, a_2, \ldots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geq 0} t^n \, S_n(A) = \overrightarrow{\prod_{i \geq 1}} \, (1 - t\, a_i)^{-1} \, . \tag{1}$$

The noncommutative *ribbon Schur functions* R_I can be defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} \, S^J \, . \tag{2}$$

The R_I form a basis of **Sym**. In the realization of **Sym** given by equation (1), R_I reduces to the sum of all words of shape I [8].

The algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions introduced by Gessel in [10] (*cf.* [8, 23]). Let $X = \{ x_1, x_2, \ldots \}$ be an infinite totally ordered set of commutative indeterminates. An element $f \in \mathbb{C}[X]$ is said to be a *quasi-symmetric function* iff for each composition $K = (k_1, \ldots, k_r)$ all monomials $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$ with $i_1 < i_2 < \ldots < i_r$ have the same coefficient in f. The quasi-symmetric functions form a subalgebra of $\mathbb{C}[X]$ denoted by $QSym$. The *quasi-monomial functions*

$$M_I = \sum_{j_1 < \ldots < j_r} x_{j_1}^{i_1} \ldots x_{j_r}^{i_r}$$

(labelled by compositions I) form a basis of $QSym$. Another important basis of $QSym$ is given by the *quasi-ribbon functions*

$$F_I = \sum_{I \preceq J} M_J \ ,$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I. The pairing $\langle \cdot, \cdot \rangle$ between **Sym** and $QSym$ is defined by $\langle S^I, M_J \rangle = \delta_{IJ}$ or equivalently $\langle R_I, F_J \rangle = \delta_{IJ}$ (*cf.* [23, 8]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras [6, 17].

3 The hypoplactic algebra

The *plactic algebra* [20] on the totally ordered alphabet A is the $\mathbb{C}$-algebra $Pl(A)$ defined as the quotient of $\mathbb{C}\langle A \rangle$ by the relations

$$\begin{cases} aba = baa \ , \ bba = bab & \text{for } a < b \ , \\ acb = cab \ , \ bca = bac & \text{for } a < b < c \ . \end{cases}$$

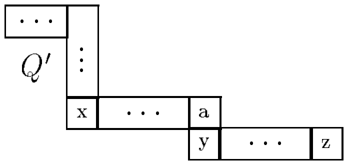
These relations, due to Knuth [18], generate the equivalence relation that identifies two words which have the same P-symbol under the Robinson-Schensted correspondence. The associated monoid structure on the set of tableaux was studied by Lascoux and Schützenberger [20] under the name "plactic monoid". Their point of view is now illuminated by Kashiwara's theory of crystal bases [13, 14] which also leads to the definition of plactic algebras associated with the classical Lie algebras [19, 22]. Other interpretations of the plactic relations in terms of quantum groups can be found in [2, 21].

The hypoplactic algebra is a quotient of the plactic algebra where ribbon quasi-tableaux play the same rôle as Young tableaux in the plactic case. This algebra was introduced in [17] where it is shown that a remarkable commutative subalgebra of the hypoplactic algebra is isomorphic to the character ring of a degenerate quantum group and provides a hypoplactic realization of the algebra of quasi-symmetric functions analogous to the plactic interpretation of ordinary symmetric functions discovered by Lascoux and Schützenberger [20]. We will see in Section 8 that the same result holds when one rather considers polynomial modules over the crystalization of the version of $\mathcal{U}_q(gl_N)$ introduced in Section 4.

Definition 3.1 *The* hypoplactic algebra $HPl(A)$ *is the quotient of the plactic algebra* $Pl(A)$ *by the quartic relations*

$$\begin{cases} baba = abab \ , \ baca = abac & \text{for } a < b < c \ , \\ cacb = acbc \ , \ cbab = bacb & \text{for } a < b < c \ , \\ badc = dbca \ , \ acbd = cdab & \text{for } a < b < c < d \ . \end{cases}$$

The hypoplactic monoid has the same properties as the plactic monoid with respect to a Robinson-Schensted type correspondence where tableaux are replaced by ribbon quasi-tableaux [17]. The insertion algorithm is as follows. Let Q be a ribbon quasi-tableau and let $a \in A$. Denote by Q' be the quasi-tableau obtained from Q by deleting its last row and let x (resp. z) be the first (resp. last) letter of the last row of Q. The result $\mathcal{Q}$ of the insertion of a in Q is defined by the following rules: 1) If $z \leq a$, $\mathcal{Q}$ is obtained by adding a box containing a at the end of the last row of Q; 2) if $x \leq a < z$, let y be the first letter of the last row of Q which is strictly greater than a. The quasi-tableau $\mathcal{Q}$ is then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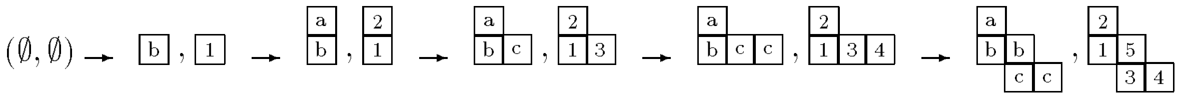
3) if $a < x$, $\mathcal{Q}$ is obtained by inserting a in Q' and glueing under the quasi-ribbon tableau obtained in this way the last row of Q.

Let now $w = a_1 \ldots a_n$ be a word of length n. One associates with w a pair (Q, R), where Q is a quasi-tableau of ribbon shape I and where R is a standard ribbon tableau of the same shape. Q is obtained by inserting the letters of w (from left to right) starting from an empty diagram and R is constructed by putting at each step $i \in [1, n]$ of the insertion algorithm the number i in the box that contains at this moment in Q the letter a_i inserted at this step. Let us illustrate this process on $w = baccb$.



The correspondence $w \to (Q, R)$ is a bijection, and we have the following quasi-ribbon analogue of Knuth's theorem (*cf.* [18]).

Proposition 3.2 [17] *Two words* $u, v \in \mathbb{A}^*$ *are mapped to the same quasi-tableau* Q *by the above insertion process iff* $u \equiv v$ *with respect to the hypoplactic congruence.*

We also recall the quasi-ribbon version of the cross-section theorem of [20].

Theorem 3.3 [17] *The classes of quasi-ribbon words form a linear basis of the hypoplactic algebra.*

Other combinatorial properties of the hypoplactic monoid have been investigated by Novelli [25].

4 The quantum enveloping algebra $\mathcal{U}_q(gl_N)$

In this section, we introduce a variant of the quantized enveloping algebra $U_q(gl_n)$ which displays the same behaviour for generic q, but can also be specialized at $q = 0$. To allow this specialization, we have to modify a little Takeuchi's $U_{q,1}(gl_N)$, that is, to use non-invertible generators for the Cartan part, and to define it only as a bialgebra (no antipode). This algebra, as well as other multiparameter deformations, can be realized by specialization of the difference operators of [9].

Definition 4.1 *Let q be an indeterminate or a complex parameter. The* crystalizable quantum analogue $\mathcal{U}_q(gl_N)$ *of the universal enveloping algebra of the Lie algebra gl_N is the algebra over $\mathbb{C}(q)$ generated by the elements $(e_i)_{1\le i\le N-1}$, $(f_i)_{1\le i\le N-1}$ and $(k_i)_{1\le i\le N}$ with relations :*

$$k_i\,k_j = k_j\,k_i \qquad \text{for } 1 \le i,j \le N\ , \tag{3}$$

$$\left\{\begin{array}{rcll} q\,k_i\,e_{i-1} &=& e_{i-1}\,k_i & \text{for } 2 \le i \le N-1\ ,\\ k_i\,e_i &=& q\,e_i\,k_i & \text{for } 1 \le i \le N-1\ ,\\ k_i\,e_j &=& e_j\,k_i & \text{for } j \ne i-1, i\ , \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{rcll} k_i\,f_{i-1} &=& q\,f_{i-1}\,k_i & \text{for } 2 \le i \le N-1\ ,\\ q\,k_i\,f_i &=& f_i\,k_i & \text{for } 1 \le i \le N-1\ ,\\ k_i\,f_j &=& f_j\,k_i & \text{for } j \ne i-1, i\ , \end{array}\right. \tag{5}$$

$$[e_i, f_j] = \delta_{ij}\,\frac{k_i - k_{i+1}}{q-1} \qquad \text{for } 1 \le i,j \le N\ , \tag{6}$$

$$\left\{\begin{array}{rcll} q\,e_{i+1}\,e_i^2 - (1+q)\,e_i\,e_{i+1}\,e_i + e_i^2\,e_{i+1} &=& 0 & \text{for } 1 \le i \le N-2\ ,\\ q\,e_{i+1}^2\,e_i - (1+q)\,e_{i+1}\,e_i\,e_{i+1} + e_i\,e_{i+1}^2 &=& 0 & \text{for } 1 \le i \le N-2\ , \end{array}\right. \tag{7}$$

$$\left\{\begin{array}{rcll} f_{i+1}\,f_i^2 - (1+q)\,f_i\,f_{i+1}\,f_i + q\,f_i^2\,f_{i+1} &=& 0 & \text{for } 1 \le i \le N-2\ ,\\ f_{i+1}^2\,f_i - (1+q)\,f_{i+1}\,f_i\,f_{i+1} + q\,f_i\,f_{i+1}^2 &=& 0 & \text{for } 1 \le i \le N-2\ , \end{array}\right. \tag{8}$$

$$\left\{\begin{array}{rcll} [e_i, e_j] &=& 0 & \text{for } |i-j| > 1\ ,\\ [f_i, f_j] &=& 0 & \text{for } |i-j| > 1\ , \end{array}\right. \tag{9}$$

$$\left\{\begin{array}{rcll} e_i\,e_{i+1}\,k_{i+1} &=& k_{i+1}\,e_i\,e_{i+1} & \text{for } 1 \le i \le N-2\ ,\\ f_{i+1}\,f_i\,k_{i+1} &=& k_{i+1}\,f_{i+1}\,f_i & \text{for } 1 \le i \le N-2\ . \end{array}\right. \tag{10}$$

Relations (10) are consequences of the other relations when $q \neq 0$ and do not play any rôle in the generic case. One adds them in order to endow $\mathcal{U}_q(gl_N)$ with a bialgebra structure at $q=0$. Not also that the q-Serre relations (7) are exactly those obtained by Ringel [26] in his construction of $U_q(\mathfrak{n}_+)$ via the Hall algebras associated to quivers. In this construction, the parameter q is the cardinality of a finite field.

Proposition 4.2 *Let $q \in \mathbb{C}$. Then $\mathcal{U}_q(gl_N)$ is a $\mathbb{C}$-bialgebra for the comultiplication Δ, with counity ε defined by*

$$\begin{cases} \Delta(e_i) &=& 1 \otimes e_i + e_i \otimes k_i, & \varepsilon(e_i) &=& 0 & \textit{for } 1 \leq i \leq N-1 \ , \\ \Delta(f_i) &=& k_{i+1} \otimes f_i + f_i \otimes 1, & \varepsilon(f_i) &=& 0 & \textit{for } 1 \leq i \leq N-1 \ , \\ \Delta(k_i) &=& k_i \otimes k_i, & \varepsilon(k_i) &=& 1 & \textit{for } 1 \leq i \leq N \ . \end{cases}$$

Let $(\xi_i)_{1 \leq i \leq N}$ be the canonical basis of $V = \mathbb{C}^N$, and let E_{ij} be the endomorphism $E_{ij}\xi_k = \delta_{jk}\xi_i$. One can define an algebra morphism ρ_V of $\mathcal{U}_q(gl_N)$ in $End_{\mathbb{C}}(V)$ by setting

$$\begin{cases} \rho_V(e_i) &=& E_{i,i+1} & \text{for } 1 \leq i \leq N-1 \ , \\ \rho_V(f_i) &=& E_{i+1,i} & \text{for } 1 \leq i \leq N-1 \ , \\ \rho_V(k_i) &=& q\, E_{i,i} + \sum_{j \neq i} E_{j,j} & \text{for } 1 \leq i \leq N \ . \end{cases}$$

The pair (ρ_V, V) is called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_q(gl_N)$. Since $\mathcal{U}_q(gl_N)$ is a bialgebra, one can define for every $n \geq 2$ its n-th tensor power $(\rho_{n,N}, V^{\otimes n})$ by $\rho_{n,N} = \rho_V^{\otimes n} \circ \Delta^{(n)} \ : \ \mathcal{U}_q(gl_N) \longrightarrow End_{\mathbb{C}}(V^{\otimes n})$, where $\Delta^{(n)}$ denotes the n-fold comultiplication $\mathcal{U}_q(gl_N) \to \mathcal{U}_q(gl_N)^{\otimes n}$.

5 The quantum matrix algebra $A_q(N)$

We denote by $A_q(N)$ the $\mathbb{C}(q)$-algebra generated by N^2 elements $(x_{ij})_{1 \leq i,j \leq N}$ subject to the defining relations:

$$\begin{cases} x_{il}\, x_{jk} &=& q\, x_{jk}\, x_{il} & \text{for } 1 \leq i \leq j \leq N, \ 1 \leq k < l \leq N \ , \\ x_{ik}\, x_{jk} &=& x_{jk}\, x_{ik} & \text{for } 1 \leq i, j, k \leq N \ , \\ x_{jl}\, x_{ik} - x_{ik}\, x_{jl} &=& (q-1)\, x_{jk}\, x_{il} & \text{for } 1 \leq i < j \leq N, \ 1 \leq k < l \leq N \ . \end{cases} \tag{11}$$

This algebra is a quantization of the bialgebra of polynomial functions on the variety of $N \times N$ matrices over $\mathbb{C}$. It is the specialization $GL_{q,1}(N)$ of Takeuchi's algebra [27]. Up to the symmetry $x_{ij} \leftrightarrow x_{ji}$, $A_q(N)$ is the quantization introduced by Dipper and Donkin in [3]. It is not isomorphic to the more familiar quantization of Faddev-Reshetikhin-Takhtadzhyan [7]. Although for generic values of q both versions have the same representation theory, an essential difference is that $A_q(N)$ is defined for $q = 0$.

One can endow $A_q(N)$ with a bialgebra structure whose comultiplication $\overline{\Delta}$ and counity η are defined by

$$\overline{\Delta}(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj} \qquad \text{and} \qquad \eta(x_{ij}) = \delta_{ij} \ .$$

An important subalgebra of $A_q(N)$ is the quantum diagonal algebra (or quantum torus) $\Delta_q(N)$. It is a q-analogue of the algebra of polynomial functions on a maximal torus of GL_N. As such, it occurs in the character theory of Hecke algebras, of $A_q(N)$ and of $\mathcal{U}_q(gl_N)$ (*cf.* Section 7 and [17]).

Definition 5.1 *The* quantum diagonal algebra $\Delta_q(N)$ *is the subalgebra of* $A_q(N)$ *generated by the diagonal elements* $x_{11}, \ldots, x_{NN}$.

We propose a conjecture that would give a presentation of $\Delta_q(N)$ in the generic case. Let $(A, <)$ be a totally ordered alphabet. We define the quantum pseudoplactic algebra $PPl_q(A)$ as the quotient of $\mathbb{C}(q)\langle A\rangle$ by the relations

$$\begin{cases} q\,a\,a\,b - (q+1)\,a\,b\,a + b\,a\,a &= 0 \qquad \text{for } a < b\,, \\ q\,a\,b\,b - (q+1)\,b\,a\,a + b\,b\,a &= 0 \qquad \text{for } a < b\,, \\ c\,a\,b - a\,c\,b - b\,c\,a + b\,a\,c &= 0 \qquad \text{for } a < b < c\,. \end{cases}$$

Conjecture 5.2 [17] *Let* $A = \{\, a_1 < \ldots < a_N \,\}$. *For* $q \notin \{0, 1\}$, *the mapping* $\varphi : a_i \to x_{ii}$ *induces an isomorphism between* $PPl_q(A)$ *and* $\Delta_q(N)$.

For $q = 1$ the diagonal algebra $\Delta_1(N)$ is just an algebra of commutative polynomials. The case $q = 0$ is more interesting:

Theorem 5.3 [17] *The ring homomorphism defined by* $\varphi : a_i \longrightarrow x_{ii}$ *is an isomorphism between the hypoplactic algebra* $HPl(A)$ *and the quantum diagonal algebra* $\Delta_0(N)$ *at* $q = 0$.

For any $q \in \mathbb{C}$, the bialgebras $A_q(N)$ and $\mathcal{U}_q(gl_N)$ are in duality, in a sense to be made precise below. According to the general theory of bialgebras (*cf.* [15]), the dual $\mathcal{U}_q(gl_N)^*$ of $\mathcal{U}_q(gl_N)$ has a canonical algebra structure, defined by the convolution product

$$\varphi \cdot \psi = \mu \circ (\varphi \otimes \psi) \circ \Delta$$

for $\varphi, \psi \in \mathcal{U}_q(gl_N)^*$, where μ is the multiplication map $\mu(g \otimes h) = g\,h$, $g, h \in \mathcal{U}_q(gl_N)$.

Define linear functionals $(a_{ij})_{1\le i,j\le N} \in \mathcal{U}_q(gl_N)^*$ as the matrix coefficients of the vector representation, *i.e.* $\rho_V(g) = (a_{ij}(g))_{1\le i,j\le N}$ for $g \in \mathcal{U}_q(gl_N)$.

Theorem 5.4 [27] *The* N^2 *linear functionals* $(a_{ij})_{1\le i,j\le N}$ *of* $\mathcal{U}_q(gl_N)^*$ *satisfy the quantum relations (11).*

Therefore, the map $\Psi: \; A_q(N) \longrightarrow \mathcal{U}_q(gl_N)^*$ defined by $\Psi(x_{ij}) = a_{ij}$ for $1 \le i, j \le N$ is an algebra morphism. The duality of bialgebras between $A_q(N)$ and $\mathcal{U}_q(gl_N)$ is given by the pairing $\langle x, g\rangle = \Psi(x)(g)$ ($x \in A_q(N)$, $g \in \mathcal{U}_q(gl_N)$).

6 Quantum Schur-Weyl duality

A fundamental property of the standard version of $U_q(gl_N)$ is the existence of a q-analogue of the Schur-Weyl duality involving the Hecke algebra instead of the symmetric group [12]. Such a duality can also be worked out for $\mathcal{U}_q(gl_N)$. The double commutant theorem will break down at $q = 0$, but one can still give a weaker statement, which will allow us to interpret quasi-symmetric functions and noncommutative symmetric functions as characters of $\mathcal{U}_0(gl_N)$.

The (Iwahori) Hecke algebra $H_n(q)$ of type A_{n-1} is the $\mathbb{C}(q)$-algebra generated by the $n-1$ elements $(T_i)_{1\le i\le n-1}$ with relations

$$\begin{cases} T_i^2 &= (q-1)\,T_i + q & \text{for } 1 \le i \le n-1 \ , \\ T_i\,T_{i+1}\,T_i &= T_{i+1}\,T_i\,T_{i+1} & \text{for } 1 \le i \le n-2 \ , \\ T_i\,T_j &= T_j\,T_i & \text{for } |i-j| > 1 \ . \end{cases}$$

For generic complex values of q, the Hecke algebra $H_n(q)$ is isomorphic to $\mathbb{C}[\mathfrak{S}_n]$ (the standard presentation is obtained for $q = 1$) and hence semi-simple, except when $q = 0$ or when q is a root of unity.

Following [3, 5], we introduce a variant of Jimbo's right action of $H_n(q)$ on $V^{\otimes n}$ (*cf.* [12]) which is defined by setting

$$\begin{cases} \mathbf{v}\cdot T_i &= \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \ , \\ \mathbf{v}\cdot T_i &= q\,\mathbf{v} & \text{if } k_i = k_{i+1} \ , \\ \mathbf{v}\cdot T_i &= q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \ , \end{cases}$$

for $\mathbf{v} = \xi_{k_1} \otimes \cdots \otimes \xi_{k_n} \in V^{\otimes n}$ (where $\mathbf{v}^{\sigma_i}$ denotes the tensor obtained from $\mathbf{v}$ by permuting the i and $i{+}1$-th factors). We denote by $\pi_{n,N}$ the representation of $H_n(q)$ in $End_{\mathbb{C}}(V^{\otimes n})$ defined by this action. One can check that $\rho_{n,N}$ commutes with $\pi_{n,N}$. For generic q, one has a stronger property:

Theorem 6.1 *Let q be a nonzero complex number which is not a non trivial k-th root of unity for some $k \le n$. The two subalgebras $\pi_{n,N}(H_n(q))$ and $\rho_{n,N}(\mathcal{U}_q(gl_N))$ of $End_{\mathbb{C}}(V^{\otimes n})$ are then the commutant of each other.*

7 Polynomial $\mathcal{U}_q(gl_N)$-modules and their characters

In the classical case, the character of a polynomial representation of GL_N can be regarded as a polynomial function on the group GL_N. In the quantum case, it is natural to define the character as an element of the deformed function algebra $A_q(N)$.

Definition 7.1 *A* polynomial $\mathcal{U}_q(gl_N)$-module *of degree n is a submodule of $V^{\otimes n}$.*

In this section we assume that q is a complex number which is neither zero nor a non trivial k-th root of unity for some $k \le n$. For a partition λ of n, let $\mathcal{Y}_\lambda(q) \in H_n(q)$ be a q-Young idempotent. It will be convenient to use the version introduced in [5].

The family $(H_n(q)\,\mathcal{Y}_\lambda(q))_{\lambda\vdash n}$ is a complete system of $H_n(q)$-irreducible modules. Since $H_n(q)$ is semi-simple, it follows from Theorem 6.1 that the family of $\mathcal{U}_q(gl_N)$-modules

$$\mathbb{S}_\lambda(V;q) = V^{\otimes n}\cdot\mathcal{Y}_\lambda(q)\ ,$$

where λ runs over all partitions of n, is a complete system of irreducible polynomial $\mathcal{U}_q(gl_N)$-modules of degree n.

Let now M be a polynomial $\mathcal{U}_q(gl_N)$-module of degree n and let $(m_i)_{1\le i\le m}$ be a $\mathbb{C}(q)$-linear basis of M. Since M is a $\mathcal{U}_q(gl_N)$-module, there exists a family $(\mu_{ij})_{1\le i,j\le m}$ of linear functionals of $\mathcal{U}_q(gl_N)^*$ such that for $g\in\mathcal{U}_q(gl_N)$

$$g\cdot m_i = \sum_{j=1}^{m}\ \mu_{ij}(g)\,m_j\ .$$

One can check that the trace of the action of $\mathcal{U}_q(gl_N)$ on M, *i.e.* the element

$$c(M) = \sum_{i=1}^{m}\ \mu_{ii}\in\mathcal{U}_q(gl_N)^*$$

is independent on the choice of the basis. Moreover $c(M)$ belongs to the subalgebra generated by the $(a_{ij})_{1\le i,j\le N}$. Indeed, for $I\in[1,N]^n$,

$$g\cdot\xi^I = \sum_{J\in[1,N]^n}\ a_{JI}(g)\,\xi^J$$

where $a_{JI}=a_{j_1i_1}\,\ldots\,a_{j_ni_n}$. It follows that there exists elements $p_{IJ}(q)$ of $\mathbb{C}(q)$ such that

$$c(M) = \sum_{I,J\in[1,N]^n}\ p_{IJ}(q)\,a_{IJ}\ .$$

Since the map Ψ (defined at the end of Section 5) is injective on the linear subspace of $A_q(N)$ spanned by homogeneous elements of fixed length, we can lift this formula to $A_q(N)$ (according to Theorem 5.4). Let now x_{IJ} be the monomial $x_{i_1j_1}\,\ldots\,x_{i_nj_n}\in A_q(N)$. The element

$$\chi(M) = \sum_{I,J\in[1,N]^n}\ p_{IJ}(q)\,x_{IJ}$$

of $A_q(N)$ will be called the *character* of the polynomial $\mathcal{U}_q(gl_N)$-module M.

Theorem 7.2 [17] *For $q\ne 1$, $\chi(M)$ belongs to the diagonal algebra $\Delta_q(n)$.*

The proof of Theorem 7.2 relies on an interesting formula which expresses the quantum determinant in $A_q(N)$ in terms of the diagonal generators x_{ii} of $A_q(N)$:

$$\left|\begin{matrix} x_{11} & x_{12} & \ldots & x_{1N}\\ x_{21} & x_{22} & \ldots & x_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ x_{N1} & x_{n2} & \ldots & x_{NN}\end{matrix}\right|_q \overset{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_N}\ \varepsilon(\sigma)\,x_{\sigma(1)1}\,\ldots\,x_{\sigma(N)N} = \frac{1}{(1-q)^{N-1}}\,[x_{NN},[\ldots,[x_{22},x_{11}]_q\ldots]_q]_q\ ,$$

where $\varepsilon(\sigma)$ is the signature of σ and $[a,b]_q = ab - qba$.

One can show that the commutative character of $\mathbb{S}_\lambda(V;q)$ is the Schur function s_λ. Hence the characters in our sense are quantum analogues of Schur functions. One can check that

$$\chi(\mathbb{S}_n(V;q)) = \sum_{i_1+\ldots+i_N=n} x_{11}^{i_1} \ldots x_{NN}^{i_N} ,$$

$$\chi(\mathbb{S}_{1^n}(V;q)) = \frac{1}{(1-q)^{n-1}} \sum_{1\le i_1<\ldots<i_n\le N} [x_{i_ni_n}, [\ldots, [x_{i_2i_2}, x_{i_1i_1}]_q \ldots]_q]_q$$

for all n. These quantum Schur functions form a linear basis of a commutative subalgebra of $\Delta_q(N)$ which is isomorphic to the algebra of ordinary symmetric functions [17].

8 Polynomial $\mathcal{U}_0(gl_N)$-modules

From now on, we set $q=0$. It follows from Theorem 7.2 and from Theorem 5.3 that the character of a polynomial $\mathcal{U}_0(gl_N)$-module is an element of the hypoplactic algebra. However, $\rho_{n,N}(\mathcal{U}_0(gl_N))$ is not semisimple, the modules $\mathbb{S}_\lambda(V;0)$ of the preceding section are in general no longer irreducible, and we have to look for the simple composition factors of $V^{\otimes n}$ (the irreducible polynomial modules) and its indecomposable direct summands. These modules can be constructed in the same way as the irreducible modules and the principal indecomposable projective modules of the 0-Hecke algebra.

For $i \in [1, n-1]$, let $\square_i = 1 + T_i$ considered here as an element of $H_n(0)$. These elements satisfy the braid relations, together with $\square_i^2 = \square_i$ so that one can define, for $\sigma \in \mathfrak{S}_n$, the element $\square_\sigma = \square_{i_1} \ldots \square_{i_r}$ where $\sigma_{i_1} \ldots \sigma_{i_r}$ is any reduced decomposition of σ.

Let us now see how to construct a complete family of irreducible polynomial $\mathcal{U}_0(gl_N)$-modules. For I a composition of n one defines the element

$$\eta_I = T_{\omega(\overline{I})} \square_{\alpha(I^\sim)}$$

of $H_n(0)$. Norton [24] showed that the $H_n(0)\,\eta_I$ form a complete family of irreducible $H_n(0)$-modules (see also [1, 17]). As the action of $\mathcal{U}_0(gl_N)$ on $V^{\otimes n}$ commutes with the right action of $H_n(0)$, one can use η_I to construct the $\mathcal{U}_0(gl_N)$-module

$$\mathbf{D}_I = V^{\otimes n} \cdot \eta_I \ .$$

Proposition 8.1 $\mathbf{D}_I$ *is an irreducible* $\mathcal{U}_0(gl_N)$*-module. Its character is equal to the sum* $F_I(x_{11}, \ldots, x_{NN})$ *of all quasi-ribbon words over* $\{ x_{11} < \ldots < x_{NN} \}$.

The character of $\mathbf{D}_I$ is therefore a hypoplactic analogue of the quasisymmetric function F_I. The modules $\mathbf{D}_I$ have a simple combinatorial structure, described by a *quasi-crystal graph*, whose vertices are quasi-ribbons. It has been recently shown by Hivert [11] that the ordinary characters $F_I(x_1, \ldots, x_n)$ can be obtained by a version of the Weyl character formula, refinable into a Demazure type formula. This formalism also provides quasi-symmetric and noncommutative analogues of Hall-Littlewood functions.

To construct the indecomposable summand of $V^{\otimes n}$ labelled by the composition I of n, we consider the element

$$\nu_I = T_{\alpha(I)} \square_{\alpha(\overline{I}^\sim)} \ .$$

These elements generate a complete family $(H_n(0)\,\nu_I)$ of indecomposable projective left $H_n(0)$-modules [1, 24]. We can construct the $\mathcal{U}_0(gl_N)$-module

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I \ .$$

These modules are indecomposable, and their characters are hypoplactic analogues of ribbon Schur functions.

Proposition 8.2 *$\mathbf{N}_I$ is an indecomposable $\mathcal{U}_0(gl_N)$-module. Its character is the sum $R_I(x_{11}, \ldots, x_{NN})$ of all words of ribbon shape I over $\{x_{11} < \ldots < x_{NN}\}$.*

Using the decomposition

$$V^{\otimes n} = \bigoplus_{I \vdash n} \mathbf{N}_I \tag{12}$$

as $\mathcal{U}_0(gl_N)$-modules, we can now give the complete classification of irreductible polynomial $\mathcal{U}_0(gl_N)$-modules and of the indecomposable direct summands of $V^{\otimes n}$.

Proposition 8.3 (i) *The $\mathbf{D}_I$ form a complete family of irreducible polynomial $\mathcal{U}_0(gl_N)$-modules.*
(ii) *The $\mathbf{N}_I$ form a complete family of indecomposable polynomial $\mathcal{U}_0(gl_N)$-modules which are direct summands of some $V^{\otimes n}$.*

The noncommutative analogues of quasi-ribbon functions arising in Proposition 8.3 form a linear basis of a commutative subalgebra of the hypoplactic algebra. This subalgebra is isomorphic to the algebra of quasi-symmetric functions and plays therefore the rôle of the character ring of polynomial $\mathcal{U}_0(gl_N)$-modules. Similarly, the algebra of noncommutative symmetric functions is a Grothendieck ring for "projective" polynomial $\mathcal{U}_0(gl_N)$-modules (*i.e.* direct summands of some $V^{\otimes n}$). The noncommutativity of this ring reflects the fact that at $q = 0$, the tensor products $M \otimes N$ and $N \otimes M$ are actually not isomorphic.

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