# $q$-Deformed Fock spaces and modular representations of spin symmetric groups

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**Abstract.** We use the Fock space representation of the quantum affine algebra of type $A_{2n}^{(2)}$ to obtain a description of the global crystal basis of its basic level 1 module. We formulate a conjecture relating this basis to decomposition matrices of spin symmetric groups in characteristic $2n+1$.

## 1. Introduction

The Fock space representation of the quantum affine algebra $U_q(\widehat{sl}_n) = U_q(A_{n-1}^{(1)})$ was constructed by Hayashi [14]. A combinatorial version of this construction was then used by Misra and Miwa [27] to describe Kashiwara's crystal basis of the basic representation $V(\Lambda_0)$. This made it possible to compute the global crystal basis of $V(\Lambda_0)$ [24]. Then, it was conjectured that the transition matrices giving the coefficients of the global basis on the natural basis of the Fock space were $q$-analogues of the decomposition matrices of the type $A$ Hecke algebras $H_m$ at an $n$th root of unity. According to a conjecture of James [15], these should coincide, for $n$ prime and large enough, with the decomposition matrices of symmetric groups $\mathrm{S}_m$ over a field of characteristic $n$. The conjecture of [24] has been proved by Ariki [2], and by Grojnowski [13] using the results of [12].

The results of [24, 2, 13] have been applied recently by Foda *et al.* [10] to determine which simple $H_m$-modules remain simple by restriction to $H_{m-1}$ and to show that this problem is equivalent to the decomposition of a tensor product of level 1 $A_{n-1}^{(1)}$-modules. This provided an explanation for an intriguing correspondence previously observed in [11] between a class of RSOS models and modular representations of symmetric groups.

Another description of the $U_q(A_{n-1}^{(1)})$-Fock space, as a deformation of the infinite wedge realization of the fermionic Fock space, was obtained by Stern [34]. In [23], the $q$-bosons needed for the decomposition of the Fock space into irreducible $U_q(A_{n-1}^{(1)})$-modules were introduced. This construction was used in [25] to give a combinatorial formula for the highest weight vectors, and in [26] to define a canonical basis of the whole Fock space which was conjectured to yield the decomposition matrices of $q$-Schur algebras at roots of unity. Moreover, strong support in favor of this conjecture was obtained by establishing its compatibility with a version of the Steinberg tensor product theorem proved by James in this context [15, 26].

Recently, the theory of perfect crystals [19, 20] allowed Kashiwara *et al.* [22] to define a general notion of $q$-Fock space, extending the results of [23] to several series of affine algebras. Their results apply in particular to the twisted affine algebra of type $A_{2n}^{(2)}$, which is the case considered in this note.

It has been noticed by Nakajima and Yamada [31] that the combinatorics of the basic representation $V(\Lambda_n)$ of $A_{2n}^{(2)}$ was similar to the one encountered in the $(2n+1)$-modular representation theory of the spin symmetric groups $\widehat{\mathrm{S}}_m$ by Morris [28] as early as 1965. This can be explained to a certain extent by observing that the $(r, \bar{r})$-inducing operators of Morris and Yaseen [30] coincide with the Chevalley lowering operators of the Fock space representation of $A_{2n}^{(2)}$. This provides a further example of the phenomenon observed in [24] in the case of symmetric groups and $A_{n-1}^{(1)}$-algebras.

In this note, we give the analogues for $U_q(A_{2n}^{(2)})$ of the results of [24]. Using the level 1 $q$-Fock spaces of [22], we describe an algorithm for computing the canonical basis of the basic representation $V(\Lambda_n)$, which allows us to prove that this basis is in the $\mathbf{Z}[q]$-lattice spanned by the natural basis of the $q$-Fock space, and that the transition matrices have an upper triangle of zeros. We conjecture that the specialization $q = 1$ gives, up to splitting of rows and columns for pairs of associate characters, and for sufficiently large primes $p = 2n + 1$, the decomposition matrices of spin symmetric groups. However, the reduction $q = 1$ is more tricky than in the $A_{n-1}^{(1)}$ case. Indeed, the $q$-Fock space of $A_{2n}^{(2)}$ is strictly larger than the classical one, and one has to factor out the null space of a certain quadratic form [22] to recover the usual description.

This should have implications on the way of labelling the irreducible modular spin representations of $\widehat{\mathrm{S}}_m$. Up to now, a coherent labelling scheme has been found only for $p = 3$ [6] and $p = 5$ [1]. The case $p \geq 7$ led to formidable difficulties. To overcome this problem, we propose to use the labels of the crystal graph of $V(\Lambda_n)$, which may contain partitions with repeated parts not arising in the representation theory of $\widehat{\mathrm{S}}_m$, and corresponding to ghost vectors of the $q$-Fock space at $q = 1$.

## 2. The Fock space representation of $U_q(A_{2n}^{(2)})$

The Fock space representation of the affine Lie algebra $A_{2n}^{(2)}$ can be constructed by means of its embedding in $b_\infty = \widehat{go}_\infty$, the completed infinite rank affine Lie algebra of type $B$ [8, 9].

The (bosonic) Fock space of type $B$ is the polynomial algebra $\mathcal{F} = \mathbf{C}[p_{2j+1}, j \geq 0]$ in an infinite number of generators $p_{2j+1}$ of odd degree $2j+1$. If one identifies $p_k$ with the power sum symmetric function $p_k = \sum_i x_i^k$ in some infinite set of variables, the natural basis of weight vectors for $b_\infty$ is given by Schur's $P$-functions $P_\lambda$ (where $\lambda$ runs over the set DP of partitions into distinct parts) [8, 36, 17].

The Chevalley generators $e_i^\infty$, $f_i^\infty$ $(i \geq 0)$ of $b_\infty$ act on $P_\lambda$ by

$$e_i^\infty P_\lambda = P_\mu \ , \qquad f_i^\infty P_\lambda = P_\nu \tag{1}$$

where $\mu$ (resp. $\nu$) is obtained from $\lambda$ by replacing its part $i+1$ by $i$ (resp. its part $i$ by $i+1$), the result being 0 if $i+1$ (resp. $i$) is not a part of $\lambda$. Also, it is understood that $P_\mu = 0$ as soon as $\mu$ has a multiple part. For example, $f_0^\infty P_{32} = P_{321}$, $f_3^\infty P_{32} = P_{42}$, $e_1^\infty P_{32} = P_{31}$ and $e_2^\infty P_{32} = P_{22} = 0$.

Let $h = 2n+1$. The Chevalley generators $e_i$, $f_i$ of $A_{2n}^{(2)}$ will be realized as

$$f_i = \sum_{j \equiv n \pm i} f_j^\infty \qquad (i = 0, \ldots, n)\,, \tag{2}$$

$$e_i = \sum_{j \equiv n \pm i} e_j^\infty \qquad (i = 0, \ldots, n-1)\,, \qquad e_n = e_0^\infty + 2 \sum_{\substack{j>0 \\ j \equiv 0,-1}} e_j^\infty \,, \tag{3}$$

where all congruences are taken modulo $h$. Let $A_{2n}^{(2)\prime}$ be the derived algebra of $A_{2n}^{(2)}$ (obtained by omitting the degree operator $d$). The action of $A_{2n}^{(2)\prime}$ on $\mathcal{F}$ is centralized by the Heisenberg algebra generated by the operators $\dfrac{\partial}{\partial p_{hs}}$ and $p_{hs}$ for odd $s \geq 1$. This implies that the Fock space decomposes under $A_{2n}^{(2)}$ as

$$\mathcal{F} = \bigoplus_{k \geq 0} V(\Lambda_n - k\delta)^{\oplus p^*(k)} \tag{4}$$

where $p^*(k)$ is the number of partitions of $k$ into odd parts. In particular, the subrepresentation generated by the vacuum vector $|0\rangle = P_0 = 1$ is the basic representation $V(\Lambda_n)$ of $A_{2n}^{(2)}$, and its principally specialized character is [18]

$$\mathrm{ch}_t\, V(\Lambda_n) = \sum_{m \geq 0} \dim V(\Lambda_n)_m \, t^m = \prod_{\substack{i \ \mathrm{odd} \\ i \not\equiv 0 \mod h}} \frac{1}{1-t^i}\,. \tag{5}$$

The $q$-deformation of this situation has been discovered by Kashiwara et al. [22]. Contrary to the case of $A_{n-1}^{(1)}$, the $q$-Fock space is strictly larger than the classical one. We recall here briefly their construction, referring to [22] for details and notation.

Let $\mathrm{DP}_h(m)$ be the set of partitions $\lambda = (1^{m_1} 2^{m_2} \ldots r^{m_r})$ of $m$ for which $m_i \leq 1$ when $i \not\equiv 0 \mod h$. For example, $\mathrm{DP}_3(7) = \{(7), (61), (52), (43), (421), (331)\}$. Set $\mathrm{DP}_h = \bigcup_m \mathrm{DP}_h(m)$. Then, the $q$-Fock space of type $A_{2n}^{(2)}$ is

$$\mathcal{F}_q = \bigoplus_{\lambda \in \mathrm{DP}_h} \mathbf{Q}(q)\, |\lambda\rangle \tag{6}$$

where for $\lambda = (\lambda_1, \dots, \lambda_r)$, $|\lambda\rangle$ denotes the infinite $q$-wedge product

$$|\lambda\rangle = u_\lambda = u_{\lambda_1} \wedge_q u_{\lambda_2} \wedge_q \cdots \wedge_q u_{\lambda_r} \wedge_q u_0 \wedge_q u_0 \wedge_q \cdots$$

of basis vectors $u_i$ of the representation $V_{\mathrm{aff}}$. The quantum affine algebra $U_q(A_{2n}^{(2)})$ acts on $V_{\mathrm{aff}} = \bigoplus_{i\in\mathbf{Z}} \mathbf{Q}(q)u_i$ by

$$f_i u_j = \begin{cases} u_{j+1} & \text{if } j \equiv n \pm i \bmod h \\ 0 & \text{otherwise} \end{cases} \qquad (i = 0, \dots, n-1) \tag{7}$$

$$f_n u_j = \begin{cases} u_{j+1} & \text{if } j \equiv -1 \bmod h \\ (q + q^{-1})u_{j+1} & \text{if } j \equiv 0 \bmod h \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$e_i u_j = \begin{cases} u_{j-1} & \text{if } j \equiv n + 1 \pm i \bmod h \\ 0 & \text{otherwise} \end{cases} \qquad (i = 0, \dots, n-1) \tag{9}$$

$$e_n u_j = \begin{cases} u_{j-1} & \text{if } j \equiv 1 \bmod h \\ (q + q^{-1})u_{j-1} & \text{if } j \equiv 0 \bmod h \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$t_0 u_j = \begin{cases} q^4 u_j & \text{if } j \equiv n \bmod h \\ q^{-4} u_j & \text{if } j \equiv n+1 \bmod h \\ u_j & \text{otherwise} \end{cases} \tag{11}$$

$$t_i u_j = \begin{cases} q^2 u_j & \text{if } j \equiv n \pm i \bmod h \\ q^{-2} u_j & \text{if } j \equiv n + 1 \pm i \bmod h \\ u_j & \text{otherwise} \end{cases} \qquad (i = 1, \dots, n-1) \tag{12}$$

$$t_n u_j = \begin{cases} q^2 u_j & \text{if } j \equiv -1 \bmod h \\ q^{-2} u_j & \text{if } j \equiv 1 \bmod h \\ u_j & \text{otherwise} \end{cases} \tag{13}$$

The only commutation rules we will need to describe the action of $e_i$ and $f_i$ on $\mathcal{F}_q$ are:

$$u_j \wedge_q u_j \quad = 0 \text{ if } j \not\equiv 0 \bmod h \tag{14}$$

$$u_j \wedge_q u_{j+1} = -q^2 u_{j+1} \wedge_q u_j \text{ if} j \equiv 0, -1 \bmod h \ . \tag{15}$$

The action on the vacuum vector $|0\rangle = u_0 \wedge_q u_0 \wedge_q \cdots$ is given by

$$e_i|0\rangle = 0, \qquad f_i|0\rangle = \delta_{in}|1\rangle, \qquad t_i|0\rangle = q^{\delta_{in}}|0\rangle, \tag{16}$$

and on a $q$-wedge $|\lambda\rangle = u_{\lambda_1} \wedge_q \cdots \wedge_q u_{\lambda_r} \wedge_q |0\rangle$,

$$\begin{aligned} f_i|\lambda\rangle = {} & f_i u_{\lambda_1} \wedge_q t_i u_{\lambda_2} \wedge_q \cdots t_i u_{\lambda_r} \wedge_q t_i|0\rangle \\ & + u_{\lambda_1} \wedge_q f_i u_{\lambda_2} \wedge_q \cdots t_i u_{\lambda_r} \wedge_q t_i|0\rangle \\ & + \cdots + u_{\lambda_1} \wedge_q u_{\lambda_2} \wedge_q \cdots u_{\lambda_r} \wedge_q f_i|0\rangle \end{aligned} \tag{17}$$

$$
\begin{aligned}
e_i|\lambda\rangle &= t_i^{-1}u_{\lambda_1}\wedge_q t_i^{-1}u_{\lambda_2}\wedge_q\cdots t_i^{-1}u_{\lambda_r}\wedge_q e_i|0\rangle\\
&\quad + t_i^{-1}u_{\lambda_1}\wedge_q t_i^{-1}u_{\lambda_2}\wedge_q\cdots e_iu_{\lambda_r}\wedge_q |0\rangle\\
&\quad + \cdots + e_iu_{\lambda_1}\wedge_q u_{\lambda_2}\wedge_q\cdots u_{\lambda_r}\wedge_q |0\rangle
\end{aligned}
\tag{18}
$$

$$
t_i|\lambda\rangle = t_iu_{\lambda_1}\wedge_q t_iu_{\lambda_2}\wedge_q\cdots\wedge_q t_iu_{\lambda_r}\wedge_q t_i|0\rangle\ . \tag{19}
$$

For example, with $n=2$, one has

$$
f_2|542\rangle = (q^4+q^2)|642\rangle + q|552\rangle + |5421\rangle,
$$

and

$$
f_2|552\rangle = (q^2+1)(|652\rangle+|562\rangle) + |5521\rangle = (1-q^4)|652\rangle + |5521\rangle,
$$

the last equality resulting from (15).

It is proved in [22] that $\mathcal{F}_q$ is an integrable highest weight $U_q(A_{2n}^{(2)})$-module whose decomposition into irreducible components, obtained by means of $q$-bosons, is

$$
\mathcal{F}_q = \bigoplus_{k\geq 0} V(\Lambda_n - k\delta)^{\oplus p(k)} \tag{20}
$$

where $p(k)$ is now the number of all partitions of $k$ (compare (4)). Thus, the submodule $U_q(A_{2n}^{(2)})\,|0\rangle$ is a realization of the basic representation $V(\Lambda_n)$.

## 3. The crystal graph of the $q$-Fock space

The first step in computing the global basis of $V(\Lambda_n)\subset\mathcal{F}_q$ is to determine the crystal basis of $\mathcal{F}_q$ whose description follows from [22, 19, 20]. Let $A$ denote the subring of $\mathbf{Q}(q)$ consisting of rational functions without pole at $q=0$. The crystal lattice of $\mathcal{F}_q$ is $L=\bigoplus_{\lambda\in\mathrm{DP}_h} A\,|\lambda\rangle$, and the crystal basis of the $\mathbf{Q}$-vector space $L/qL$ is $B=\{|\lambda\rangle \bmod qL, \lambda\in\mathrm{DP}_h\}$. We shall write $\lambda$ instead of $|\lambda\rangle \bmod qL$.

The Kashiwara operators $\tilde{f}_i$ act on $B$ in a simple way recorded on the crystal graph $\Gamma(\mathcal{F}_q)$. To describe this graph, one starts with the crystal graph $\Gamma(V_{\mathrm{aff}})$ of $V_{\mathrm{aff}}$. This is the graph with vertices $j\in\mathbf{Z}$, whose arrows labelled by $i\in\{0,1,\ldots,n\}$ are given, for $i\neq n$, by

$$
j\overset{i}{\longrightarrow} j+1 \quad\Longleftrightarrow\quad j\equiv n\pm i \bmod h\,,
$$

and for $i=n$ by

$$
j\overset{n}{\longrightarrow} j+1 \quad\Longleftrightarrow\quad j\equiv -1,0 \bmod h\,.
$$

Thus for $n=2$ this graph is

$$
\cdots\overset{1}{\longrightarrow} -1 \overset{2}{\longrightarrow} 0 \overset{2}{\longrightarrow} 1 \overset{1}{\longrightarrow} 2 \overset{0}{\longrightarrow} 3 \overset{1}{\longrightarrow} 4 \overset{2}{\longrightarrow} 5 \overset{2}{\longrightarrow} 6 \overset{1}{\longrightarrow} 7 \overset{0}{\longrightarrow}\cdots
$$

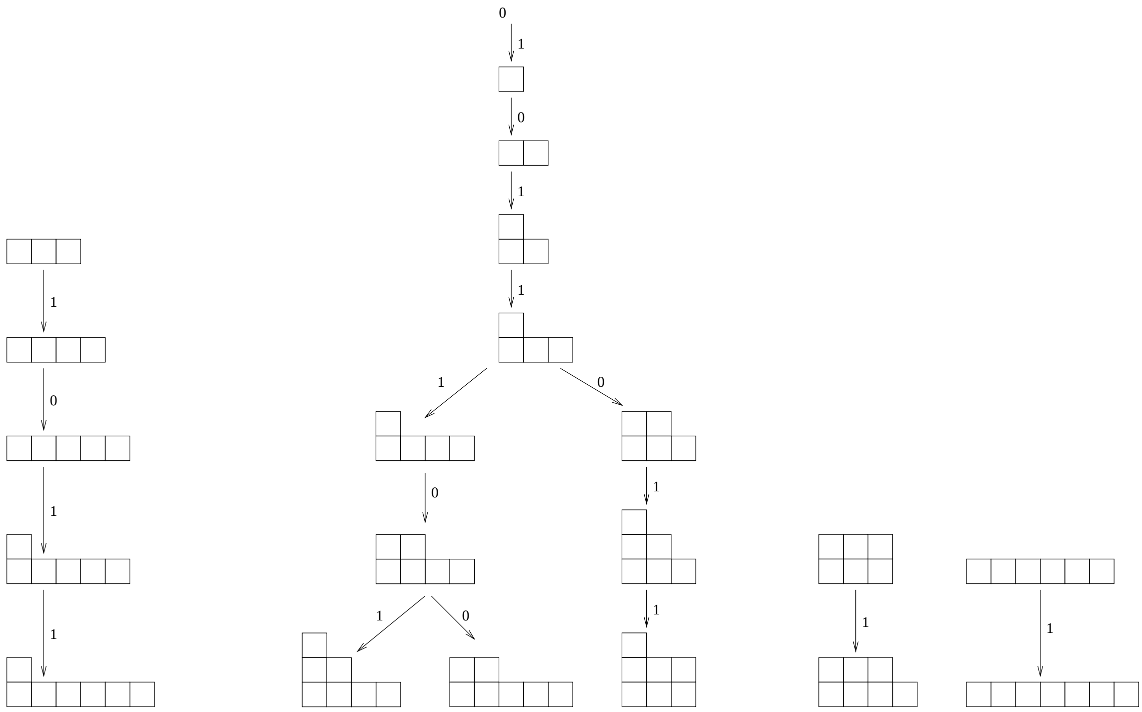


**Figure 1.** The graph $\Gamma(\mathcal{F}_q)$ for $A_2^{(2)}$ up to degree 7

The graph $\Gamma(\mathcal{F}_q)$ is obtained inductively from $\Gamma(V_{\text{aff}})$ using the following rules. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in B$, and write $\lambda = (\lambda_1, \lambda^*)$ where $\lambda^* = (\lambda_2, \ldots, \lambda_r)$. Then one has $\tilde{f}_i(0) = \delta_{in}(1)$, $\varphi_i(0) = \delta_{in}$, and

$$\tilde{f}_i\lambda = \begin{cases} (\tilde{f}_i\lambda_1, \lambda^*) \text{ if } \varepsilon_i(\lambda_1) \geq \varphi_i(\lambda^*), \\ (\lambda_1, \tilde{f}_i\lambda^*) \text{ if } \varepsilon_i(\lambda_1) < \varphi_i(\lambda^*). \end{cases}$$

Here, $\varepsilon_i(\lambda_1)$ means the distance in $\Gamma(V_{\text{aff}})$ from $\lambda_1$ to the origin of its $i$-string, and $\varphi_i(\lambda^*)$ means the distance in $\Gamma(\mathcal{F}_q)$ from $\lambda^*$ to the end of its $i$-string.

Thus for $n = 1$ one computes successively the following 1-strings of $\Gamma(\mathcal{F}_q)$

$$(0) \xrightarrow{1} (1)$$

$$(2) = (2, 0) \xrightarrow{1} (2, 1) \xrightarrow{1} (3, 1) \xrightarrow{1} (4, 1)$$

$$(3, 2) = (3, 2, 0) \xrightarrow{1} (3, 2, 1) \xrightarrow{1} (3, 3, 1) \xrightarrow{1} (4, 3, 1)$$

from which one deduces that $\tilde{f}_1(3, 3, 1) = (4, 3, 1)$ and $\varphi_1(3, 3, 1) = 1$.

The first layers of the crystal $\Gamma(\mathcal{F}_q)$ for $n = 1$ are shown in Fig. 1. One can observe that the decomposition of $\Gamma(\mathcal{F}_q)$ into connected components reflects the decomposition (20) of $\mathcal{F}_q$ into simple modules. More precisely, the connected components of $\Gamma(\mathcal{F}_q)$

are all isomorphic as colored graphs to the component $\Gamma(\Lambda_n)$ containing the empty partition. Their highest vertices are the partitions $\nu$ whose parts are all divisible by $h$. This follows from the fact, easily deduced from the rules we have just explained, that if $\nu = h\mu = (h\mu_1, \ldots, h\mu_r)$ is such a partition, then the map

$$\lambda \mapsto \lambda + \nu = (\lambda_1 + h\mu_1, \lambda_2 + h\mu_2, \ldots) \tag{21}$$

is a bijection from $\Gamma(\Lambda_n)$ onto the connected component of $\Gamma(\mathcal{F}_q)$ containing $\nu$, and this bijection commutes with the operators $\tilde{e}_i$ and $\tilde{f}_i$. This implies that the vertices of $\Gamma(\Lambda_n)$ are the partitions $\lambda = (\lambda_1, \ldots, \lambda_r, 0) \in \mathrm{DP}_h$ such that for $i = 1, 2, \ldots, r$, one has $\lambda_i - \lambda_{i+1} \le h$ and $\lambda_i - \lambda_{i+1} < h$ if $\lambda_i \equiv 0 \bmod h$. We shall call $h$-regular a partition satisfying these conditions. The set of $h$-regular partitions of $m$ will be denoted by $\mathrm{DPR}_h(m)$, and we shall write $\mathrm{DPR}_h = \bigcup_m \mathrm{DPR}_h(m)$.

For example,

$$\mathrm{DPR}_3(10) = \{(3331), (4321), (532), (541)\}\,.$$

## 4. The canonical basis of $V(\Lambda_n)$

In this section, we describe an algorithm for computing the canonical basis (global lower crystal basis) of the basic representation $V(\Lambda_n) = U_q(A_{2n}^{(2)})|0\rangle$ in terms of the natural basis $|\lambda\rangle$ of the $q$-Fock space. To characterize the canonical basis, we need the following notations

$$q_i = \begin{cases} q & \text{if } i = n \\ q^2 & \text{if } 1 \le i < n \\ q^4 & \text{if } i = 0 \end{cases} \qquad t_i = \begin{cases} q^{h_n} & \text{if } i = n \\ q^{2h_i} & \text{if } 1 \le i < n \\ q^{4h_0} & \text{if } i = 0 \end{cases} \tag{22}$$

and

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}\,, \qquad [k]_i! = [k]_i[k-1]_i \cdots [1]_i\,. \tag{23}$$

The $q$-divided powers of the Chevalley generators are defined by

$$e_i^{(k)} = \frac{e_i^k}{[k]_i!}\,, \qquad f_i^{(k)} = \frac{f_i^k}{[k]_i!}\,. \tag{24}$$

The canonical basis is defined in terms of an involution $v \mapsto \overline{v}$ of $V(\Lambda_n)$. Let $x \mapsto \overline{x}$ be the ring automorphism of $U_q(A_{2n}^{(2)})$ such that $\overline{q} = q^{-1}$, $\overline{q^h} = q^{-h}$ for $h$ in the Cartan subalgebra of $A_{2n}^{(2)}$, and $\overline{e_i} = e_i$, $\overline{f_i} = f_i$. Then, for $v = x|0\rangle \in V(\Lambda_n)$, define $\overline{v} = \overline{x}|0\rangle$.

We denote by $U_{\mathbf{Q}}^-$ the sub-$\mathbf{Q}[q, q^{-1}]$-algebra of $U_q(A_{2n}^{(2)})$ generated by the $f_i^{(k)}$ and set $V_{\mathbf{Q}}(\Lambda_n) = U_{\mathbf{Q}}^-|0\rangle$. Then, as shown by Kashiwara [21], there exists a unique $\mathbf{Q}[q, q^{-1}]$-basis $\{G(\mu), \mu \in \mathrm{DPR}_h\}$ of $V_{\mathbf{Q}}(\Lambda_n)$, such that

(G1) $G(\mu) \equiv |\mu\rangle \bmod qL$

(G2) $\overline{G(\mu)} = G(\mu)$.

To compute $G(\mu)$, we follow the same strategy as in [24]. We first introduce an auxiliary basis $A(\mu)$ satisfying (G2), from which we manage to construct combinations satisfying also (G1). More precisely, let $\mathcal{F}_q^m$ be the subspace of $\mathcal{F}_q$ spanned by $|\lambda\rangle$ for $\lambda \in \mathrm{DP}_h(m)$ and set $V(\Lambda_n)_m = \mathcal{F}_q^m \cap V(\Lambda_n)$. Denote by $\trianglelefteq$ the natural order on partitions. Then, the auxiliary basis will satisfy

(A0) $\{A(\mu), \mu \in \mathrm{DPR}_h(m)\}$ is a $\mathbf{Q}[q,q^{-1}]$-basis of $V_{\mathbf{Q}}(\Lambda_n)_m$,

(A1) $A(\mu) = \sum_\lambda a_{\lambda\mu}(q)|\lambda\rangle$, where $a_{\lambda\mu}(q) = 0$ unless $\lambda \trianglerighteq \mu$, $a_{\mu\mu}(q) = 1$ and $a_{\lambda\mu}(q) \in \mathbf{Z}[q,q^{-1}]$,

(A2) $\overline{A(\mu)} = A(\mu)$.

The basis $A(\mu)$ is obtained by applying monomials in the $f_i^{(k)}$ to the highest weight vector, that is, $A(\mu)$ is of the form

$$A(\mu) = f_{r_s}^{(k_s)} f_{r_{s-1}}^{(k_{s-1})} \cdots f_{r_1}^{(k_1)} |0\rangle \tag{25}$$

so that (A2) is certainly verified.

The two sequences $(r_1, \ldots, r_s)$ and $(k_1, \ldots, k_s)$ are, as in [24], obtained by peeling off the $A_{2n}^{(2)}$-ladders of the partition $\mu$, which are defined as follows. We first fill the cells of the Young diagram $Y$ of $\mu$ with integers (called residues), constant in each column of $Y$. If $j \equiv n \pm i \bmod h$ $(0 \le i \le n)$, the numbers filling the $j$-th column of $Y$ will be equal to $i$. A ladder of $\mu$ is then a sequence of cells with the same residue, located in consecutive rows at horizontal distance $h$, except when the residue is $n$, in which case two consecutive $n$-cells in a row belong also to the same ladder. For example, with $n = 3$ and $\mu = (11, 7, 7, 4)$, one finds 22 ladders (indicated by subscripts), the longest one being the 7th, containing three 3-cells:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $3_{19}$ | $2_{20}$ | $1_{21}$ | $0_{22}$ | | | | | | | |
| $3_{13}$ | $2_{14}$ | $1_{15}$ | $0_{16}$ | $1_{17}$ | $2_{18}$ | $3_{19}$ | | | | |
| $3_7$ | $2_8$ | $1_9$ | $0_{10}$ | $1_{11}$ | $2_{12}$ | $3_{13}$ | | | | |
| $3_1$ | $2_2$ | $1_3$ | $0_4$ | $1_5$ | $2_6$ | $3_7$ | $3_7$ | $2_8$ | $1_9$ | $0_{10}$ |

Note that this definition of ladders agrees with that of [6] for $n = 1$, but differs from that of [1] for $n = 2$.

Then, in (25), $s$ is the number of ladders, $r_i$ the residue of the $i$th ladder, and $k_i$ the number of its cells. Thus, proceeding with our example,

$$A(11,7,7,4) = f_0 f_1 f_2 f_3^{(2)} f_2 f_1 f_0 f_1 f_2 f_3^{(2)} f_2 f_1 f_0^{(2)} f_1^{(2)} f_2^{(2)} f_3^{(3)} f_2 f_1 f_0 f_1 f_2 f_3 |0\rangle \ .$$

The proof of (A0) and (A1) can be readily adapted from [24]. In particular, (A1) follows from the fact that a partition $\lambda$ belongs to $\mathrm{DPR}_h$ if and only if all cells of a given ladder intersecting $\lambda$ occupy the highest possible positions on this ladder.

Another choice of an intermediate basis, more efficient for practical computations, would be to use inductively the vectors $G(\nu)$ already computed and to set $A(\mu) = f_{r_s}^{(k_s)} G(\nu)$, where $\nu$ is the partition obtained from $\mu$ by removing its outer ladder.

Define now the coefficients $b_{\nu\mu}(q)$ by

$$G(\mu) = \sum_{\nu} b_{\nu\mu}(q) A(\nu) \ . \tag{26}$$

Still following [24], one can check that $b_{\nu\mu}(q) = 0$ unless $\nu \geq \mu$, where $\geq$ denote the lexicographic ordering on partitions, and that $b_{\mu\mu}(q) = 1$. Therefore, one can apply the triangular process of [24] as follows.

Let $\mu^{(1)} < \mu^{(2)} < \ldots < \mu^{(t)}$ be the set $\mathrm{DPR}_h(m)$ sorted in lexicographic order, so that $A(\mu^{(t)}) = G(\mu^{(t)})$. Suppose that the expansion on the basis $|\lambda\rangle$ of $G(\mu^{(i+1)}), \ldots, G(\mu^{(t)})$ has already been calculated. Then,

$$G(\mu^{(i)}) = A(\mu^{(i)}) - \gamma_{i+1}(q) G(\mu^{(i+1)}) - \cdots - \gamma_t(q) G(\mu^{(t)}) \ , \tag{27}$$

where the coefficients are determined by the conditions

$$\gamma_s(q^{-1}) = \gamma_s(q), \qquad G(\mu^{(i)}) \equiv |\mu^{(i)}\rangle \bmod qL.$$

Thus, for $n = 1$, the first partition for which $A(\mu) \neq G(\mu)$ is $\mu = (3321)$ and

$$\begin{aligned} A(3321) = {} & |3321\rangle + q|333\rangle + (q^2 - q^6)|432\rangle + (1 + 2q^2)|531\rangle + (q^2 + q^4)|54\rangle \\ & + (2q^2 + q^4)|621\rangle + 2q^3|63\rangle + (q^4 + q^6)|72\rangle + q^4|81\rangle + q^5|9\rangle \end{aligned} \tag{28}$$

Indeed, $A(3321) \equiv |3321\rangle + |531\rangle \bmod qL$. On the other hand, $A(531) = |531\rangle + q^2|54\rangle + q^2|621\rangle + q^3|63\rangle + q^6|72\rangle$ is equal to $G(531)$, and one finds by subtracting this from $A(3321)$ that

$$G(3321) = |3321\rangle + q|333\rangle + (q^2 - q^6)|432\rangle + 2q^2|531\rangle + q^4|54\rangle \tag{29}$$

$$+ (q^2 + q^4)|621\rangle + q^3|63\rangle + q^4|72\rangle + q^4|81\rangle + q^5|9\rangle \ . \tag{30}$$

Since $A(432) = |432\rangle + q^4|531\rangle + q^2|72\rangle + q^6|81\rangle$ satisfies (G1) and (G2), it has to be equal to $G(432)$, which completes the determination of the canonical basis for $m = 9$. For $m = 10$, the results are displayed as the columns of Table 1.

In the Fock space representation of $A_{n-1}^{(1)}$, the weight of a basis vector $|\lambda\rangle$ is determined by the $n$-core of the partition $\lambda$ (and its degree) [3, 24]. There is a similar result of Nakajima and Yamada [31] for $A_{2n}^{(2)}$, in terms of the notion of $\overline{h}$-core of a strict partition introduced by Morris [28] in the context of the modular representation theory of spin symmetric groups.

**Table 1.** The canonical basis for $n = 1$ and $m = 10$.

| | (3331) | (4321) | (532) | (541) |
|---|---|---|---|---|
| (3331) | $1$ | $0$ | $0$ | $0$ |
| (4321) | $q - q^5$ | $1$ | $0$ | $0$ |
| (433) | $q^2$ | $q$ | $0$ | $0$ |
| (532) | $0$ | $0$ | $1$ | $0$ |
| (541) | $q + q^3$ | $q^2 + q^4$ | $0$ | $1$ |
| (631) | $2\,q^2$ | $q^3$ | $0$ | $q$ |
| (64) | $q^4$ | $0$ | $0$ | $q^3$ |
| (721) | $q^3 + q^5$ | $q^2$ | $0$ | $q^4$ |
| (73) | $q^4$ | $q^3$ | $0$ | $q^5$ |
| (82) | $0$ | $0$ | $q^2$ | $0$ |
| (91) | $q^4$ | $q^5$ | $0$ | $0$ |
| (10) | $q^6$ | $0$ | $0$ | $0$ |

One way to see this is to use a theorem of [29] according to which $\lambda, \mu \in \mathrm{DP}(m)$ have the same $\overline{h}$-core if and only if they have, for each $i$, the same number $n_i$ of nodes of residue $i$. On the other hand, it follows from the implementation of the Chevalley generators that $|\lambda\rangle$ has $A_{2n}^{(2)}$-weight $\Lambda_n - \sum_{0\le i\le n} n_i\alpha_i$, and the statement follows.

The definition of $\overline{h}$-cores can be extended to $\mathrm{DP}_h$ by deciding that if $\lambda$ has repeated parts, its $\overline{h}$-core is equal to that of the partition obtained by removing those repeated parts. Then it is clear that if $|\lambda\rangle$ and $\mu\rangle$ have the same $U_q(A_{2n}^{(2)})$-weight, the two partitions $\lambda$ and $\mu$ have the same $\overline{h}$-core. It follows, since $G(\mu)$ is obviously a weight vector, that its expansion on the basis $|\lambda\rangle$ involves only partitions $\lambda$ with the same $\overline{h}$-core as $\mu$.

Summarizing the discussion, we have:

*Theorem 4.1.* For $\mu \in \mathrm{DPR}_h(m)$, define $d_{\lambda\mu}(q)$ by $G(\mu) = \sum_{\lambda\in \mathrm{DP}_h(m)} d_{\lambda\mu}(q)|\lambda\rangle$. Then,

(i) $d_{\lambda\mu}(q) \in \mathbf{Z}[q]$,
(ii) $d_{\lambda\mu}(q) = 0$ unless $\lambda \trianglerighteq \mu$, and $d_{\mu\mu}(q) = 1$,
(iii) $d_{\lambda\mu}(q) = 0$ unless $\lambda$ and $\mu$ have the same $\overline{h}$-core.

## 5. The reduction $q = 1$

As observed by Kashiwara *et al.* [22], to recover the classical Fock space representation $\mathcal{F}$ of $A_{2n}^{(2)}$, one has to introduce the inner product on $\mathcal{F}_q$ for which the vectors $|\lambda\rangle$ are orthogonal and the adjoint operators of the Chevalley generators are

$$f_i^\dagger = q_i e_i t_i, \qquad e_i^\dagger = q_i f_i t_i^{-1}, \qquad t_i^\dagger = t_i. \tag{31}$$

It can be checked that, for $\lambda \in \mathrm{DP}_h$,

$$\langle \lambda | \lambda \rangle = \prod_{k>0} \prod_{i=1}^{m_{kh}} (1 - (-q^2)^i), \tag{32}$$

where $m_{kh}$ is the multiplicity of the part $kh$ in $\lambda$.

Let $\mathcal{F}_1$ denote the $A_{2n}^{(2)}$-module obtained by specializing $q$ to 1 as in [22]. This space is strictly larger than the classical Fock space $\mathcal{F}$, since the dimension of its $m$th homogeneous component (in the principal gradation) is $|\mathrm{DP}_h(m)|$ whereas that of $\mathcal{F}$ is only $|\mathrm{DP}(m)|$. Let $\mathcal{N} = \mathcal{F}_1^\perp$ denote the nullspace. It follows from (31) that $\mathcal{N}$ is a $A_{2n}^{(2)}$-module, and from (32) that $\mathcal{N}$ is the subspace of $\mathcal{F}_1$ spanned by the wedge products $|\lambda\rangle$ labelled by $\lambda \in \mathrm{DP}_h - \mathrm{DP}$. Therefore $\mathcal{F}_1/\mathcal{N}$ is a $A_{2n}^{(2)}$-module that can be identified to $\mathcal{F}$.

In this identification one has, for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathrm{DP}$,

$$P_\lambda = 2^{\sum_{i=1}^r \lfloor (\lambda_i - 1)/h \rfloor} |\lambda\rangle. \tag{33}$$

The power of 2 comes from the fact that if $\lambda_i = kh$ for $k > 0$, and $\mu$ denotes the partition obtained from $\lambda$ by replacing $\lambda_i$ by $\mu_i = \lambda_i + 1$, then it follows from (1), (2) that $f_n P_\lambda$ contains $P_\mu$ with coefficient 1, while $f_n|\lambda\rangle$ contains $|\mu\rangle$ with coefficient 2 by (8). For later use we set

$$a_h(\lambda) = \sum_{i=1}^r \left\lfloor \frac{\lambda_i - 1}{h} \right\rfloor . \tag{34}$$

## 6. Modular representations of $\widehat{\mathrm{S}}_m$

We refer the reader to [5] for an up to date review of the representation theory of the spin symmetric groups and their combinatorics.

Let $\widehat{\mathrm{S}}_m$ be the spin symmetric group as defined by Schur [33], that is, the group of order $2\,m!$ with generators $z, s_1, \ldots, s_{m-1}$ and relations $z^2 = 1$, $zs_i = s_i z$ $(1 \le i \le m-1)$, $s_i s_j = z s_j s_i$ $(|i-j| \ge 2)$ and $(s_i s_{i+1})^3 = z$ $(1 \le i \le m-2)$.

On an irreducible representation of $\widehat{\mathrm{S}}_m$, the central element $z$ has to act by $+1$ or by $-1$. The representations for which $z = 1$ are actually linear representations of the symmetric group $\mathrm{S}_m$, and those with $z = -1$, called spin representations correspond to two-valued representations of $\mathrm{S}_m$. The irreducible spin representations over a field of characteristic 0 are labelled, up to association, by strict partitions $\lambda \in \mathrm{DP}(m)$. More precisely, let $\mathrm{DP}_+(m)$ (resp. $\mathrm{DP}_-(m)$) be the set of strict partitions of $m$ having an even (resp. odd) number of even parts. Then, to each $\lambda \in \mathrm{DP}_+(m)$ corresponds a self-associated irreducible spin character $\langle\lambda\rangle$, and to each $\lambda \in \mathrm{DP}_-(m)$ a pair of associated irreducible spin characters denoted by $\langle\lambda\rangle$ and $\langle\lambda\rangle'$.

According to Schur [33], the values $\langle\lambda\rangle(\rho)$ of the spin character $\langle\lambda\rangle$ on conjugacy classes of cycle-type $\rho = (1^{m_1}, 3^{m_3}, \ldots)$ are given by the expansion of the symmetric function $P_\lambda$ on the basis of power sums, namely

$$P_\lambda = \sum_\rho 2^{\lceil(\ell(\rho)-\ell(\lambda))/2\rceil} \langle\lambda\rangle(\rho) \frac{p_\rho}{z_\rho} \tag{35}$$

where $z_\rho = \prod_j j^{m_j} m_j!$ and $\ell(\lambda)$ stands for the length, that is the number of parts, of the partition $\lambda$.

For $\lambda \in \mathrm{DP}(m)$, one introduces the self-associated spin character

$$\langle\widehat{\lambda}\rangle = \begin{cases} \langle\lambda\rangle & \text{if } \lambda \in \mathrm{DP}_+(m), \\ \langle\lambda\rangle + \langle\lambda\rangle' & \text{if } \lambda \in \mathrm{DP}_-(m). \end{cases} \tag{36}$$

The branching theorem for spin characters of Morris [28] implies that if $\langle\widehat{\lambda}\rangle$ gets identified with a weight vector of $\mathcal{F}$ by setting

$$P_\lambda = 2^{\lfloor(m-\ell(\lambda))/2\rfloor} \langle\widehat{\lambda}\rangle, \tag{37}$$

then the $b_\infty$-operator $f = \sum_{i\geq 0} f_i^\infty$ implements the induction of self-associated spin characters from $\widehat{\mathrm{S}}_m$ to $\widehat{\mathrm{S}}_{m+1}$. Similarly, $e = e_0^\infty + 2\sum_{i>0} e_i^\infty$ implements the restriction from $\widehat{\mathrm{S}}_m$ to $\widehat{\mathrm{S}}_{m-1}$. Thus, the Fock space representation of $b_\infty$ may be viewed as the sum $\mathcal{F} = \bigoplus_m \mathcal{C}(m)$ of additive groups generated by self-associated spin characters of $\widehat{\mathrm{S}}_m$ in characteristic 0. In this setting, the Chevalley generators of $b_\infty$ act as refined induction and restriction operators.

Now, similarly to the case $A_{n-1}^{(1)}$, the reduction from $b_\infty$ to $A_{2n}^{(2)}$ parallels the reduction modulo $p = h = 2n+1$ of representations of $\widehat{\mathrm{S}}_m$ (from now on we assume that $h$ is an odd prime).

More precisely, using (1) (2) (37), one sees immediately that the Chevalley generators $f_i$ of $A_{2n}^{(2)}$ act on $\langle\widehat{\lambda}\rangle$ as the $(r,\overline{r})$-induction operators of Morris and Yaseen ($r = n+1-i$) [30]. Hence the vectors of degree $m$ of $V(\Lambda_n) = U(A_{2n}^{(2)})^- \,|0\rangle$ can be identified with linear combinations of self-associated spin characters obtained by a sequence of $(r,\overline{r})$-inductions. It is known from modular representation theory that the maximal number of linearly independent self-associated projective spin characters of $\widehat{\mathrm{S}}_m$ in characteristic $p$ is equal to the number of partitions of $m$ into odd summands prime to $p$. Therefore the following results follows at once from (5).

*Theorem 6.1.* The self-associated projective spin characters of $\widehat{\mathrm{S}}_m$ in characteristic $p$ are linear combinations of characters obtained by a sequence of $(r,\overline{r})$-inductions.

This seemed to be an open question [5]. Moreover, the construction of Section 4 gives an explicit basis for the space spanned by such characters. Denote by $\underline{A}(\mu)$ the column vector obtained from $A(\mu)$ by reduction $q=1$ and expansion on the basis $\langle\widehat{\lambda}\rangle$. Then $\{\underline{A}(\mu) \mid \mu \in \mathrm{DPR}_p(m)\}$ is a basis of the $\mathbf{Q}$-vector space of self-associated projective spin characters of $\widehat{\mathrm{S}}_m$ in characteristic $p$.

These remarks and the results of [24, 2, 13, 26] lead us to formulate a conjecture relating the global basis of $V(\Lambda_n)$ and the decomposition matrices for spin characters of the groups $\widehat{\mathrm{S}}_m$.

Let $\mu \in \mathrm{DPR}_p(m)$ and let $\underline{G}(\mu)$ stand for the image of the global basis $G(\mu)$ in $\mathcal{F} = \mathcal{F}_1/\mathcal{N}$, that is,

$$\underline{G}(\mu) = \sum_{\lambda \in \mathrm{DP}(m)} 2^{b(\lambda)-a_p(\lambda)} d_{\lambda\mu}(1)\langle\widehat{\lambda}\rangle\,, \tag{38}$$

where $a_p(\lambda)$ is given by (34) and

$$b(\lambda) = \left\lfloor \frac{m-\ell(\lambda)}{2} \right\rfloor\,. \tag{39}$$

Then denote by $\underline{\underline{G}}(\mu)$ the vector obtained by factoring out the largest power of 2 dividing the coefficients of $\underline{G}(\mu)$ on the basis $\langle\widehat{\lambda}\rangle$. For simplicity of notation, we shall identify $\underline{\underline{G}}(\mu)$ with the column vector of its coordinates on $\langle\widehat{\lambda}\rangle$.

Finally, let us call reduced decomposition matrix of $\widehat{\mathrm{S}}_m$ in characteristic $p$ the matrix obtained from the usual decomposition matrix for spin characters by adding up pairs of associated columns and expanding the column vectors so obtained on the basis $\langle\widehat{\lambda}\rangle$. This is a matrix with $|\mathrm{DP}(m)|$ rows and $|\mathrm{DPR}_p(m)|$ columns. The definition is illustrated in Table 2 and Table 3. (Table 2 is taken from [30], except for the column labels which are ours and will be explained in the next section.)

**Table 2.** The decomposition matrix of $\widehat{\mathrm{S}}_{10}$ in characteristic 3.

| | (3331) | (3331)' | (4321) | (4321)' | (532) | (541) | (541)' |
|---|---|---|---|---|---|---|---|
| $\langle 4321\rangle$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| $\langle 532\rangle$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $\langle 532\rangle'$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $\langle 541\rangle$ | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| $\langle 541\rangle'$ | 1 | 1 | 1 | 1 | 0 | 1 | 0 |
| $\langle 631\rangle$ | 2 | 2 | 1 | 1 | 0 | 1 | 1 |
| $\langle 631\rangle'$ | 2 | 2 | 1 | 1 | 0 | 1 | 1 |
| $\langle 64\rangle$ | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $\langle 721\rangle$ | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| $\langle 721\rangle'$ | 1 | 1 | 1 | 0 | 0 | 1 | 0 |
| $\langle 73\rangle$ | 1 | 1 | 1 | 1 | 0 | 1 | 1 |
| $\langle 82\rangle$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $\langle 91\rangle$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\langle 10\rangle$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\langle 10\rangle'$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 3.** The reduced decomposition matrix of $\widehat{\mathrm{S}}_{10}$ in characteristic 3.

| | (3331) | (4321) | (532) | (541) |
|---|---|---|---|---|
| $\langle\widehat{4321}\rangle$ | 0 | 2 | 0 | 0 |
| $\langle\widehat{532}\rangle$ | 0 | 0 | 1 | 0 |
| $\langle\widehat{541}\rangle$ | 2 | 2 | 0 | 1 |
| $\langle\widehat{631}\rangle$ | 4 | 2 | 0 | 2 |
| $\langle\widehat{64}\rangle$ | 2 | 0 | 0 | 2 |
| $\langle\widehat{721}\rangle$ | 2 | 1 | 0 | 1 |
| $\langle\widehat{73}\rangle$ | 2 | 2 | 0 | 2 |
| $\langle\widehat{82}\rangle$ | 0 | 0 | 1 | 0 |
| $\langle\widehat{91}\rangle$ | 2 | 2 | 0 | 0 |
| $\langle\widehat{10}\rangle$ | 1 | 0 | 0 | 0 |

*Conjecture 6.2.* (i) The set of column vectors of the reduced decomposition matrix of $\widehat{\mathrm{S}}_m$ in odd characteristic $p > m^2$ coincides with $\{\underline{\underline{G}}(\mu) \mid \mu \in \mathrm{DPR}_p(m)\}$.

(ii) For $p \leq m^2$, the reduced decomposition matrix of $\widehat{\mathrm{S}}_m$ is obtained by postmultiplying the matrix whose columns are $\underline{\underline{G}}(\mu)$ by a unitriangular matrix with nonnegative entries.

Our conjecture has been checked on the numerical tables computed by Morris and Yaseen ($p = 3$) [30] and Yaseen ($p = 5, 7, 11$) [35]. Thus, for $p = 3$, $m = 11$, the columns of the reduced decomposition matrix are

$$\underline{\underline{G}}(3332),\ \underline{\underline{G}}(4331) + \underline{\underline{G}}(641),\ \underline{\underline{G}}(5321),\ \underline{\underline{G}}(542),\ \underline{\underline{G}}(641).$$

## 7. Labels for irreducible modular spin characters and partition identities

The natural labels for irreducible modular representations of symmetric groups form a subset of the ordinary labels [16]. It was therefore natural to look for a labelling scheme for irreducible modular spin representations of $\widehat{\mathrm{S}}_m$ using a subset of $\mathrm{DP}(m)$. This was accomplished for $p = 3$ in [6], where such labels were found to be the Schur regular partitions of $m$. These are the partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_i - \lambda_{i+1} \geq 3$ for $i = 1, \ldots, r-1$, and $\lambda_i - \lambda_{i+1} > 3$ whenever $\lambda_i \equiv 0 \bmod 3$.

In [6], it was also conjectured that for $p = 5$, the labels should be the partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ satisfying the following conditions: (1) $\lambda_i > \lambda_{i+1}$ for $i \leq r-1$, (2) $\lambda_i - \lambda_{i+2} \geq 5$ for $i \leq r-2$, (3) $\lambda_i - \lambda_{i+2} > 5$ if $\lambda_i \equiv 0 \bmod 5$ or if $\lambda_i + \lambda_{i+1} \equiv 0 \bmod 5$ for $i \leq r-2$, and (4) there are no subsequences of the following types (for some $j \geq 0$): $(5j+3, 5j+2)$, $(5j+6, 5j+4, 5j)$, $(5j+5, 5j+1, 5j-1)$, $(5j+6, 5j+5, 5j, 5j-1)$. This conjecture turned out to be equivalent to a $q$-series identity conjectured long ago by Andrews in the context of extensions of the Rogers-Ramanujan identities, and was eventually proved by Andrews *et al.* [1]. The authors of [1] observed however that such a labelling scheme could not be extended to $p = 7, 11, 13$ (see also [5]).

In terms of canonical bases, the obstruction can be understood as follows. Assuming our conjecture and using the results of [6, 1], one can see that for $p = 3, 5$, the labels of [6] and [1] are exactly the partitions indexing the lowest nonzero entries in the columns of the matrices $D_m(q) = [d_{\lambda\mu}(q)]_{\lambda,\mu\vdash m}$. For example, in Table 1, these are $(10), (91), (82)$ and $(73)$, which are indeed the Schur regular partitions of 10. The problem is that for $p \geq 7$, it can happen that two columns have the same partition indexing the lowest nonzero entry. For example, with $p = 7$ $(n = 3)$ and $m = 21$, the two canonical basis vectors

$G(75432) = |75432\rangle + q^2|76431\rangle + q|7752\rangle + q^3|7761\rangle + q^2|8643\rangle + (q^2 + q^4)\,|8652\rangle + q^3|876\rangle + q^4|9543\rangle + (q^4 + q^6)\,|9651\rangle + q^5|975\rangle$

and

$G(654321) = |654321\rangle + q|75432\rangle + q|76431\rangle + q|76521\rangle + q^2|7743\rangle + q^2|7752\rangle + q^2|7761\rangle + q^3|777\rangle + (q^3 + q^5)\,|8643\rangle + (q^3 + q^5)\,|8652\rangle + (q^4 - q^8)\,|876\rangle + (q^3 + q^5)\,|9651\rangle + (q^4 + q^6)\,|975\rangle$

have the same bottom partition $(975)$ (compare [5], end of Section 3).

On the other hand the partitions indexing the highest nonzero entries in the columns of $D_m(q)$ are the labels of the crystal graph (by Theorem 4.1(ii)), so that they are necessarily distinct. Therefore, we propose to use the set

$$\begin{aligned}\mathrm{DPR}_p(m) = \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash m \mid 0 < \lambda_i - \lambda_{i+1} \leq p \text{ if } \lambda_i \not\equiv 0 \bmod p,\\ 0 \leq \lambda_i - \lambda_{i+1} < p \text{ if } \lambda_i \equiv 0 \bmod p, (1 \leq i \leq r)\}\end{aligned}$$

for labelling the irreducible spin representations of $\widehat{\mathrm{S}}_m$ in characteristic $p$. Indeed its definition is equally simple for all $p$. Moreover, because of Theorem 4.1(iii), this labelling would be compatible with the $p$-block structure, which can be read on the $\overline{p}$-cores. Also, it is adapted to the calculation of the vectors $\underline{A}(\mu)$ which give an approximation to the reduced decomposition matrix.

Finally, we note that since $\mathrm{DPR}_p$ provides the right number of labels we have the following partition identity

$$\sum_{m\geq 0} |\mathrm{DPR}_p(m)| t^m = \prod_{\substack{i \text{ odd} \\ i \not\equiv 0 \bmod p}} \frac{1}{1 - t^i} \tag{40}$$

which for $p = 3, 5$ is a pendant to the Schur and Andrews-Bessenrodt-Olsson identities. It can be derived without using representation theory by simply considering the bijections (21).

## 8. Discussion

We have used the level 1 $q$-deformed Fock spaces of Kashiwara *et al.* to compute the canonical basis of the basic representation of $U_q(A_{2n}^{(2)})$, and we have formulated a conjectural relation with the decomposition matrices of the spin symmetric groups in odd characteristic $p = 2n+1$.

As in the case of $A_{n-1}^{(1)}$, it is reasonable to expect that the canonical basis gives the exact decomposition matrices of a certain family of Hecke algebras at $(2n+1)$th roots of unity. A good candidate might be the Hecke-Clifford superalgebra introduced by Olshanski [32].

The case of $2n$th roots of unity should then be related in a similar way to the Fock space representation of the affine Lie algebras of type $D_{n+1}^{(2)}$. In particular we believe that the fact exploited by Benson [4] and Bessenrodt-Olsson [7] that the 2-modular irreducible characters of $\widehat{\mathrm{S}}_m$ can be identified with the 2-modular irreducible characters of $\mathrm{S}_m$ corresponds in the realm of affine Lie algebras to the isomorphism $D_2^{(2)} \simeq A_1^{(1)}$.

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