Ribbon tableaux and q-analogues of fusion rules in WZW conformal field theories

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Abstract. Starting from known q-analogues of ordinary $SU(n)$ tensor products multiplicities, we introduce q-analogues of the fusion coefficients of the WZW conformal field theories associated with $SU(n)$. We conjecture combinatorial interpretations of these polynomials, which can be proved in special cases. This allows us to derive in a simple way various kinds of branching functions, the simplest ones being the characters of the minimal unitary series of the Virasoro algebra. We also obtain q-analogues of the dimensions of spaces of nonabelian theta functions.

1. Introduction

The Littlewood-Richardson coefficient $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$, where $\lambda, \mu^{(1)}, \ldots, \mu^{(r)}$ are partitions of $N, m_1, \ldots, m_r$ respectively, and $N = m_1 + \cdots + m_r$, is defined as the multiplicity of the irreducible representation V_λ of $U(n)$ (or of $gl(n, \mathbf{C})$) in the tensor product $V_{\mu^{(1)}} \otimes \cdots \otimes V_{\mu^{(r)}}$, or equivalently, as the coefficient of the Schur function s_λ in the product $s_{\mu^{(1)}} \cdots s_{\mu^{(r)}}$. Also, by Schur-Weyl duality, $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ is the multiplicity of the irreducible representation S_λ of $\mathfrak{S}_N$ in the representation induced from the representation $S_{\mu^{(1)}} \otimes \cdots \otimes S_{\mu^{(r)}}$ of the Young subgroup $\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_r}$.

In recent years, two kinds of generalizations of these numbers have been considered:

1) Polynomials $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}(q)$ with integer coefficients, reducing to Littlewood-Richardson multiplicites for $q = 1$. As a special case, one finds the Kostka-Foulkes polynomials, which can also be interpreted as q-analogues of weight multiplicities. The most general q-analogues are defined in terms of certain generalized Young tableaux,

called *ribbon tableaux* [12]. These polynomials are known to be related to the quantum affine algebras $U_q(\widehat{sl}_n)$, and have been recently identified as a special family of Kazhdan-Lusztig polynomials for affine symmetric groups [13]. Their specialization at roots of unity are believed to be relevant to the calculation of certain plethysms [12]. Another definition of q-analogues of LR coefficients appears in [19]. It seems to coincide with those of [12] in most cases but the reason for this is still unclear.

2) Restricted Littlewood-Richardson coefficients $\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$, which are the structure constants of the fusion algebras of Wess-Zumino-Witten conformal field theories associated to $\widehat{sl}_n$ at level l. These numbers can be computed as alternating sums of ordinary Littlewood-Richardson coefficients (the Kac-Walton formula [23, 9, 3]). They are known to be nonnegative, but a combinatorial interpretation is available only in special cases.

The aim of this paper is to propose a common generalization of 1) and 2), that is, to define q-analogues of the restricted Littlewood-Richardson coefficients in terms of ribbon tableaux. Our definition proceeds by introducing appropriate powers of q in the Kac-Walton formula, so that the replacement of the ordinary multiplicities by their q-analogues yields a nonnegative polynomial. We are able to prove this only in special cases, but the simplest examples are already of interest, since they lead to simple formulae for various kinds of branching functions, such as the characters of the representations of unitary minimal series of the Virasoro algebra. We also obtain q-analogues of the dimensions of the spaces of nonabelian theta functions.

2. Fusion rules and vertex operators

Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$ be the fundamental weights of the affine Lie algebra $\widehat{sl}_n$ of type $A^{(1)}_{n-1}$, and let δ be the null root. The Chevalley generators are denoted by e_i, h_i, f_i, $i = 0, \ldots, n-1$ and the degree generator by d. The weight lattice is $P = \mathbf{Z}\Lambda_0 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}\delta$. The set of dominant integral weights is $P^+ = \mathbf{N}\Lambda_0 \oplus \cdots \oplus \mathbf{N}\Lambda_{n-1} \oplus \mathbf{Z}\delta$. The Weyl vector is $\rho = \Lambda_0 + \cdots + \Lambda_{n-1}$. The level of a weight $\Lambda = \sum_{i=0}^{n-1} l_i\Lambda_i + z\delta \in P^+$ is $\mathrm{lev}\,(\Lambda) = \sum_{i=0}^{n-1} l_i$. We denote by P^+_l the subset of level l dominant integral weights of the form $\Lambda = \sum_{i=0}^{n-1} l_i\Lambda_i$ (no δ) with $l_i \in \mathbf{N}$ and $l_0 + \cdots + l_{n-1} = l$. For each $\lambda \in P^+$ there is a unique integrable highest weight module $L(\Lambda)$ with highest weight Λ. Any weight $\mu \in P^+$ of the form $\mu = m_1\Lambda_1 + \cdots + m_{n-1}\Lambda_{n-1}$ can be interpreted as a dominant integral weight of the finite dimensional Lie algebra $sl_n \subset \widehat{sl}_n$. For any choice of a complex number z, the irreducible representation V_μ of sl_n can be extended to a level 0 representation $V_\mu(z)$ of $\widehat{sl}_n'$ (evaluation modules).

The classical part $\lambda = \bar{\Lambda}$ of a weight $\Lambda \in P$ is defined by means of the linear map $\bar{\Lambda}_i = \Lambda_i - \Lambda_0$, $\bar{\delta} = 0$. If we fix a level $l \geq 1$, a dominant integral weight $\Lambda \in P^+_l$ is specified by its classical part $\lambda = \bar{\Lambda}$, which can be identified with a partition whose Young diagram is contained in the rectangle (l^{n-1}), the fundamental weight Λ_i being represented by a column of height i. Conversely, we set $\mathrm{wt}\,(\lambda) = \Lambda$.

For such partitions $\lambda, \mu^{(1)}, \ldots, \mu^{(r)}$, let $\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ denote the dimension of the space of intertwining operators

$$\mathrm{Hom}_{\widehat{sl}'_n}\left(L(\Lambda),\, L(l\Lambda_0) \otimes V_{\mu^{(1)}}(z_1) \otimes \cdots \otimes V_{\mu^{(r)}}(z_r)\right) \tag{1}$$

(called chiral vertex operators) where $(z_1 \ldots, z_r) \in \mathbf{C}^r$ is generic. It is known that the numbers $\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ are the structure constants of an associative algebra $\mathcal{F}^{(n,l)}$, the fusion algebra of the Wess-Zumino-Witten model associated to $\widehat{sl}_n$ at level l (see [21]). This algebra has several (non obviously equivalent) interpretations. The most elementary one, on which we will rely in this paper, is due to Goodman and Wenzl [6].

Let $H_N(q)$ be the Hecke algebra of type A_{N-1}, *i.e.* the $\mathbf{C}[q, q^{-1}]$-algebra generated by elements $T_1, \ldots, T_{N-1}$ verifying the braid relations together with $T_i^2 = (q-1)T_i + q$. This algebra is a q-analogue of the group algebra of the symmetric group $\mathfrak{S}_N$, and there exist for it q-analogues of the various realizations of the irreducible representations of the symmetric group. In particular, there is a q-analogue π^q_λ of Young's orthogonal form. The representation space $V_\lambda(q)$ has an orthonormal basis $|t\rangle$ labelled by standard Young tableaux of shape λ.

Recall that a standard tableau t can be interpreted as a chain of partitions $\lambda^{(1)} \subset \lambda^{(2)} \subset \ldots \subset \lambda^{(N)} = \lambda$, where $\lambda^{(k)}$ is obtained by adding to the diagram of $\lambda^{(k-1)}$ the box containing the entry k in t. Let us say that a partition λ is (n, l)-restricted if it has at most n parts and $\lambda_1 - \lambda_n \le l$. Let $\Pi^{(n,l)}$ be the set of such partitions. The tableau t is said to be (n, l)-restricted if all the intermediate partitions $\lambda^{(k)} \in \Pi^{(n,l)}$. Denote by $\mathrm{STab}^{(n,l)}(\lambda)$ the set of (n, l)-restricted tableaux of shape λ.

Suppose now that $q = \zeta$, a primitive Lth root of unity, where $L = n + l$. Then, $H_N(\zeta)$ is not semisimple for $N \ge L$, and $V_\lambda(\zeta)$ is not irreducible, nor even semisimple in general. Moreover only the integral form of the irreducible representations can be specialized at $q = \zeta$. For $\lambda \in \Pi^{(n,l)}$, Wenzl [24] showed how to construct from the orthogonal form $V_\lambda(\zeta)$ an irreducible representation D_λ. The point is that the matrix elements $\langle t'|\pi^q_\lambda(T_w)|t''\rangle$ have no pole at $q = \zeta$ as soon as t' and t'' are (n, l)-retricted. Moreover, the representation matrices restricted to the subspace

$$V^{(n,l)}_\lambda(\zeta) = \bigoplus_{t \in \mathrm{STab}^{(n,l)}(\lambda)} \mathbf{C}|t\rangle \tag{2}$$

define an irreducible representation D_λ of $H_N(\zeta)$. Denote by $\pi^{(n,l)}_\lambda$ the representation on this space, and let

$$H^{(n,l)}_N = \bigoplus_{\lambda \in \Pi^{(n,l)}_N} \pi^{(n,l)}_\lambda(H_N(\zeta))\,. \tag{3}$$

Then, $H_N^{(n,l)}$ is a semi-simple quotient of $H_N(\zeta)$, and its irreducible representations are exactly the $\pi_\lambda^{(n,l)}$.

Let $\mathcal{R}_N^{(n,l)} = R(H_N^{(n,l)})$ be the Grothendieck group of $H_N^{(n,l)}$, *i.e.* the free abelian group generated by the isomorphism classes $[D_\lambda]$ of irreducible representations, with addition corresponding to direct sum. The sum

$$\mathcal{R}^{(n,l)} = \bigoplus_{N\geq 0} \mathcal{R}_N^{(n,l)} \tag{4}$$

can be endowed with a ring structure by setting $[D_\lambda]\cdot[D_\mu] = \left[D_\lambda \hat{\otimes} D_\mu\right]$ where $\hat{\otimes}$ is the external tensor product, obtained by inducing the $H_N \otimes H_M$-module $D_\lambda \otimes D_\mu$ to H_{N+M}. Then [6], $\mathcal{R}^{(n,l)}$ is isomorphic to a quotient $Sym(n)/\mathcal{J}^{n,l}$ of the ring of symmetric polynomials in n variables $Sym(n) = \mathbf{Z}[x_1,\ldots,x_n]^{\mathfrak{S}_n}$, the ideal $\mathcal{J}^{n,l}$ being generated by the Schur functions s_λ labelled by partitions such that $\lambda_1 - \lambda_n = l+1$. The fusion algebra $\mathcal{F}^{(n,l)}$ is isomorphic to the quotient of $\mathcal{R}^{(n,l)}$ by the single relation $s_{(1^n)} \equiv 1$.

3. Crystal base and q-vertex operators

If we consider q as an indeterminate, we can replace $\widehat{sl}_n'$ in (1) by $U_q'(\widehat{sl}_n)$. According to [5, 4] the space of q-vertex operators $\mathrm{Hom}_{U_q'(\widehat{sl}_n)}(L(\Lambda), L(l\Lambda_0)\otimes V_{\mu^{(1)}}(z_1)\otimes\cdots\otimes V_{\mu^{(r)}}(z_r))$ is identified with the following vector space:

$$\bigoplus \bigotimes_{j=1}^{r} \mathbf{Q}(q)\langle v \in V_{\mu^{(j)}} \mid \mathrm{wt}\, v = \Lambda^{(j)} - \Lambda^{(j-1)}, e_i^{\langle h_i, \Lambda^{(j-1)}\rangle+1} v = 0\, \forall i\rangle, \tag{5}$$

where the direct sum is taken over all sequences $(\Lambda^{(0)},\ldots,\Lambda^{(r)}) \in (P_l^+)^{r+1}$ such that $\Lambda^{(0)} = l\Lambda_0, \Lambda^{(r)} = \Lambda$.

We would like to relate our restricted LR coefficient to the crystal base theory introduced by Kashiwara [10]. For this purpose we prepare some notations. Any integrable $U_q(\widehat{sl}_n)$-module $L(\Lambda)$ has a crystal base $(\mathcal{L}(\Lambda), B(\Lambda))$ [10]. But the finite-dimensional $U_q'(\widehat{sl}_n)$-module V_λ does not necessarily have a crystal base. From this reason we restrict ourselves later in this section to the cases when all $\mu^{(j)}$'s are of rectangular shape, and let $(\mathcal{L}_{\mu^{(j)}}, B_{\mu^{(j)}})$ be the crystal base of $V_{\mu^{(j)}}$. As is well known, modified Chevalley generators (or Kashiwara operators) $\tilde{e}_i, \tilde{f}_i$ act on $B(\Lambda)$ or $B_{\mu^{(j)}}$.

Let $\Phi(z)$ be an appropriately normalized q-vertex operator from $L(\Lambda')$ to $L(\Lambda)\otimes V_\lambda(z)$. It is known [4] that it preserves the crystal lattice, *i.e.*

$$\Phi(z)\mathcal{L}(\Lambda') \subset \mathcal{L}(\Lambda)\otimes \mathcal{L}_\lambda(z).$$

Therefore, counting the dimensionality of the space of vertex operators is reduced to the combinatorics of crystals. Let us define the set of restricted paths.

$$\mathcal{P}^\lambda_{\mu^{(1)},\ldots,\mu^{(r)}} = \left\{ p = b_1\otimes\cdots\otimes b_r \;\middle|\; \begin{array}{l} b_j \in B_{\mu^{(j)}}, \mathrm{wt}\, b_j = \Lambda^{(j)} - \Lambda^{(j-1)}, \\ \tilde{e}_i^{\langle h_i,\Lambda^{(j-1)}\rangle+1} b_j = 0\, \forall j = 1,\ldots,r \end{array} \right\},$$

where $\Lambda^{(j)}$'s are as in (5). Thus we have

$$\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}} = \left|\mathcal{P}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}\right|.$$

We shall introduce a statistic on the set of restricted paths. First we introduce a $\mathbf{Z}$-valued function $H_{\mu^{(j)}\mu^{(k)}}$ on $B_{\mu^{(j)}} \otimes B_{\mu^{(k)}}$ called the energy function. It is determined from the combinatorics of crystal graphs. For an element $p = b_1 \otimes \cdots \otimes b_r$ of $B_{\mu^{(1)}} \otimes \cdots \otimes B_{\mu^{(r)}}$, we define the energy of p by

$$E(p) = \sum_{1 \le j < k \le r} H_{\mu^{(j)}\mu^{(k)}}(b_j \otimes b_k^{(j+1)}),$$

where $b_k^{(j+1)}$ is determined under some isomorphism of crystals (see [15, 17, 18] for details). Using this statistic we define a q-analogue $\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}(q)$ of $\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ by

$$\bar{c}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}(q) = \sum_{p \in \mathcal{P}^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}} q^{E(p)}. \tag{6}$$

4. Ribbon tableaux and q-analogues of LR coefficients

Recall that a Schur function $s_\lambda(X)$ can be expressed as a sum over semi-standard Young tableaux t of shape λ

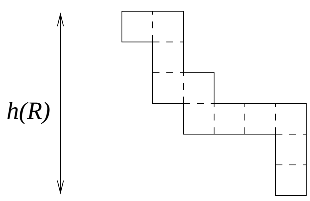
$$s_\lambda(X) = \sum_{t \in \mathrm{Tab}\,(\lambda)} X^t \tag{7}$$

where $X^t = \prod_i x_i^{m_i}$, m_i being the number of occurences of the integer i in t. Therefore, a product of r Schur functions $s_{\mu^{(i)}}$ is a sum over r-tuples of tableaux

$$s_{\mu^{(1)}} s_{\mu^{(r)}} \cdots s_{\mu^{(r)}} = \sum_{(t_1,\ldots,t_r)} X^{t_1} X^{t_2} \cdots X^{t_r}\,. \tag{8}$$

The point of this trivial remark is that r-tuples of tableaux are in one-to-one correspondence with a certain kind of generalized Young tableaux, the so-called r-ribbon tableaux, on which some extra combinatorial information can be read.

The notion of ribbon tableau is derived from the r-core and r-quotient algorithms, originating from the modular representation theory of symmetric groups (*cf.* [8]). An r-ribbon R is a connected skew Young diagram of r boxes, not containing a 2×2 square. Its height $h(R)$ is the number of rows occupied by the diagram. For example,

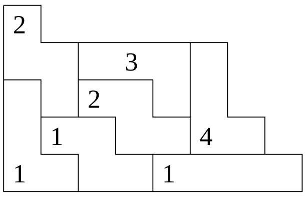


is an 11-ribbon of height $h(R) = 6$. A partition is said to be an r-core if it is not possible to peel off an r-ribbon from its diagram. The r-core $\lambda_{(r)}$ of a partition λ is the partition obtained by removing from its diagram a maximal number of r-ribbons. This can usually be done in several ways, but all of them lead to the same result. The r-core play the role of the remainder in a kind of Euclidean division of partitions. The role of the quotient is played by an r-tuple of partitions, the r-quotient $(\lambda^{(0)}, \dots, \lambda^{(r-1)})$, satisfying

$$|\lambda| = |\lambda_{(r)}| + r \sum_{i=0}^{r-1} |\lambda^{(i)}| \,. \tag{9}$$

Details can be found in [8]. This algorithm is reversible, and it provides a one-to-one correspondence between partitions with fixed r-core ν and r-tuples of partitions. In particular, any r-tuple of partitions can be interpreted as a single partition with empty r-core.

In the same way as standard tableaux can be regarded as chains of partitions whose consecutive terms differ by exactly one box, semi-standard tableaux can be interpreted as chains whose consecutive terms differ by horizontal strips of boxes. Applying the inverse r-quotient algorithm to an r-tuple of semi-standard tableaux, interpreted as a chain of r-tuples of partitions, one obtains a chain of partitions without r-core, in which the skew diagrams formed from two consecutive terms $k-1, k$ can be tiled by r-ribbons in exactly one way. These ribbons can be labelled by k, and the chain can be represented by an r-ribbon tableau, which is a tiling of a Young diagram by labelled r-ribbons, satisfying the following condition: if we define the root of a ribbon as its rightmost lowest cell, the root of a ribbon labelled k should not lie above any ribbon labelled j for $j \geq k$. The weight of a ribbon tableau is defined as for ordinary tableaux, so that the correspondence between r-tuples of ordinary tableaux and ribbon tableaux is weight preserving. The spin $s(R)$ of a ribbon R is $\frac{1}{2}(h(R)-1)$, and the spin $s(T)$ of a ribbon tableau T is the sum of the spins of its ribbons. For example,



is a 4-ribbon tableau of shape $(8, 7, 6, 6, 1)$, weight $(3, 2, 1, 1)$ and spin $9/2$.

Let T be a ribbon tableau of shape λ. Let $s_r^*(\lambda)$ be the maximal spin of an r-ribbon tableau of shape λ, and define the cospin $\tilde{s}(T)$ as the difference $s_r^*(\lambda) - s(T)$. It can be shown that this is always an integer. For a partition λ without r-core, let $\mathrm{Tab}_r(\lambda)$ be the set of semi-standard r-ribbon tableaux of shape λ. Then, the polynomial

$$\tilde{G}_\lambda^{(r)}(X;q) = \sum_{T\in \mathrm{Tab}_r(\lambda)} q^{s(T)} X^T \tag{10}$$

is a q-analogue of the product of Schur functions $s_{\lambda^{(0)}} s_{\lambda^{(1)}} \cdots s_{\lambda^{(r-1)}}$. It can be shown that it is actually a symmetric polynomial, though this is not obvious from the definition. The polynomials

$$c^{\nu}_{\lambda^{(0)},\ldots,\lambda^{(r-1)}}(q) = \langle s_\nu, \tilde{G}_\lambda^{(r)}(X;q)\rangle \tag{11}$$

are therefore q-analogues of Littlewood-Richardson coefficients. It is conjectured that their coefficients are non-negative (this is also unclear from their definition). For $r = 2$ this has been proved by a combinatorial argument [2]. This is also known when all the $\lambda^{(i)}$ are row (resp. column) diagrams. For the general case, the positivity should follow from a recent expression of these polynomials as parabolic Kazhdan-Lusztig polynomials of the affine symmetric group [13], but the relevant geometrical interpretation does not seem to be available in the literature.

An interesting particular case is obtained when λ is of the form $\lambda = r\mu = (r\mu_1, \ldots, r\mu_m)$, μ being an arbitrary partition. Then λ has no r-core, and one can define

$$\tilde{H}_\mu^{(r)}(X;q) = \tilde{G}_{r\mu}^{(r)}(X;q)\,, \tag{12}$$

$$H_\mu^{(r)}(X;q) = q^{s_r^*(r\mu)} \tilde{H}_\mu^{(r)}(X;q^{-1})\,. \tag{13}$$

For r sufficiently large ($r \geq m = \ell(\mu)$), it is known that $H_\mu^{(r)}$ is equal to the Hall-Littlewood function

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q) s_\lambda \tag{14}$$

where the $K_{\lambda\mu}(q)$ are the Kostka-Foulkes polynomials. Moreover, it is conjectured that the difference between two consecutive H-functions $H_\mu^{(j+1)} - H_\mu^{(j)}$ is nonnegative on the Schur basis.

5. The (n,l)-restricted q-analogues

One can compute in fusion algebras $\mathcal{F}^{(n,l)}$ by an algorithm due (independently) to Kac and Walton. In the case of $\widehat{sl}_n$, this algorithm turns out to be identical to the one devised by Goodmann and Wenzl for the algebras $\mathcal{R}^{(n,l)}$, and it is this coincidence which proves the relation between these algebras.

The Goodman-Wenzl algorithm for computing in $\mathcal{R}^{(n,l)}$ can be described as follows. To compute the product $\bar{f}_1 \cdots \bar{f}_r$ of elements $\bar{f}_i \in \mathcal{R}^{(n,l)}$, one first evaluates the corresponding product of symmetric functions

$$f_1 \cdots f_r = \sum_\lambda c_\lambda s_\lambda$$

in terms of Schur functions. Next, one replaces each s_λ by its class $\bar{s}_\lambda$, taking into account the equivalences

$$\bar{s}_\lambda = \begin{cases} \varepsilon(w)\bar{s}_{w\circ\lambda} & \text{if there is a } w \in \hat{W} \text{ such that } w\circ\lambda \in \Pi^{(n,l)} \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where the action of the affine Weyl group $\hat{W} = \langle \sigma_0, \ldots, \sigma_{n-1}\rangle$ of type $A^{(1)}_{n-1}$ on $\mathbf{Z}^n$ is defined by

$$\sigma_i(\lambda) = \begin{cases} (\lambda_n + L, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 - L) & \text{for } i = 0 \\ (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n) & \text{for } i = 1, \ldots, n-1 \end{cases} \tag{16}$$

and $w\circ\lambda = w(\lambda+\bar{\rho}) - \bar{\rho}$, where $\bar{\rho} = (n-1, n-2, \ldots, 1, 0)$.

Let $\lambda \in \Pi_N^{(n,l)}$ and $\Lambda = \mathrm{wt}\,(\lambda)$. If $\mu = w\circ\lambda$ is a partition, it is the unique one such that $\mathrm{wt}\,(\mu) = w\cdot\Lambda$ and $|\mu| = |\lambda|$. So the $\circ$ action corresponds to the projection of the dot action on the classical part of the weight lattice. If one forgets about columns of height n, one obtains the true fusion algebra. In this context, the above algorithm is due to Kac [9] and Walton [23].

The intersection of the orbit of a partition $\lambda \in \Pi^{(n,l)}$ under $\hat{W}$ with the dominant chamber admits a convenient graphical description. It can be generated by sliding in the diagram of λ the L-ribbons whose root lies in the first row (if there is no such ribbon, then $\bar{s}_\lambda \equiv 0$ unless λ is a (n,l)-regular L-core). The sign of an element of the orbit is the product of the signs $(-1)^{h(R)-h(R')}$ for all ribbons R of λ where R' is the new ribbon obtained by sliding R and $h(R)$ denotes the heigth of R [6, 3].

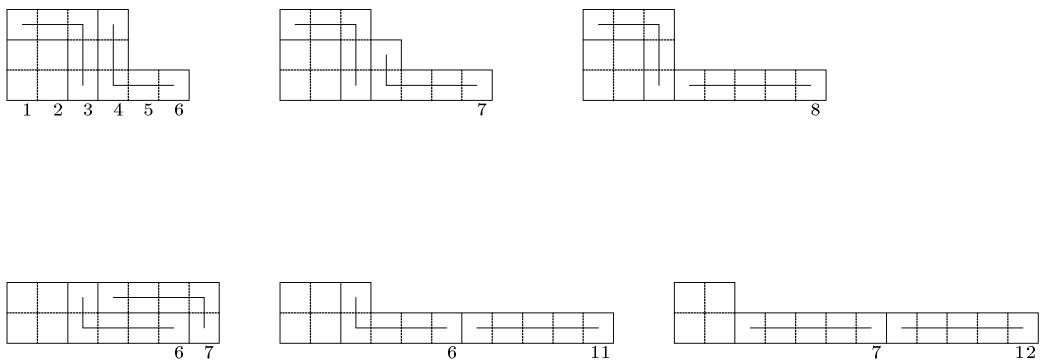
Let $r(R)$ be the position of the root of R, as in the picture below. If μ is obtained from λ by transforming R into R', we define the t-equivalence by

$$s_\lambda \equiv_t (-1)^{h(R)-h(R')} t^{r(R')-r(R)} s_\mu \ . \tag{17}$$

For example, with $n=3$ and $l=2$, s_{644} is t-equivalent to

$$-t^1 s_{743},\ t^2 s_{833},\ t^4 s_{77}, -t^8 s_{11,3},\ t^{10} s_{12,2}$$

as illustrated below:



The power of t is just the opposite of the coefficient of the null root δ in $w\cdot\Lambda$, where $\Lambda = \mathrm{wt}\,(\lambda)$. In our example, we start from $\lambda = (6,4,4)$. We have $n = 3$ and $L = 5$ so that we use weights of level $5-3 = 2$. Therefore, $\mathrm{wt}\,(\Lambda) = 2\Lambda_1$. The orbit of Λ is

$$\begin{array}{ll}
\Lambda = 2\Lambda_1 & \leftrightarrow s_{644} \\
s_0\cdot\Lambda = -2\Lambda_0 + 3\Lambda_1 + \Lambda_2 - \delta & \leftrightarrow\ -ts_{743} \\
s_0s_2\cdot\Lambda = -3\Lambda_0 + 5\Lambda_1 - 2\delta & \leftrightarrow t^2 s_{833} \\
s_0s_1\cdot\Lambda = -5\Lambda_0 + 7\Lambda_2 - 4\delta & \leftrightarrow t^4 s_{77} \\
s_0s_2s_1\cdot\Lambda = -8\Lambda_0 + 7\Lambda_1 + 3\Lambda_2 - 8\delta & \leftrightarrow\ -t^8 s_{11,3} \\
s_0s_2s_1s_0\cdot\Lambda = -10\Lambda_0 + 10\Lambda_1 + 2\Lambda_2 - 10\delta & \leftrightarrow t^{10} s_{12,2}\ .
\end{array}$$

The q-fusion coefficients are defined by applying this t-reduction algorithm to the q-analogues of unrestricted products of Schur functions, with $t = q$ for the spin/charge/energy q-analogues, and $t = 1/q$ for the cospin/cocharge q-analogues.

For example, the cospin q-analogue of the cube $(s_{21})^3$ of the Schur function s_{21} (restricted to partitions of length ≤ 4) is equal to

$$\begin{aligned}
\tilde G^{(3)}_{666333} = \quad & s_{63} + (q+q^2)s_{621} + q^3 s_{6111} + (q+q^2)s_{54} \\
&+ (q+2q^2+2q^3+q^4)s_{531} + (q^2+2q^3+q^4)s_{522} + (q^2+2q^3+2q^4+q^5)s_{5211} \\
&+ (q^2+2q^3+q^4)s_{441} + (q^2+2q^3+3q^4+2q^5)s_{432} + (2q^3+3q^4+3q^5+q^6)s_{4311} \\
&+ (q^3+3q^4+3q^5+2q^6)s_{4221} + (q^3+q^6)s_{333} + (2q^4+3q^5+2q^6+q^7)s_{3321} \\
&+ (q^5+2q^6+q^7)s_{3222}\,.
\end{aligned}$$

Now, take $n = 4$ and $l = 2$, so that only the partitions (3321) and (3222) are $(4,2)$-restricted. The q-reduction algorithm produces as q-analogue of $(\bar s_{21})^3$ in $\mathcal{F}^{(4,2)}$

$$q^7\bar s_{3321} + (q^6+q^7)\bar s_{3222}\,. \tag{18}$$

We conjecture that our restricted q-analogues are always positive, and that for a product of rectangular Schur functions, the spin q-analogues coincide (up to an overall power of q) with the ones defined by equation (6). We can prove this from the results of [11] in the case where all the $\mu^{(i)}$ are equal to the same row (or column) partition.

The modified Hall-Littlewood function Q'_μ is a q-analogue of the product $h_{\mu_1}h_{\mu_2}\dots h_{\mu_r}$. In this case, our conjecture states that the restricted Kostka polynomial $\bar K_{\lambda\mu}(q)$ (coefficient of $\bar s_\lambda$ in $\bar Q'_\mu$) is equal to the restricted 1d-configuration sum $X^{(l-k)}_\mu(\lambda)$ of [7]. For example, in Example 3.1 of [7] it is found that $X^{(2)}_{2211}(321) = q$. On the other hand, the above algorithm gives for the restricted Kostka polynomial

$$\begin{aligned}
\bar K_{321,2211}(q) \quad &= K_{321,2211}(q) - q^{-1}K_{42,2211}(q) + q^{-3}K_{6,2211}(q) \\
&= (q+2q^2+q^3) - q^{-1}(2q^3+q^4+q^5) + q^{-3}(q^7) = q\,.
\end{aligned}$$

6. Examples

6.1. *Virasoro characters*

We consider the $\widehat{sl}_2$ case ($L = l + 2$). Assume $\mu^{(1)} = \cdots = \mu^{(r)} = (1)$. We set

$$\bar{k}_{(N+b,N)}(q) = \bar{c}^{(N+b,N)}_{(1),\dots,(1)}(q) = \bar{K}_{(N+b,N),(1^{2N+b})}(q).$$

The intersection of the orbit of a dominant integral weight of $A_1^{(1)}$ under the affine Weyl group with the dominant chamber can be explicitly computed. As a result, we obtain an explicit formula for the restricted Kostka polynomial

$$\bar{k}_{(N+b,N)}(q) = \sum_{n\geq 0} q^{-n(Ln+b+1)} k_{(N+Ln+b,N-Ln)}(q) - \sum_{n\geq 1} q^{-n(Ln-b-1)} k_{(N+Ln-1,N-Ln+b+1)}(q)\ .$$

Taking into account the expression

$$k_{(\lambda_1,\lambda_2)}(q) = q^{\binom{\lambda_1}{2}+\binom{\lambda_2}{2}} \begin{bmatrix} \lambda_1+\lambda_2 \\ \lambda_1 \end{bmatrix}_q - q^{\binom{\lambda_1+1}{2}+\binom{\lambda_2-1}{2}} \begin{bmatrix} \lambda_1+\lambda_2 \\ \lambda_1+1 \end{bmatrix}_q\ ,$$

one obtains

$$\bar{k}_{(N+b,N)}(q) = q^{N(N+b-1)+b(b-1)/2} \left(\sum_{n=-\infty}^{\infty} q^{n[L(L-1)n+Lb-b-1]} \begin{bmatrix} 2N+b \\ N-Ln \end{bmatrix}_q \right.$$
$$\left. - \sum_{n=-\infty}^{\infty} q^{n[L(L-1)n+Lb+2L-b-1]+b+1} \begin{bmatrix} 2N+b \\ N-Ln-1 \end{bmatrix}_q \right)\ . \tag{19}$$

A restricted path in $\mathcal{P}^{(N+b,N)}_{(1),\dots,(1)}$ can be interpreted as an oriented path on P_l^+ starting from $l\Lambda_0$ and arriving at $(l-b)\Lambda_0 + b\Lambda_1$. More generally, paths of length $2N+b-a$ starting from $(l-a)\Lambda_0 + a\Lambda_1$ and arriving to $(l-b)\Lambda_0 + b\Lambda_1$ are encoded by the skew diagram of shape $(N+b,N)/(a)$. The reduced skew Kostka polynomials are obtained in a similar way.

$$\bar{k}_{(N+b,N)/(a)}(q) = q^{N(N+b-a-1)+(b-a)(b-a-1)/2} \sum_{n=-\infty}^{\infty} \left(q^{n[L(L-1)n+(L-1)b-La-1]} \begin{bmatrix} 2N+b-a \\ N-Ln \end{bmatrix}_q \right.$$
$$\left. - \sum_{n=-\infty}^{\infty} q^{[(L-1)n+a+1][Ln+b+1]} \begin{bmatrix} 2N+b-a \\ N+Ln+b+1 \end{bmatrix}_q \right)\ . \tag{20}$$

In the limit $N \to \infty$, we get

$$\lim_{N\to\infty} q^{-N(N+b-a-1)-(b-a)(b-a-1)/2} \bar{k}_{(N+b,N)/(a)}(q)$$
$$= \frac{1}{\varphi(q)} \sum_{n=-\infty}^{\infty} \left(q^{n[L(L-1)n+(L-1)b-La-1]} - q^{[(L-1)n+a+1][Ln+b+1]} \right)\ . \tag{21}$$

We shall relate this series to a Virasoro charcter. We use the following conventions for the Virasoro characters. The unitary series minimal models are parametrized by three integers giving their central charges

$$c = 1 - \frac{6}{m(m+1)} \qquad (m \in \mathbf{N},\ m \geq 2) \tag{22}$$

and conformal weights

$$h = h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \qquad r = 1, \ldots, m-1,\ s = 1, \ldots, r\ . \tag{23}$$

The character of the module $M^{(m)}(r,s) = L(c,h)$ with c and h as above is given by the Rocha-Caridi formula [16]

$$\chi^{(m)}_{r,s}(q) = \frac{1}{\varphi(q)} \sum_{n=-\infty}^{\infty} \left(q^{a_-(n)} - q^{a_+(n)} \right) \tag{24}$$

where

$$a_\pm(n) = \frac{[2m(m+1)n + (m+1)r \pm ms]^2 - 1}{4m(m+1)}\,. \tag{25}$$

Taking $m = L-1$, $r = a+1$, $s = b+1$, we find

$$\lim_{N\to\infty} q^{-N(N+b-a-1)-(b-a)(b-a-1)/2}\,\bar{k}_{(N+b,N)/(a)}(q) = q^{-h}\chi^{(L-1)}_{a+1,b+1}(q)\,. \tag{26}$$

6.2. Nonabelian theta functions

Let $\mathcal{SU}_\Sigma(n)$ be the moduli space of semi-stable holomorphic vector bundles with trivial determinant over a compact Riemann surface Σ of genus g (*cf.* [1]). All line bundles on $\mathcal{SU}_\Sigma(n)$ are powers of the so-called determinant bundle $\mathcal{L}$. The sections of $\mathcal{L}^l$ are called nonabelian theta functions of rank n and level l on Σ. The dimension $h^0(\mathcal{L}^l)$ of $H^0(\mathcal{SU}_\Sigma(n), \mathcal{L}^l)$ is obtained by a calculation in the fusion algebra $\mathcal{F}^{(n,l)}$ as follows [22, 1]. For a partition λ interpreted as a dominant weight of sl_n, let $\lambda^* = -w_0(\lambda)$ be the highest weight of the dual representation $(V_\lambda)^*$. Now, let

$$\omega = \sum_{\lambda \subset (l^{n-1})} \bar{s}_\lambda \bar{s}_{\lambda^*} \quad \in \quad \mathcal{F}^{(n,l)}\,. \tag{27}$$

Then, $h^0(\mathcal{L}^l)$ is equal to the constant term in ω^g. For example, with $n=2$, $h^0(\mathcal{L}) = 2^g$, $h^0(\mathcal{L}^2) = 2^{g-1}(2^g+1)$, and so on. If we compute ω^g by using the cospin q-analogues of the fusion coefficients, we obtain q-analogues $h^0_q(\mathcal{L}^l)$ of these numbers. With $n=2$ as above, one finds $h^0_q(\mathcal{L}) = (1+q)^g$ and $h^0_q(\mathcal{L}^2) = \frac{1}{2}[(1+q^2)^g + (1+q)^{2g}]$. This suggests the existence of some natural filtration of the space of nonabelian theta functions.

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