Noncommutative symmetric functions V: A degenerate version of $U_q(gl_N)$ *

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Dedicated to the memory of Marcel–Paul Schützenberger

Abstract: We interpret quasi-symmetric functions and noncommutative symmetric functions as characters of a degenerate quantum group obtained by putting $q = 0$ in a variant of $U_q(gl_N)$.

1 Introduction

This paper is a continuation of the series [9, 7, 21, 35] devoted to the representation theoretical interpretations of the algebras of quasi-symmetric functions (see [13]) and of noncommutative symmetric functions [11], two different generalizations of the classical theory of symmetric functions which eventually turned out to be dual to each other [29].

The new theories are very similar to the classical one, the main difference being that the rôle of partitions is in both cases played by compositions. Both algebras possess distinguished bases which can be regarded as analogues of Schur functions. In particular, the structure constants in these bases are nonnegative integers, which suggests the existence of representation theoretical interpretations similar to the relation between symmetric functions, symmetric groups and general linear groups.

The analogue of the relation between Schur functions and representations of the symmetric group was found to be provided by the 0-Hecke algebra $H_n(0)$ of type A [9, 7, 21]. The known q-analogue of the Schur-Weyl duality between generic Hecke algebras $H_n(q)$ and the quantum groups associated with general linear groups [16] suggested the existence of an interpretation of quasi-symmetric functions and noncommutative symmetric functions as characters of some quantum group at $q = 0$.

However one has to be careful since the standard quantum groups are not defined for $q = 0$. Indeed, their presentations are generally symmetric in q and q^{-1}, and have no limit for $q \to 0$. The theory of crystal bases [17] which is sometimes loosely

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referred to as "the theory of quantum groups at $q = 0$" allows only to set $q = 0$ in certain formulas, obtained from the generic case, and involving only polynomials after some normalization. That is to say, the theory of crystal bases does not deal with a degenerate case, but rather describes the combinatorial aspects of the generic case (e.g. the combinatorics of Young tableaux in the case of gl_N). The relation between quantum groups at $q = 0$ and classical combinatorics was first pointed out in [4].

One can associate two kinds of quantum algebras to a classical group G. First a q analogue $F_q(G)$ of the algebra of polynomial functions on the group (*cf.* [10]) whose comodules are in duality with polynomial representations of the group. Second a q-analogue $U_q(\mathfrak{g})$ of the universal enveloping algebra of its Lie algebra [16].

There exists a non-standard version of $F_q(GL_N)$, due to Dipper and Donkin [5] in which the specialization $q = 0$ is possible. This allowed us to intepret quasi-symmetric functions and noncommutative symmetric functions as characters of $F_0(GL_N)$ comodules, an exact analogue of the relation between Schur functions and polynomial representations of GL_N [21]. Moreover this point of view leads to an interpretation of quasi-symmetric functions similar to the plactic interpretation of symmetric functions introduced by Lascoux and Schützenberger in [25].

The Dipper-Donkin version of $F_q(GL_N)$ and the standard one are in fact specializations of a two-parameter quantization discovered by Takeuchi [34], who also gives the corresponding family $U_{\alpha,\beta}(gl_N)$ of quantized enveloping algebras. Jimbo's quantization corresponds to $U_{q,q}(gl_N)$, and the one appropriate to the Dipper-Donkin algebra is $U_{q,1}(gl_N)$.

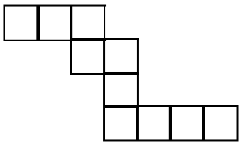
The specialization related to quasi-symmetric functions and noncommutative symmetric functions is $\mathcal{U}_0(gl_N) = U_{0,1}(gl_N)$. It is this algebra that we study in this paper. We describe the basic properties of $U_{q,1}(gl_N)$, including the quantum Schur-Weyl duality in the generic case, and study the degeneracy at $q = 0$. We can then prove that distinguished bases of quasi-symmetric functions and of noncommutative symmetric functions are actually characters of irreducible and of indecomposable polynomial modules over the degenerate algebra and we give the classification of such modules.

2 Notations and background

Our notations will be essentially those of [11, 20, 7, 21]. We recall here the most important ones to make this paper self-contained.

2.1 Compositions and permutations

A composition of n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers whose sum is equal to n. The length of I is $\ell(I) = r$. A composition can be represented by a skew Young diagram called a *ribbon diagram* of shape I. For example, the ribbon diagram of shape $I = (3, 2, 1, 4)$ is



The *mirror image* of I is $\overline{I} = (i_r, \ldots, i_1)$. The *conjugate* composition $I^\sim$ of I is obtained by reading from right to left the heights of the columns of the ribbon diagram of I. For example, $(3,2,1,4)^\sim = (1,1,1,3,2,1,1)$.

One associates with a composition $I = (i_1, \ldots, i_r)$ of n the subset $D(I)$ of $[1, n-1]$ defined by $D(I) = \{\, i_1, i_1+i_2, \ldots, i_1+\ldots+i_{r-1} \,\}$. The *reverse refinement order*, denoted $\preceq$, on the set of all compositions of the same integer n is defined by $I \preceq J$ iff $D(I) \subseteq D(J)$. For example $(3,2,6) \preceq (2,1,2,3,1,2)$. One associates with a permutation σ of $\mathfrak{S}_n$ a composition $C(\sigma)$ of n called the *descent composition* of σ, whose ribbon diagram is called the shape of σ. The descent composition of σ encodes its *descent set* $\mathcal{D}(\sigma) = \{i \in [1, n-1] \mid \sigma(i) > \sigma(i+1)\}$. That is, $D(C(\sigma)) = \mathcal{D}(\sigma)$. For example, $C(245316) = (3,1,2)$ and $\mathcal{D}(245316) = \{3,4\}$..

The set D_I of all permutations $\sigma \in \mathfrak{S}_n$ having I as descent composition is an interval $[\alpha(I), \omega(I)]$ for the weak order (permutohedron) of $\mathfrak{S}_n$, where $\alpha(I)$ is the permutation obtained by filling the columns of the ribbon diagram of shape I from bottom to top and from left to right with $1, 2, \ldots, n$ and $\omega(I)$ is the permutation obtained by filling the rows of the ribbon diagram of shape I from left to right and from bottom to top with $1, 2, \ldots, n$. For example, $\alpha(2,2,1,1,3) = 132765489$ and $\omega(2,2,1,1,3) = 896754123$.

2.2 Tableaux and quasi-tableaux

Let $(A, <)$ be a totally ordered alphabet. A *tableau* of ribbon shape I is a skew Young tableau obtained by filling a ribbon diagram R of shape I by letters of A in such a way that each row of r is nondecreasing from left to right and each column of r is strictly increasing from bottom to top. A word w is said to be of *ribbon shape I* if it can be obtained by reading from top to bottom and from left to right the columns of a tableau of ribbon shape I, which amounts to say that the descent composition of w is I. For example, the ribbon shape of $w = aacbacdcd$ is $(3,1,3,2)$.

A *quasi-tableau* of ribbon shape I is an object obtained by filling a ribbon diagram r of shape I by letters of A in such a way that each row of r is nondecreasing from left to right and each column of r is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-tableau of shape I. For example, the word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-tableau. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3,1,3,2)$.

2.3 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* [11] is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \ldots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \ldots, i_r)$, one sets $S^I = S_{i_1} S_{i_2} \ldots S_{i_r}$. The family (S^I) is a linear basis of **Sym**. Although it is convenient to define **Sym** as an abstract algebra, a useful realisation can be obtained by taking an infinite alphabet $A = \{ a_1, a_2, \ldots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geq 0} t^n S_n(A) = \overrightarrow{\prod_{i \geq 1}} (1 - t\, a_i)^{-1} \ . \tag{1}$$

The noncommutative *ribbon Schur functions* R_I can be defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} S^J \ . \tag{2}$$

The R_I form a basis of **Sym**. In the realization of **Sym** given by equation (1), R_I reduces to the sum of all words of shape I [11].

2.4 Quasi-symmetric functions

The algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions introduced by Gessel in [13] (*cf.* [11, 29]). Let $X = \{ x_1, x_2, \ldots \}$ be an infinite totally ordered set of commutative indeterminates. An element $f \in \mathbb{C}[X]$ is said to be a *quasi-symmetric function* iff for each composition $K = (k_1, \ldots, k_r)$ all monomials $x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$ with $i_1 < i_2 < \ldots < i_r$ have the same coefficient in f. The quasi-symmetric functions form a subalgebra of $\mathbb{C}[X]$ denoted by $QSym$. The *quasi-monomial functions*

$$M_I = \sum_{j_1 < \ldots < j_r} x_{j_1}^{i_1} \ldots x_{j_r}^{i_r}$$

(labelled by compositions I) form a basis of $QSym$. Another important basis of $QSym$ is given by the *quasi-ribbon functions*

$$F_I = \sum_{I \preceq J} M_J \ ,$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I. The pairing $\langle \cdot, \cdot \rangle$ between **Sym** and $QSym$ is defined by $\langle S^I, M_J \rangle = \delta_{IJ}$ or equivalently $\langle R_I, F_J \rangle = \delta_{IJ}$ (*cf.* [29, 11]). This duality can be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras [9, 21].

2.5 The hypoplactic algebra

The *plactic algebra* [25] on the totally ordered alphabet A is the $\mathbb{C}$-algebra $Pl(A)$ defined as the quotient of $\mathbb{C}\langle A \rangle$ by the relations

$$\begin{cases} aba = baa \ , \quad bba = bab & \text{for } a < b \ , \\ acb = cab \ , \quad bca = bac & \text{for } a < b < c \ . \end{cases}$$

These relations, due to Knuth [22], generate the equivalence relation that identifies two words which have the same P-symbol under the Robinson-Schensted correspondence. The associated monoid structure on the set of tableaux was studied by Lascoux and Schützenberger [25] under the name "plactic monoid". Their point of view is now illuminated by Kashiwara's theory of crystal bases [17, 18] which also leads to the definition of plactic algebras associated with the classical Lie algebras [24, 28]. Other interpretations of the plactic relations in terms of quantum groups can be found in [4, 27].

The hypoplactic algebra is a quotient of the plactic algebra where ribbon quasi-tableaux play the same rôle as Young tableaux in the plactic case. This algebra was introduced in [21] where it is shown that a remarkable commutative subalgebra of the hypoplactic algebra is isomorphic to the character ring of a degenerate quantum group and provides a hypoplactic realization of the algebra of quasi-symmetric functions analogous to the plactic interpretation of ordinary symmetric functions discovered by Lascoux and Schützenberger [25]. We will see in Section 7 that the same result holds when one rather considers polynomial modules over the crystalization of the version of $\mathcal{U}_q(gl_N)$ introduced in Section 3.

Definition 2.1 *The* hypoplactic algebra $HPl(A)$ *is the quotient of the plactic algebra* $Pl(A)$ *by the quartic relations*

$$\begin{cases} baba = abab \quad , \quad baca = abac & \text{for } a < b < c \ , \\ cacb = acbc \quad , \quad cbab = bacb & \text{for } a < b < c \ , \\ badc = dbca \quad , \quad acbd = cdab & \text{for } a < b < c < d \ . \end{cases}$$

The hypoplactic monoid has the same properties as the plactic monoid with respect to a Robinson-Schensted type correspondence where tableaux are replaced by ribbon quasi-tableaux [21]. The insertion algorithm, described in [21], associates with a word w a ribbon quasi-tableau $Q(w)$ and a standard ribbon tableau $R(w)$ of the same shape. The correspondence $w \to (Q, R)$ is a bijection, and we have the following quasi-ribbon analogue of Knuth's theorem (*cf.* [22]).

Proposition 2.2 [21] *Two words* $u, v \in \mathbb{A}^*$ *are mapped to the same quasi-tableau* Q *by the above insertion process iff* $u \equiv v$ *with respect to the hypoplactic congruence.*

We also recall the quasi-ribbon version of the cross-section theorem of [25].

Theorem 2.3 [21] *The classes of quasi-ribbon words form a linear basis of the hypoplactic algebra.*

Other combinatorial properties of the hypoplactic monoid have been investigated by Novelli [31].

3 The quantum enveloping algebra $\mathcal{U}_q(gl_N)$

In this section, we introduce the main object of our study, a variant of the quantized enveloping algebra $U_q(gl_n)$ which displays the same behaviour for generic q, but can also be specialized at $q=0$. This algebra is essentially Takeuchi's $U_{q,1}(gl_N)$ [34], with a few modifications. To allow the specialization $q=0$, we have to use non-invertible generators for the Cartan part, to introduce extra relations, and to define it only as a bialgebra (no antipode). This algebra, as well as other multiparameter deformations, can be realized by specialization of the difference operators of [12].

3.1 The bialgebra $\mathcal{U}_q(gl_N)$

Definition 3.1 *Let q be an indeterminate or a complex parameter. The* crystalizable quantum analogue $\mathcal{U}_q(gl_N)$ *of the universal enveloping algebra of the Lie algebra gl_N is the algebra over $\mathbb{C}(q)$ generated by the elements $(e_i)_{1\le i\le N-1}$, $(f_i)_{1\le i\le N-1}$ and $(k_i)_{1\le i\le N}$ with relations:*

$$k_i\,k_j \;=\; k_j\,k_i \qquad \text{for } 1\le i,j\le N\ , \tag{3}$$

$$\left\{\begin{array}{rcll} q\,k_i\,e_{i-1} &=& e_{i-1}\,k_i & \text{for } 2\le i\le N-1\ ,\\ k_i\,e_i &=& q\,e_i\,k_i & \text{for } 1\le i\le N-1\ ,\\ k_i\,e_j &=& e_j\,k_i & \text{for } j\ne i-1,i\ , \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{rcll} k_i\,f_{i-1} &=& q\,f_{i-1}\,k_i & \text{for } 2\le i\le N-1\ ,\\ q\,k_i\,f_i &=& f_i\,k_i & \text{for } 1\le i\le N-1\ ,\\ k_i\,f_j &=& f_j\,k_i & \text{for } j\ne i-1,i\ , \end{array}\right. \tag{5}$$

$$[e_i,f_j] \;=\; \delta_{ij}\,\frac{k_i-k_{i+1}}{q-1} \qquad \text{for } 1\le i,j\le N\ , \tag{6}$$

$$\left\{\begin{array}{rcll} q\,e_{i+1}\,e_i^2-(1+q)\,e_i\,e_{i+1}\,e_i+e_i^2\,e_{i+1} &=& 0 & \text{for } 1\le i\le N-2\ ,\\ q\,e_{i+1}^2\,e_i-(1+q)\,e_{i+1}\,e_i\,e_{i+1}+e_i\,e_{i+1}^2 &=& 0 & \text{for } 1\le i\le N-2\ , \end{array}\right. \tag{7}$$

$$\left\{\begin{array}{rcll} f_{i+1}\,f_i^2-(1+q)\,f_i\,f_{i+1}\,f_i+q\,f_i^2\,f_{i+1} &=& 0 & \text{for } 1\le i\le N-2\ ,\\ f_{i+1}^2\,f_i-(1+q)\,f_{i+1}\,f_i\,f_{i+1}+q\,f_i\,f_{i+1}^2 &=& 0 & \text{for } 1\le i\le N-2\ , \end{array}\right. \tag{8}$$

$$\left\{\begin{array}{rcll} [e_i,e_j] &=& 0 & \text{for } |i-j|>1\ ,\\ [f_i,f_j] &=& 0 & \text{for } |i-j|>1\ , \end{array}\right. \tag{9}$$

$$\left\{\begin{array}{rcll} e_i\,e_{i+1}\,k_{i+1} &=& k_{i+1}\,e_i\,e_{i+1} & \text{for } 1\le i\le N-2\ ,\\ f_{i+1}\,f_i\,k_{i+1} &=& k_{i+1}\,f_{i+1}\,f_i & \text{for } 1\le i\le N-2\ . \end{array}\right. \tag{10}$$

Relations (10) are consequences of the other relations when $q\ne 0$ and do not play any rôle in the generic case. One has to add them in order to endow $\mathcal{U}_q(gl_N)$ with a bialgebra structure at $q=0$. Note also that the q-Serre relations (7) are exactly those obtained by Ringel [32] in his construction of $U_q(\mathfrak{n}_+)$ via the Hall algebras associated to quivers. In this construction, the parameter q is the cardinality of a finite field.

The following property is easily checked by a direct calculation.

Proposition 3.2 *Let $q \in \mathbb{C}$. Then $\mathcal{U}_q(gl_N)$ is a $\mathbb{C}$-bialgebra for the comultiplication Δ and the counity ε defined by*

$$\left\{\begin{array}{lllllll} \Delta(e_i) & = & 1 \otimes e_i + e_i \otimes k_i, & \varepsilon(e_i) & = & 0 & \text{for } 1 \le i \le N-1 \ , \\ \Delta(f_i) & = & k_{i+1} \otimes f_i + f_i \otimes 1, & \varepsilon(f_i) & = & 0 & \text{for } 1 \le i \le N-1 \ , \\ \Delta(k_i) & = & k_i \otimes k_i, & \varepsilon(k_i) & = & 1 & \text{for } 1 \le i \le N \ . \end{array}\right.$$

3.2 The vector representation of $\mathcal{U}_q(gl_N)$

Let $(\xi_i)_{1\le i\le N}$ be the canonical basis of $V = \mathbb{C}^N$, and let E_{ij} be the endomorphism defined by $E_{ij}\xi_k = \delta_{jk}\xi_i$. One can define an algebra morphism ρ_V of $\mathcal{U}_q(gl_N)$ in $End_{\mathbb{C}}(V)$ by setting

$$\left\{\begin{array}{llll} \rho_V(e_i) & = & E_{i,i+1} & \text{for } 1 \le i \le N-1 \ , \\ \rho_V(f_i) & = & E_{i+1,i} & \text{for } 1 \le i \le N-1 \ , \\ \rho_V(k_i) & = & q\, E_{i,i} + \sum\limits_{j\neq i} E_{j,j} & \text{for } 1 \le i \le N \ . \end{array}\right.$$

The pair (ρ_V, V) is called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_q(gl_N)$. Since $\mathcal{U}_q(gl_N)$ is a bialgebra, one can define for every $n \ge 2$ its n-th tensor power $(\rho_{n,N}, V^{\otimes n})$ by $\rho_{n,N} = \rho_V^{\otimes n} \circ \Delta^{(n)} \ : \ \mathcal{U}_q(gl_N) \longrightarrow End_{\mathbb{C}}(V^{\otimes n})$, where $\Delta^{(n)}$ denotes the n-fold comultiplication $\mathcal{U}_q(gl_N) \to \mathcal{U}_q(gl_N)^{\otimes n}$.

4 The quantum matrix algebra $A_q(N)$

4.1 The bialgebra $A_q(N)$

We denote by $A_q(N)$ the $\mathbb{C}(q)$-algebra generated by the N^2 elements $(x_{ij})_{1\le i,j\le N}$ subject to the defining relations:

$$\left\{\begin{array}{rcll} x_{il}\, x_{jk} & = & q\, x_{jk}\, x_{il} & \text{for } 1 \le i \le j \le N,\ 1 \le k < l \le N \ , \\ x_{ik}\, x_{jk} & = & x_{jk}\, x_{ik} & \text{for } 1 \le i, j, k \le N \ , \\ x_{jl}\, x_{ik} - x_{ik}\, x_{jl} & = & (q-1)\, x_{jk}\, x_{il} & \text{for } 1 \le i < j \le N,\ 1 \le k < l \le N \ . \end{array}\right. \tag{11}$$

This algebra is a quantization of the bialgebra of polynomial functions on the variety of $N \times N$ matrices over $\mathbb{C}$. It is the specialization $GL_{q,1}(N)$ of Takeuchi's algebra [34]. Up to the symmetry $x_{ij} \leftrightarrow x_{ji}$, $A_q(N)$ is the quantization introduced by Dipper and Donkin in [5] and used in [21]. It is not isomorphic to the more familiar quantization of Faddeev-Reshetikhin-Takhtadzhyan [10]. For generic values of q both versions have the same representation theory, but an essential difference is that $A_q(N)$ is defined for $q = 0$.

One can endow $A_q(N)$ with a bialgebra structure whose comultiplication $\overline{\Delta}$ and counity η are defined by

$$\overline{\Delta}(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj} \qquad \text{and} \qquad \eta(x_{ij}) = \delta_{ij} \ .$$

4.2 The quantum diagonal algebra

An important subalgebra of $A_q(N)$ is the quantum diagonal algebra (or quantum torus) $\Delta_q(N)$. It is a q-analogue of the algebra of polynomial functions on a maximal torus of GL_N. As such, it occurs in the character theory of Hecke algebras, of $A_q(N)$ and of $\mathcal{U}_q(gl_N)$ (*cf.* Section 6.2 and [21]).

Definition 4.1 *The* quantum diagonal algebra *(or* quantum torus*)* $\Delta_q(N)$ *is the subalgebra of* $A_q(N)$ *generated by the diagonal elements* $x_{11}, \ldots, x_{NN}$.

In [21], we conjectured the following presentation of $\Delta_q(N)$ for $q \neq 0, 1$:

$$\left\{ \begin{array}{rcll} q\,a\,a\,b-(q+1)\,a\,b\,a+b\,a\,a & = & 0 & \text{for } a<b\,, \\ q\,a\,b\,b-(q+1)\,b\,a\,a+b\,b\,a & = & 0 & \text{for } a<b\,, \\ c\,a\,b-a\,c\,b-b\,c\,a+b\,a\,c & = & 0 & \text{for } a<b<c\,, \end{array} \right.$$

where $a, b, c \in A = \{x_{11} < x_{22} < \ldots < x_{NN}\}$. The algebra defined by these relations has been called the quantum pseudoplactic algebra.

For $q = 1$, the diagonal algebra $\Delta_1(N)$ is just an algebra of commutative polynomials. The case $q = 0$ is more interesting:

Theorem 4.2 [21] *The ring homomorphism defined by* $\varphi : a_i \longrightarrow x_{ii}$ *is an isomorphism between the hypoplactic algebra* $HPl(A)$ *and the quantum diagonal algebra* $\Delta_0(N)$ *at* $q = 0$.

4.3 Duality between $A_q(N)$ and $\mathcal{U}_q(gl_N)$

For any $q \in \mathbb{C}$, the bialgebras $A_q(N)$ and $\mathcal{U}_q(gl_N)$ are in duality, in a sense to be made precise below. According to the general theory of bialgebras (*cf.* [19]), the dual $\mathcal{U}_q(gl_N)^*$ of $\mathcal{U}_q(gl_N)$ has a canonical algebra structure, defined by the convolution product

$$\varphi \cdot \psi = \mu \circ (\varphi \otimes \psi) \circ \Delta$$

for $\varphi, \psi \in \mathcal{U}_q(gl_N)^*$, where μ is the multiplication map $\mu(g \otimes h) = g\,h$, $g, h \in \mathcal{U}_q(gl_N)$.

Define linear functionals $(a_{ij})_{1\le i,j\le N} \in \mathcal{U}_q(gl_N)^*$ as the matrix coefficients of the vector representation, *i.e.* $\rho_V(g) = (a_{ij}(g))_{1\le i,j\le N}$ for $g \in \mathcal{U}_q(gl_N)$. The following result is stated in [34].

Theorem 4.3 *The* N^2 *linear functionals* $(a_{ij})_{1\le i,j\le N}$ *of* $\mathcal{U}_q(gl_N)^*$ *satisfy the quantum relations (11).*

Proof – One has to prove that the relations

$$\left\{ \begin{array}{rcll} (a_{il}\,a_{jk})(g) & = & q\,(a_{jk}\,a_{il})(g) & \text{for } i \le j,\ k<l\,, \\ (a_{ik}\,a_{jk})(g) & = & (a_{jk}\,a_{ik})(g) & \text{for every } i,j,k\,, \\ (a_{jl}\,a_{ik})(g)-(a_{ik}\,a_{jl})(g) & = & (q-1)\,(a_{jk}\,a_{il})(g) & \text{for } i<j,\ k<l\,. \end{array} \right. \qquad (\mathcal{QR}(g))$$

hold for every $g \in \mathcal{U}_q(gl_N)$. For $g = 1$ or $g = a_{rs}$, this is easily checked by direct calculations. Now, one has

$$(\mathcal{QR}(g)) \ \wedge \ (\mathcal{QR}(h)) \implies (\mathcal{QR}(a\,g + b\,h))$$

for every $g, h \in \mathcal{U}_q(gl_N)$ and $a, b \in \mathbb{C}(q)$, so that it suffices to check that $(\mathcal{QR}(g))$ holds for g a product of generators of $\mathcal{U}_q(gl_N)$. This is done by induction on the degree of g by means of the following lemma.

Lemma 4.4 *Let $1 \leq i, j, k, l \leq N$. Then,*

$$\left\{ \begin{array}{lcl} (a_{ik}\,a_{jl})(g\,e_I) & = & (a_{ik}\,a_{j,l-1})(g)\,a_{l-1,l}(e_I) + (a_{i,k-1}\,a_{jl})(g)\,a_{ll}(k_I)\,a_{k-1,k}(e_I) \ , \\ (a_{ik}\,a_{jl})(g\,f_I) & = & (a_{ik}\,a_{j,l+1})(g)\,a_{l+1,l}(f_I)\,a_{kk}(k_{I+1}) + (a_{i,k+1}\,a_{jl})(g)\,a_{k+1,k}(f_I) \ , \\ (a_{ik}\,a_{jl})(g\,k_I) & = & (a_{ik}\,a_{jl})(g)\,a_{kk}(k_I)\,a_{ll}(k_I) \ , \end{array} \right. \tag{12}$$

for every $g \in \mathcal{U}_q(gl_N)$ and $I \in [1, N]$ (when it makes sense).

Proof of the lemma – The three equations (12) being obtained in the same way, we only show the derivation of the first one. We have $\rho_V(gh) = \rho_V(g)\,\rho_V(h)$ for every g, h in $\mathcal{U}_q(gl_N)$. From the definition of the a_{ij}, we see that

$$a_{ij}(gh) = \sum_{k=1}^{N} a_{ik}(g)\,a_{kj}(h) \tag{13}$$

for all $1 \leq i, j \leq N$ and $g, h \in \mathcal{U}_q(gl_N)$.

Let g be an element of $\mathcal{U}_q(gl_N)$ and let $I \in [1, N-1]$. We can then write

$$(a_{ik}\,a_{jl})(g\,e_I) = \mu \circ (a_{ik} \otimes a_{jl}) \circ \Delta(g\,e_I) = \mu \circ (a_{ik} \otimes a_{jl}) \circ (\Delta(g)\,\Delta(e_I)) \ .$$

Using Sweedler's notations [33], set

$$\Delta(g) = \sum_{(g)} g^{(1)} \otimes g^{(2)} \ .$$

From the definition of $\Delta(e_I)$, we have

$$(a_{ik}\,a_{jl})(g\,e_I) = \sum_{(g)} a_{ik}(g^{(1)})\,a_{jl}(g^{(2)}\,e_I) + \sum_{(g)} a_{ik}(g^{(1)}\,e_I)\,a_{jl}(g^{(2)}\,k_I) \ .$$

But (13) shows that

$$a_{pq}(g\,e_I) = \sum_{r=1}^{N} a_{pr}(g)\,a_{rq}(e_I) = a_{p,q-1}(g)\,a_{q-1,q}(e_I)$$

since $a_{rq}(e_I)$ is necessary zero when $r \neq q-1$. Likewise, we get

$$a_{pq}(g\,k_I) = \sum_{r=1}^{N} a_{pr}(g)\,a_{rq}(k_I) = a_{pq}(g)\,a_{qq}(k_I) \ .$$

These last two equalities imply that

$$\begin{aligned}
&(a_{ik}\,a_{jl})(g\,e_I)\\
&= \sum_{(1),(2)} a_{ik}(g^{(1)})\,a_{j,l-1}(g^{(2)})\,a_{l-1,l}(\,e_I) + \sum_{(1),(2)} a_{i,k-1}(g^{(1)})\,a_{k-1,k}(\,e_I)\,a_{jl}(g^{(2)})\,a_{ll}(\,k_I)\\
&= \Big(\sum_{(1),(2)} a_{ik}(g^{(1)})\,a_{j,l-1}(g^{(2)})\Big)\,a_{l-1,l}(\,e_I) + \Big(\sum_{(1),(2)} a_{i,k-1}(g^{(1)})\,a_{jl}(g^{(2)})\Big)\,a_{k-1,k}(\,e_I)\,a_{ll}(\,k_I)\\
&= (a_{ik}\,a_{j,l-1})(g)\,a_{l-1,l}(\,e_I) + (a_{i,k-1}\,a_{jl})(g)\,a_{k-1,k}(\,e_I)\,a_{ll}(\,k_I)\ ,
\end{aligned}$$

which is the desired identity. ■

Theorem 4.3 allows us to define a morphism of algebras Ψ from $A_q(N)$ in $\mathcal{U}_q(gl_N)^*$ by setting $\Psi(x_{ij}) = a_{ij}$ for $1 \le i, j \le N$. This morphism can then be used to define the duality of bialgebras between $A_q(N)$ and $\mathcal{U}_q(gl_N)$ by setting

$$\langle x, g\rangle = \Psi(x)(g)$$

for $x \in A_q(N)$ and $g \in \mathcal{U}_q(gl_N)$.

Note 4.5 The image by Ψ of the quantum determinant of $A_q(N)$ is the algebra morphism $\alpha : \mathcal{U}_q(gl_N) \to \mathbb{C}(q)$ such that $\alpha(e_i) = \alpha(f_i) = 0$ and $\alpha(k_i) = q$. It follows from this remark that Ψ is not injective for $q = 1$ and $q = 0$.

5 Quantum Schur-Weyl duality

5.1 Hecke algebras of type A

A fundamental property of the standard version of $U_q(gl_N)$ is the existence of a q-analogue of the Schur-Weyl duality involving the Hecke algebra instead of the symmetric group [16]. Such a duality can also be worked out for $\mathcal{U}_q(gl_N)$. The double commutant theorem will break down at $q = 0$, but one can still give a weaker statement, which will allow us to interpret quasi-symmetric functions and noncommutative symmetric functions as characters of $\mathcal{U}_0(gl_N)$.

The (Iwahori) Hecke algebra $H_n(q)$ of type A_{n-1} is the $\mathbb{C}(q)$-algebra generated by the $n-1$ elements $(T_i)_{1\le i\le n-1}$ with relations

$$\begin{cases} T_i^2 &= (q-1)\,T_i + q & \text{for } 1 \le i \le n-1\ ,\\ T_i\,T_{i+1}\,T_i &= T_{i+1}\,T_i\,T_{i+1} & \text{for } 1 \le i \le n-2\ ,\\ T_i\,T_j &= T_j\,T_i & \text{for } |i-j| > 1\ . \end{cases}$$

For generic complex values of q, the Hecke algebra $H_n(q)$ is isomorphic to $\mathbb{C}[\mathfrak{S}_n]$ (the standard presentation is obtained for $q = 1$) and hence semi-simple, except when $q = 0$ or when q is a root of unity of order $\le n$. The representation theory at $q = 0$ is described by noncommutative symmetric functions and quasi-symmetric functions [9, 7, 21], whilst the case of a primitive k-th root of unity is described by the canonical basis of the basic representation of the quantum affine algebra $U_q(\widehat{sl}_k)$ [23, 1].

5.2 Tensor representations of Hecke algebras

Following [5, 8], we introduce a variant of Jimbo's right action of $H_n(q)$ on $V^{\otimes n}$ [16], defined by

$$\left\{ \begin{array}{lcll} \mathbf{v}\cdot T_i & = & \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \ , \\ \mathbf{v}\cdot T_i & = & q\,\mathbf{v} & \text{if } k_i = k_{i+1} \ , \\ \mathbf{v}\cdot T_i & = & q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \ , \end{array} \right.$$

for $\mathbf{v} = \xi_{k_1}\otimes\cdots\otimes\xi_{k_n} \in V^{\otimes n}$ (where $\mathbf{v}^{\sigma_i}$ denotes the tensor obtained from $\mathbf{v}$ by permuting the i and $i+1$-th factors).

5.3 The commutant theorem

We denote by $\pi_{n,N}$ the representation of $H_n(q)$ in $End_{\mathbb{C}}(V^{\otimes n})$ defined by this action. One can check that $\rho_{n,N}$ commutes with $\pi_{n,N}$. For generic q, one has a stronger property:

Theorem 5.1 *Let q be a nonzero complex number which is not a non trivial k-th root of unity for some $k \le n$. The two subalgebras $\pi_{n,N}(H_n(q))$ and $\rho_{n,N}(\mathcal{U}_q(gl_N))$ of $End_{\mathbb{C}}(V^{\otimes n})$ are then commutant of each other.*

This result is well-known for the standard version of $U_q(gl_N)$ [16] and can be obtained by straightforward modifications of its standard proofs. However, since the details are not easily available in the literature, we provide below a complete proof, relying on some calculations of independent interest.

Let $A^!$ denote the commutant of a subalgebra A of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$. Under our assuptions on q, the Hecke algebra $H_n(q)$ is semi-simple (*cf.* [6]), and so is its image under $\pi_{n,N}$. Therefore,

$$\pi_{n,N}(H_n(q))^{!!} = \pi_{n,N}(H_n(q)) \ .$$

It follows that it suffices to prove that

$$\pi_{n,N}(H_n(q))^{!} = \rho_{n,N}(\mathcal{U}_q(gl_N))$$

and since the actions of $H_n(q)$ and $\mathcal{U}_q(gl_N)$ on $V^{\otimes n}$ commute, it suffices in fact to show that

$$\pi_{n,N}(H_n(q))^{!} \subseteq \rho_{n,N}(\mathcal{U}_q(gl_N)) \ . \qquad (14)$$

Let us now investigate the structure of $\pi_{n,N}(H_n(q))^!$. Consider an arbitrary endomorphism M of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$. In the sequel, we shall denote the images under M of the elements of the canonical basis of $V^{\otimes n}$ as follows

$$M(\xi^J) = \sum_{I\in[1,N]^n} m_{I,J}\,\xi^I \ ,$$

where we set $\xi^I = \xi_{i_1}\otimes\ldots\otimes\xi_{i_n}$ for $I = (i_1,\ldots,i_n) \in [1,N]^n$. Let I^{σ_k} be the vector obtained by permuting the k and the $k+1$-th entries of I. Then, M belongs to $\pi_{n,N}(H_n(q))^!$ if and only if

$$\left\{ \begin{array}{lcll} m_{I,J} & = & m_{I^{\sigma_k},J} & \text{if } j_k = j_{k+1} \ , \\ m_{I,J} & = & q\,m_{I^{\sigma_k},J^{\sigma_k}} & \text{if } j_k > j_{k+1} \text{ and } i_k \le i_{k+1} \ , \\ m_{I,J} & = & (q-1)\,m_{I,J^{\sigma_k}} + m_{I^{\sigma_k},J^{\sigma_k}} & \text{if } j_k > j_{k+1} \text{ and } i_k > i_{k+1} \ , \end{array} \right. \qquad (15)$$

for all $1 \leq k \leq n-1$ and $I, J \in [1,N]^n$.

Let $\mathcal{C}_{n,N}$ denote the set of all pairs $(I,J) \in [1,N]^n \times [1,N]^n$ such that

- $J = (\underbrace{1,\ldots,1}_{j_1 \text{ times}},\ldots,\underbrace{N,\ldots,N}_{j_N \text{ times}})$ is a non decreasing vector,
- I is increasing on all the intervals $[1,j_1]$, $[j_1+1, j_1+j_2]$, $\ldots$, $[j_1+\ldots+j_{N-1}+1, j_1+\ldots+j_N]$ on which J is constant.

Let (I,J) be such a pair of compositions. We can then write $I = (I_1, I_2, \ldots, I_N)$ where I_k denotes the (possibly empty) non decreasing composition obtained by restriction of I to the (possibly empty) interval $[j_1+\ldots+j_{k-1}+1, j_1+\ldots+j_k]$. If $J = (1,1,1,3,3,4,4)$ and $I = (1,2,2,1,1,1,3)$, we have for instance $I_1 = (1,2,2)$, $I_2 = ()$, $I_3 = (1,1)$ and $I_4 = (1,3)$. We also denote by

$$\{\, I_1 \,\} \{\, I_2 \,\} \ldots \{\, I_N \,\}$$

the sum of all tensors $\xi^{K_1} \otimes \ldots \otimes \xi^{K_N}$ where K_i is an arbitrary permutation of I_i. Continuing the previous example, we have for instance

$$\{122\}\,\{\,\}\,\{11\}\,\{13\} = (\xi^{122} + \xi^{212} + \xi^{221}) \otimes \xi^{11} \otimes (\xi^{13} + \xi^{31}) \ .$$

Let us now suppose that M belongs to $\pi_{n,N}(H_n(q))^!$. Arguing by induction on the number of inversions (*i.e.* the numbers of pairs $1 \leq i < j \leq n$ such that $l_i > l_j$) of the composition L, one deduces from (15) that there exists polynomials $p_{(K,L);(I,J)}(q) \in \mathbb{Z}[q]$ such that

$$m_{K,L} = \sum_{(I,J) \in \mathcal{C}_{n,N}} p_{(K,L);(I,J)}(q)\, m_{I,J} \ .$$

It follows that a basis of $\pi_{n,N}(H_n(q))^!$ is provided by its elements $M_{I,J}$ such that $m_{I,J} = 1$ for some $(I,J) \in \mathcal{C}_{n,N}$ and $m_{K,L} = 0$ for all others pairs (K,L) of $\mathcal{C}_{n,N}$. From (15), one sees that $M_{I,J}$ is the unique element of $\pi_{n,N}(H_n(q))^!$ such that

$$M_{I,J}(\xi^K) = \begin{cases} \{I_1\}\,\{I_2\}\,\ldots\,\{I_N\} & \text{when } K = J \ , \\ 0 & \text{when } K \neq J \ , \end{cases} \tag{16}$$

for every non decreasing $K \in [1,N]^n$. This already gives the dimension

$$\dim_{\mathbb{C}} \pi_{n,N}(H_n(q))^! = \begin{pmatrix} N^2+n-1 \\ n \end{pmatrix} \ .$$

Indeed, this dimension is just the number of elements of $\mathcal{C}_{n,N}$, and it is easy to see that the correspondence

$$(I,J) = ((i_1,\ldots,i_N),(j_1,\ldots,j_N)) \quad \longleftrightarrow \quad \Big((i_1,j_1),\ (i_2,j_2),\ \ldots\ ,\ (i_N,j_N) \Big)$$

maps bijectively an element of $\mathcal{C}_{n,N}$ to a non-decreasing sequence of length n in $[1,N]^2$ (for the lexicographic order).

Let us turn back to the proof of (14). It remains to show that all the $M_{I,J}$ belong to $\rho_{n,N}(\mathcal{U}_q(gl_N))$. For $I = (i_1, \ldots, i_n) \in [1, N]^n$, we denote by $T(I) = (p_1, \ldots, p_N)$ the vector of $\mathbb{N}^N$ defined by $p_k = \mathrm{Card}\,\{\, j \in [1, n],\ i_j = k \,\}$ Let $S(I)$ be the set formed by all the components i_k of I. With a vector $L \in \mathbb{N}^N$ such that $|L| = n$, we associate the endomorphism Id_L of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ defined by

$$Id_L(\xi^K) = \begin{cases} \xi^K & \text{if } T(K) = L \ , \\ 0 & \text{if } T(K) \neq L \ . \end{cases}$$

We associate in the same way to every non empty subset S of $[1, N]$ the endomorphism id_S of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ defined by setting

$$id_S(\xi^K) = \begin{cases} \xi^K & \text{if } S(K) = S \ , \\ 0 & \text{if } S(K) \neq S \ . \end{cases}$$

Let us now denote by $\mathfrak{H}(q)_{n,N}$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ generated by the image of the family $(k_i)_{1 \leq i \leq N}$ under $\rho_{n,N}$. The following lemma describes the structure of $\mathfrak{H}(q)_{n,N}$ in two importants cases.

Lemma 5.2 *Let $q \in \mathbb{C}$ be a complex number. Then,* (i) *if q is neither* 0*, nor a non trivial k-th root of unity for some $k \leq n$,*

$$\mathfrak{H}(q)_{n,N} = \bigoplus_{L \in \mathbb{N}^N,\ |L| = n} \mathbb{C}\, Id_L \quad .$$

(ii) *For $q = 0$,*

$$\mathfrak{H}(0)_{n,N} = \bigoplus_{\emptyset \neq S \subset [1,N]} \mathbb{C}\ id_S \quad .$$

It follows from Lemma 5.2 that Id_L belongs to $\rho_{n,N}(\mathcal{U}_q(gl_N))$ for every $L \in \mathbb{N}^N$ such that $|L| = n$. Let now (I, J) be a pair of compositions in $\mathcal{C}_{n,N}$ and let $f_{I,J}$ be an element of $\pi_{n,N}(H_n(q))^!$ mapping ξ^J to $\{I_1\} \ldots \{I_N\}$. Using the characteristic property (16), we see that

$$M_{I,J} = f_{I,J} \circ M_{T(J),T(J)} = f_{I,J} \circ Id_{T(J)} \ .$$

Hence, to prove that every $M_{I,J}$ belongs to $\rho_{n,N}(\mathcal{U}_q(gl_N))$, it suffices to show that there exists for every pair (I, J) in $\mathcal{C}_{n,N}$ an element $f_{I,J}$ of $\rho_{n,N}(\mathcal{U}_q(gl_N))$ that maps ξ^J to $\{I_1\} \ldots \{I_N\}$.

Let $|I|_i$ be the number of entries of I that are equal to i. Then, e_i and f_i act on $V^{\otimes n}$ by

$$\begin{cases} \rho_{n,N}(e_i)(\xi^{(i_1,\ldots,i_n)}) & = \displaystyle\sum_{i_k = i+1} q^{|(i_{k+1},\ldots,i_n)|_i}\, \xi^{(i_1,\ldots,i_{k-1},i,i_{k+1},\ldots,i_n)} \ , \\ \rho_{n,N}(f_i)(\xi^{(i_1,\ldots,i_n)}) & = \displaystyle\sum_{i_k = i} q^{|(i_1,\ldots,i_{k-1})|_{i+1}}\, \xi^{(i_1,\ldots,i_{k-1},i+1,i_{k+1},\ldots,i_n)} \ . \end{cases}$$

For $1 \leq i \leq N-1$, let δ_i and υ_i be the operators on $\mathbb{N}^N$ defined by

$$\delta_i : K = (k_1, \ldots, k_N) \longrightarrow K^{\delta_i} = \begin{cases} (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1} + 1, k_{i+2}, \ldots, k_N) & \text{if } k_i \geq 1 \ , \\ 0 & \text{if } k_i = 0 \ , \end{cases}$$

$$v_i : K=(k_1,\ldots,k_N) \longrightarrow K^{v_i} = \begin{cases} (k_1,\ldots,k_{i-1},k_i+1,k_{i+1}-1,k_{i+2},\ldots,k_N) & \text{if } k_{i+1}\geq 1\ , \\ 0 & \text{if } k_{i+1}=0\ , \end{cases}$$

(here 0 is a just a symbol such that $\{0\}=0$ when interpreted in $V^{\otimes n}$). We can now state:

Lemma 5.3 *Let $K=(k_1,\ldots,k_n)$ be a non decreasing composition of $[1,N]^n$. Then one has*

$$\begin{cases} \rho_{n,N}(e_i)(\{K\}) &= \ [1+k_i]_q\,\{K^{v_i}\}\ , \\ \rho_{n,N}(f_i)(\{K\}) &= \ [1+k_{i+1}]_q\,\{K^{\delta_i}\}\ , \end{cases}$$

for every $1\leq i\leq N-1$.

Using Lemma 5.3, we can now prove the following result.

Lemma 5.4 *Let $(I_j)_{1\leq j\leq N}$ be a family of non decreasing compositions. Then, for $1\leq i\leq N-1$,*

$$\begin{cases} \rho_{n,N}(e_i)(\{I_1\}\ldots\{I_N\}) &= \displaystyle\sum_{k=1}^{N}\,[1+|I_k|_i]_q\,q^{|I_{k+1}|_i+\ldots+|I_N|_i}\,\{I_1\}\,\ldots\,\{I_k^{v_i}\}\,\ldots\,\{I_N\}\ , \\ \rho_{n,N}(f_i)(\{I_1\}\ldots\{I_N\}) &= \displaystyle\sum_{k=1}^{N}\,[1+|I_k|_{i+1}]_q\,q^{|I_1|_{i+1}+\ldots+|I_{k-1}|_{i+1}}\,\{I_1\}\,\ldots\,\{I_k^{\delta_i}\}\,\ldots\,\{I_N\}\ . \end{cases}$$

Let now $K=(1^{k_1},\ldots,N^{k_N})\in[1,N]^n$ be non decreasing and let $i\in[1,N]$. Let $d_i(K)$ be the smallest r such that one can write

$$K=\rho_1\circ\rho_2\circ\ldots\circ\rho_r((\underbrace{i,\ldots,i}_{n\ \text{times}}))$$

where each ρ_j is either δ_k or v_k for some $k\in[1,N-1]$. This number is given by

$$d_i(K)=\sum_{j=1}^{i-1}\,(i-j)\,k_j\ +\ \sum_{j=i+1}^{N}\,(j-i)\,k_j\ .$$

Using this formula, one sees that

$$d_i(K^{\delta_k})=\begin{cases} d_i(K)+1 & \text{if } k\geq i\ , \\ d_i(K)-1 & \text{if } k\leq i-1\ , \end{cases} \qquad d_i(K^{v_k})=\begin{cases} d_i(K)-1 & \text{if } k\geq i\ , \\ d_i(K)+1 & \text{if } k\leq i-1\ . \end{cases} \tag{17}$$

For $I=(I_i)_{1\leq i\leq N}$ a sequence of N non decreasing compositions, set

$$d(I)=(d_1(I_1),d_2(I_2),\ldots,d_N(I_N))\in\mathbb{N}^N\ .$$

Let $L=(l_1,\ldots,l_N)\in\mathbb{N}^n$ be such that $|L|=n$. We denote by $C(L)$ the unique vector of $[1,N]^n$ such that $T(C(L))=L$. It is given by

$$C(L)=(\underbrace{1,\ldots,1}_{l_1\ \text{times}},\ldots,\underbrace{N,\ldots,N}_{l_N\ \text{times}})\ .$$

Let also $\mathcal{D}_L$ be the subset of $\mathbb{N}^N$ defined by

$$\mathcal{D}_L = \{ \, d(I), \ (I, C(L)) \in \mathcal{C}_{n,N} \, \} \, .$$

It is easy to see that there exist non negative integers D_i such that $\mathcal{D}_L = [0, D_1] \times \ldots \times [0, D_N]$. Let $U^+_{n,N}(q)$ (resp. $U^-_{n,N}(q)$ be the subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ generated by the $\rho_{n,N}(e_i)$ (resp. $\rho_{n,N}(f_i)$. We are now in a position to prove by induction on $d(I)$ (with respect to the total ordering of $\mathbb{N}^N$ given by the lexicographic order) that there exists an endomorphism $f_{I,L} \in U^{\pm}_{n,N}(q)$ such that

$$f_{I,L}(\{1^{l_1}\} \, \ldots \, \{N^{l_N}\}) = \{I_1\} \, \ldots \, \{I_N\}$$

for every I such that $(I, C(L)) \in \mathcal{C}_{n,N}$. If $d(I) = (0, \ldots, 0)$, we have $I = C(L)$ and $f_{I,L} = Id = \rho_{n,N}(1)$ is the solution to our problem. Let now $d = (d_1, \ldots, d_N) \in \mathcal{D}_L$ and let us suppose that the desired property holds for all I such that $(I, C(L)) \in \mathcal{C}_{n,N}$ and $d(I) < d$. Let d' denote the successor of d in $\mathcal{D}_L$ for the lexicographic order on $\mathbb{N}^N$ and let I' be a composition such that $(I', C(L)) \in \mathcal{C}_{n,N}$ and $d(I') = d'$. We must now prove that our result also holds for I'. Two cases are to be considered according to the form of d':

(1) if $d_N < D_N$, we have $d' = (d_1, \ldots, d_{N-1}, d_N + 1)$. Let us consider the vector $I'' = (I'_1, \ldots, I'_{N-1}, {I'_N}^{\delta_{N-1}})$. According to Lemma 5.4 and to our hypothesis, there exists a family $(a_k)_{1 \leq k \leq N}$ of non zero complex numbers such that

$$\rho_{n,N}(e_{N-1})(\{I''_1\} \, \ldots \, \{I''_N\}) = \sum_{k=1}^{N} a_k \, \{I''_1\} \, \ldots \, \{{I''_k}^{\upsilon_{N-1}}\} \, \ldots \, \{I''_N\} \, ,$$

from which we deduce that $\{I'1\} \, \ldots \, \{I'_N\} = \{I''_1\} \, \ldots \, \{I''_{N-1}\} \, \{{I''_N}^{\upsilon_{N-1}}\}$ is equal to

$$\frac{1}{a_N} \, \rho_{n,N}(e_{N-1})(\{I''_1\} \, \ldots \, \{I''_N\}) - \sum_{k=1}^{N-1} \frac{a_k}{a_N} \, \{I''_1\} \, \ldots \, \{{I''_k}^{\upsilon_{N-1}}\} \, \ldots \, \{I''_N\} \, .$$

Using relations (17), one sees that $d(J) < d$ for all the compositions J involved in the terms $\{J_1\} \, \ldots \, \{J_N\}$ of the last expression. It is now easy to see that the desired property holds for I' by applying the induction hypothesis to all these compositions and using the last identity.

(2) if $d_N = D_N$, we can write $d = (d_1, \ldots, d_i, D_{i+1}, \ldots, D_N)$ with $d_i < D_i$ and $i \in [1, N{-}1]$. Hence $d' = (d_1, \ldots, d_i{+}1, 0, \ldots, 0)$ and $I' = (I'_1, \ldots, I'_i, (i{+}1)^{l_{i+1}}, \ldots, N^{l_N})$. Let us then set $I'' = (I'_1, \ldots, {I'_i}^{\upsilon_i}, (i{+}1)^{l_{i+1}}, \ldots, N^{l_N})$. According to Lemma 5.4 and to our hypothesis, there exists a family $(a_k)_{1 \leq k \leq i}$ of non zero complex numbers such that

$$\rho_{n,N}(f_i)(\{I''_1\} \, \ldots \, \{I''_N\}) = \sum_{k=1}^{i} a_k \, \{I''_1\} \, \ldots \, \{{I''_k}^{\delta_i}\} \, \ldots \, \{I''_N\} \, .$$

It follows that $\{I'1\} \, \ldots \, \{I'_N\} = \{I''_1\} \, \ldots \, \{{I''_i}^{\delta_i}\} \, \ldots \, \{I''_N\}$ is equal to

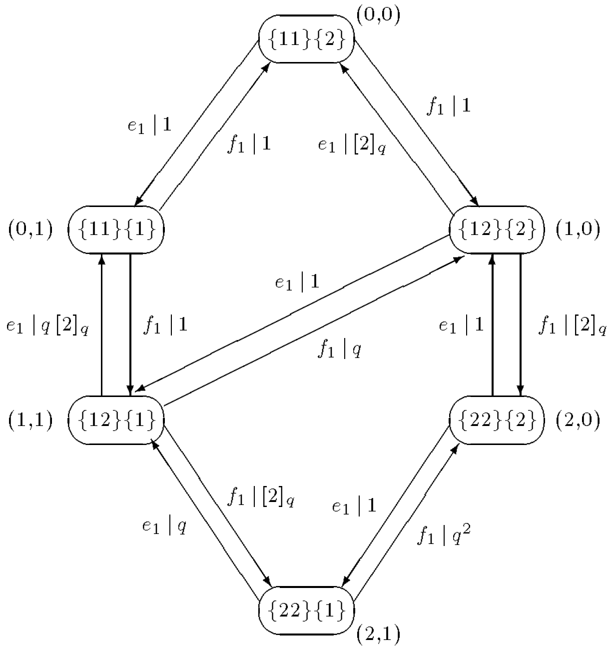
$$\frac{1}{a_i} \, \rho_{n,N}(f_i)(\{I''_1\} \, \ldots \, \{I''_N\}) - \sum_{k=1}^{i-1} \frac{a_k}{a_i} \, \{I''_1\} \, \ldots \, \{{I''_k}^{\delta_i}\} \, \ldots \, \{I''_N\} \, .$$

Using (17), one can check that $d(J) < d$ for all the compositions J involved in the terms $\{J_1\} \ldots \{J_N\}$ of this last expression. We can now conclude that the desired property holds for I' by using the same argument as in the previous case. This ends therefore the proof. ∎

Note 5.5 The proof of Theorem 5.1 can be divided into two parts: the first one is based on the introduction of the Cartan subalgebra of $\mathcal{U}_q(gl_N)$ that allows to separate the multihomogeneous components of $V^{\otimes n}$; the second one consists in the exploration of these components using the e_i and the f_i operators. It is also interesting to notice that the second part of the proof of Theorem 5.1 uses implicitly a certain $\mathcal{U}_q(gl_N)$-module that we shall explicit. For every $L = (l_1, \ldots, l_N) \in \mathbb{N}^N$ such that $|L| = n$, denote by $S(V^{\otimes n})_L$ the subspace of $V^{\otimes n}$ spanned by the tensors $\{I_1\} \ldots \{I_N\}$ where I runs over all elements of $[1, N]^n$ such that $(I, C(L)) \in \mathcal{C}_{n,N}$. Lemma 5.4 shows that this subspace is a $\mathcal{U}_q(gl_N)$-module. The second part of the proof shows that this module is cyclic and generated by $\xi^{C(L)}$, i.e. that

$$S(V^{\otimes n})_L = \mathcal{U}_q(gl_N)\, \xi^{C(L)} = \mathcal{U}_q(gl_N)\, \{1^{l_1}\} \ldots \{N^{l_N}\} .$$

We describe below the structure of $S(V^{\otimes 3})_{(2,1)} = \mathcal{U}_q(gl_2)\, \xi^{112}$ (we indicate the value of d for each basis element of this module).



The $U_q(gl_2)$-module $U_q(gl_2)\, \xi^{112}$.

This module is here represented as a $\mathbb{C}$-automaton (see [2]). We did not display the action of the k_i which is diagonal. In terms of automata theory, the last part of the proof amounts to show that the automata obtained in this way are minimal.

Note 5.6 Theorem 5.1 is not true when q is a non trivial n-th root of unity for some $k \leq n$. The smallest counterexample is found for $n = 2$, $N = 2$ and $q = -1$. One can check that the dimension of $\rho_{2,2}(\mathcal{U}_{-1}(gl_2))$ is equal to 7, whilst the dimension of the commutant of $\pi_{2,2}(H_2(-1))$ is equal to 10.

Note 5.7 When $q = 0$, the proof of Theorem 5.1 shows that the commutant of $\pi_{2,N}(H_2(0))$ is always equal to $\rho_{2,N}(\mathcal{U}_0(gl_N))$. This is not true anymore for $n \geq 3$. The smallest counterexample is found for $n = 3$ and $N = 3$. One can check that

$$\begin{cases} \dim_{\mathbb{C}}\ \rho_{2,3}(\mathcal{U}_0(gl_3)) = 19 & < \quad \dim_{\mathbb{C}} \pi_{2,3}(H_3(0))^! = 20\ , \\ \dim_{\mathbb{C}}\ \pi_{2,3}(H_3(0)) = 5 & < \quad \dim_{\mathbb{C}}\ \rho_{2,3}(\mathcal{U}_0(gl_3))^! = 7\ . \end{cases}$$

■

6 Polynomial $\mathcal{U}_q(gl_N)$-modules and their characters

6.1 Irreducible polynomial modules in the generic case

Definition 6.1 *A* polynomial $\mathcal{U}_q(gl_N)$-module *of degree n is a submodule of $V^{\otimes n}$.*

In this section we assume that q is a complex number which is neither zero nor a non trivial k-th root of unity for some $k \leq n$. For a partition λ of n, let $\mathcal{Y}_\lambda(q) \in H_n(q)$ be a q-Young symmetrizer [14] (here we use the version introduced in [8]). Let us recall briefly this construction. The Young subgroup associated with λ is the subgroup of $\mathfrak{S}_n$ denoted by $\mathfrak{S}_\lambda$ and defined by

$$\mathfrak{S}_\lambda = \mathfrak{S}_{[1,\lambda_1]} \times \mathfrak{S}_{[\lambda_1+1,\lambda_1+\lambda_2]} \times \ldots \times \mathfrak{S}_{[\lambda_1+\ldots+\lambda_{r-1}+1,\lambda_1+\ldots+\lambda_r]}\ .$$

We denote by ω_λ the maximal permutation of $\mathfrak{S}_\lambda$. One has for instance $\omega_{(1,3,4)} = 14328765$. The q-symmetrizer $\Box_\lambda$ and the q-antisymmetrizer ∇_λ associated in this way with λ are the elements of $H_n(q)$ defined by

$$\Box_\lambda = \sum_{\sigma \in \mathfrak{S}_\lambda} T_\sigma \qquad \text{and} \qquad \nabla_\lambda = \sum_{\sigma \in \mathfrak{S}_\lambda} (-q)^{\ell(\omega_\lambda)-\ell(\sigma)}\, T_\sigma\ .$$

Let t^c_λ be the Young tableau of shape λ obtained by filling all columns from bottom to top and from left to right with the numbers from 1 to n. For example, the standard Young tableau associated with $\lambda = (332)$ is

$$\begin{array}{|c|c|c} \hline 3 & 6 & \multicolumn{1}{c}{} \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 4 & \multicolumn{1}{c|}{7} \\ \hline \end{array}$$

t^c_{322}

Let $\mu(\lambda)$ be the unique permutation of $\mathfrak{S}_n$ obtained by reading from left to right and from bottom to top the rows of t^c_λ. For example, for instance $\mu(322) = 1472536$. The q-Young symmetrizer $\mathcal{Y}_\lambda(q)$ is defined by

$$\mathcal{Y}_\lambda(q) = \Box_\lambda\, T_{\mu(\lambda)}\, \nabla_{\lambda^\sim}\, T_{\mu(\lambda)^{-1}}$$

where $\lambda^\sim$ is the conjugate partition of λ. The element $\mathcal{Y}_\lambda(q)$ is an idempotent (up to a non zero element of $\mathbb{C}(q)$) of $H_n(q)$.

The family $(H_n(q)\,\mathcal{Y}_\lambda(q))_{\lambda\vdash n}$ is a complete system of $H_n(q)$-irreducible modules. Since $H_n(q)$ is semi-simple, it follows from Theorem 5.1 that the family of $\mathcal{U}_q(gl_N)$-modules

$$\mathbb{S}_\lambda(V;q) = V^{\otimes n}\cdot \mathcal{Y}_\lambda(q) \ ,$$

where λ runs over all partitions of n, is a complete system of irreducible polynomial $\mathcal{U}_q(gl_N)$-modules of degree n.

6.2 Characters of polynomial $\mathcal{U}_q(gl_N)$-modules

In the classical case, the character of a polynomial representation of GL_N can be regarded as a polynomial function on the group GL_N. In the quantum case, it is natural to define the character as an element of the deformed function algebra $A_q(N)$.

Let M be a polynomial $\mathcal{U}_q(gl_N)$-module of degree n and let $(m_i)_{1\le i\le m}$ be a $\mathbb{C}(q)$-linear basis of M. Since M is a $\mathcal{U}_q(gl_N)$-module, there exists a family $(\mu_{ij})_{1\le i,j\le m}$ of linear functionals of $\mathcal{U}_q(gl_N)^*$ such that for $g\in\mathcal{U}_q(gl_N)$

$$g\cdot m_i = \sum_{j=1}^{m} \mu_{ij}(g)\, m_j \ .$$

One can check that the trace of the action of $\mathcal{U}_q(gl_N)$ on M, *i.e.* the element

$$c(M) = \sum_{i=1}^{m} \mu_{ii} \in \mathcal{U}_q(gl_N)^*$$

is independent on the choice of the basis. Moreover $c(M)$ belongs to the subalgebra generated by the $(a_{ij})_{1\le i,j\le N}$. Indeed, for $I\in[1,N]^n$, one has

$$g\cdot\xi^I = \sum_{J\in[1,N]^n} a_{JI}(g)\,\xi^J$$

where $a_{JI} = a_{j_1 i_1}\,\ldots\,a_{j_n i_n}$. It follows that there exists elements $p_{IJ}(q)$ of $\mathbb{C}(q)$ such that

$$c(M) = \sum_{I,J\in[1,N]^n} p_{IJ}(q)\, a_{IJ} \ .$$

Since the map Ψ (defined at the end of Section 4) is injective on the linear subspace of $A_q(N)$ spanned by homogeneous elements of fixed length, we can lift this formula to $A_q(N)$ (according to Theorem 4.3). Let now x_{IJ} stand for the monomial $x_{i_1 j_1}\,\ldots\,x_{i_n j_n}$ in $A_q(N)$. The element

$$\chi(M) = \sum_{I,J\in[1,N]^n} p_{IJ}(q)\, x_{IJ}$$

of $A_q(N)$ will be called the *character* of the polynomial $\mathcal{U}_q(gl_N)$-module M. By duality, M is also an $A_q(N)$ comodule, and $\chi(M)$ is equal to its character in the sense of [21]. Therefore, we have:

Theorem 6.2 *For $q \neq 1$, $\chi(M)$ belongs to the diagonal algebra $\Delta_q(n)$.*

One can show that the commutative character of $\mathbb{S}_\lambda(V;q)$ is the Schur function s_λ. Hence the characters in our sense are quantum analogues of Schur functions. One can check that

$$\chi(\mathbb{S}_n(V;q)) = \sum_{i_1+\ldots+i_N=n} x_{11}^{i_1} \ldots x_{NN}^{i_N} ,$$

$$\chi(\mathbb{S}_{1^n}(V;q)) = \frac{1}{(1-q)^{n-1}} \sum_{1\le i_1<\ldots<i_n\le N} [x_{i_ni_n}, [\ldots, [x_{i_2i_2}, x_{i_1i_1}]_q \ldots]_q$$

for all n. These quantum Schur functions form a linear basis of a commutative subalgebra of $\Delta_q(N)$ which is isomorphic to the algebra of ordinary symmetric functions [21].

7 Polynomial $\mathcal{U}_0(gl_N)$-modules

From now on, we set $q = 0$. It follows from Theorem 6.2 and from Theorem 4.2 that the character of a polynomial $\mathcal{U}_0(gl_N)$-module is an element of the hypoplactic algebra. However, $\rho_{n,N}(\mathcal{U}_0(gl_N))$ is not semisimple, the modules $\mathbb{S}_\lambda(V;0)$ of the preceding section are in general no longer irreducible, and we have to look for the simple composition factors of $V^{\otimes n}$ (the irreducible polynomial modules) and its indecomposable direct summands. These modules can be constructed in the same way as the irreducible modules and the principal indecomposable projective modules of the 0-Hecke algebra.

For $i \in [1, n-1]$, let $\Box_i = 1 + T_i$ considered here as an element of $H_n(0)$. These elements satisfy the braid relations, together with $\Box_i^2 = \Box_i$ so that one can define, for $\sigma \in \mathfrak{S}_n$, the element $\Box_\sigma = \Box_{i_1} \ldots \Box_{i_r}$ where $\sigma_{i_1} \ldots \sigma_{i_r}$ is any reduced decomposition of σ.

Let us now see how to construct a complete family of irreducible polynomial $\mathcal{U}_0(gl_N)$-modules. For I a composition of n one defines the element

$$\eta_I = T_{\omega(\overline{I})} \, \Box_{\alpha(I^\sim)}$$

of $H_n(0)$. Norton [30] showed that the $H_n(0)\,\eta_I$ form a complete family of irreducible $H_n(0)$-modules (see also [3, 21]). As the action of $\mathcal{U}_0(gl_N)$ on $V^{\otimes n}$ commutes with the right action of $H_n(0)$, one can use η_I to construct the $\mathcal{U}_0(gl_N)$-module

$$\mathbf{D}_I = V^{\otimes n} \cdot \eta_I \ .$$

Proposition 7.1 $\mathbf{D}_I$ *is an irreducible $\mathcal{U}_0(gl_N)$-module. Its character is equal to the sum $F_I(x_{11}, \ldots, x_{NN})$ of all quasi-ribbon words over $\{ x_{11} < \ldots < x_{NN} \}$.*

Proof – For $w = w_1 \cdots w_n$ a word on the alphabet $\{1, \ldots, N\}$, denote by $\mathbf{w}$ the tensor $\xi_{w_1} \otimes \cdots \otimes \xi_{w_n}$. Let $QR(I)$ be the set of quasi-ribbon words of shape I over the same alphabet. It has been shown in [21] that the $\mathbf{d}_w = \mathbf{w}\eta_I$ for $w \in QR(I)$ form a basis of $\mathbf{D}_I$.

Now, let $\tilde{e}_i$, $\tilde{f}_i$ be Kashiwara's crystal graph operators [17], acting on words considered as vertices of the crystal graph of the n-th tensor power of the vector representation of the standard $U_q(gl_N)$ (in this case these are the same as the Lascoux-Schützenberger operators [26]). We recall their definition. To apply $\tilde{e}_i$ of $\tilde{f}_i$ to a word w, consider the subword $w^{(i)}$ obtained by restricting w to the alphabet $\{i, i+1\}$. Then erase in $w^{(i)}$ each factor $(i+1)\,i$), and repeat this procedure with the remaining word until a word $w' = i^r(i+1)^s$ is left, keeping track of the positions of the remaining letters in the original word w. Then $\tilde{e}_i$ changes the leftmost $(i+1)$ of w' into i and $\tilde{f}_i$ changes the rightmost i of w' into $i+1$. If the operation is not possible the result is zero.

We claim that

$$f_i \mathbf{d}_w = \begin{cases} \mathbf{d}_{\tilde{f}_i(w)} & \text{if } \tilde{f}_i(w) \in QR(I), \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

and similarly

$$e_i \mathbf{d}_w = \begin{cases} \mathbf{d}_{\tilde{e}_i(w)} & \text{if } \tilde{e}_i(w) \in QR(I), \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Indeed, the action of e_i or f_i on a $\mathbf{d}_w$ can only generate terms $\mathbf{d}_{w'}$ with $w' \in QR(I)$. This follows from the fact that for $w \in QR(I)$, $\mathbf{d}_w = \pm\mathbf{n}(\mathbf{w})\eta_I$, where $n(w)$ is the non-decreasing reordering of w [21]. Suppose now that w' is obtained from $n(w)$ by changing some occurence of i into $i+1$. This operation cannot create a descent at a position $j \notin \mathrm{Des}\,(I)$, since in this case one would have $\mathbf{w}'\eta_I = 0$. Also, it cannot suppress a strict rise at $j \in \mathrm{Des}\,(I)$, since it could in this case only be transformed into an equality $w'_j = w_{j+1}$, and again, this implies $\mathbf{w}'\eta_I = 0$. So the only terms which survive are of the form $n(v)$ for $v \in QR(I)$. The reasoning is similar with e_i.

Now, if $w \in QR(I)$, the only term in

$$f_i(\mathbf{w}) = \sum_{j=1}^{n} k_{i+1}^{\otimes j-1} \otimes f_i \otimes 1^{\otimes n-j}(\mathbf{w})$$

which can be a quasi-ribbon of shape I is the tensor $\mathbf{w}'$ obtained by replacing the rightmost occurence of ξ_i by ξ_{i+1}. Due to the factors k_{i+1}, this term is nonzero iff there is no previous occurence of ξ_{i+1}, in which case $w' \in QR(I)$. Then, one also has $w' = \tilde{f}_i(w)$, which proves (18). The proof of (19) is similar.

Let $\Gamma_N(I)$ be the directed graph having as vertices the quasi-ribbon words of shape I over $\{1, \ldots, N\}$, and with arrows $w \xrightarrow{f_i} w'$ and $w \xleftarrow{e_i} w'$ when $\mathbf{d}_{w'} = f_i \mathbf{d}_w$ (which is equivalent to $\mathbf{d}_w = e_i \mathbf{d}_{w'}$). This graph will be called the *quasi-crystal graph* of $\mathbf{D}_I$ (it is actually a subgraph of a crystal graph, corresponding to a quasi-symmetric character). It should be noted that although crystal graphs describe in general only the combinatorial skeleton of a generic module, the quasi-crystal graph $\Gamma_N(I)$ encode the full structure of the $\mathcal{U}_0(gl_N)$-module $\mathbf{D}_I$ (see Figure 1).

Now, it is clear from the above discussion that $\Gamma_N(I)$ is strongly connected, which proves the irreducibility of $\mathbf{D}_I$. Finally, from the definition of $\chi(\mathbf{D}_I)$ and (18), (19), we obtain

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} x_{ww} = F_I(x_{11}, \ldots, x_{NN}) \ ,$$

as required. ∎

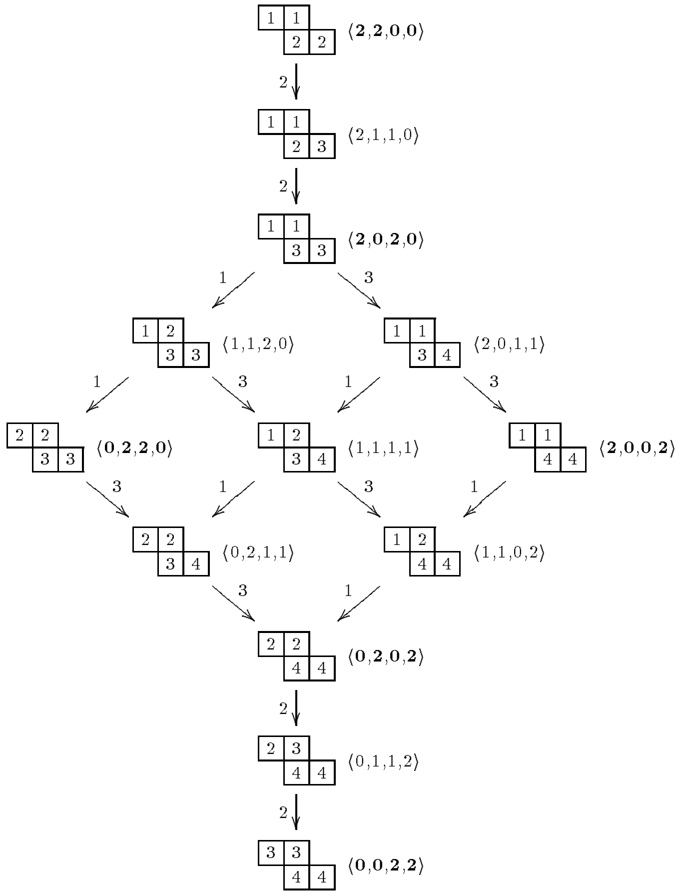


Figure 1: Quasi-crystal graph of $\mathbf{D}_{22}$ for $\mathcal{U}_0(gl_4)$ (only the arrows f_i have been represented). The symbols $\langle i_1, i_2, i_3, i_4\rangle$ are the weights. The extremal weights (corresponding to generators of Demazure modules) appear in boldtype.

The character of $\mathbf{D}_I$ is therefore a hypoplactic analogue of the quasisymmetric function F_I. It has been recently shown by Hivert [15] that the ordinary characters $F_I(x_1, \ldots, x_n)$ can be obtained by a version of the Weyl character formula, refinable into a Demazure type formula.

This formalism also provides quasi-symmetric and noncommutative analogues of Hall-Littlewood functions.

To construct the indecomposable summand of $V^{\otimes n}$ labelled by the composition I of n, we consider the element

$$\nu_I = T_{\alpha(I)} \square_{\alpha(\overline{I}^{\sim})} \ .$$

These elements generate a complete family $(H_n(0)\,\nu_I)$ of indecomposable projective left $H_n(0)$-modules [3, 30]. We can construct the $\mathcal{U}_0(gl_N)$-module

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I \ .$$

Proposition 7.2 $\mathbf{N}_I$ *is an indecomposable* $\mathcal{U}_0(gl_N)$*-module. Its character is the sum* $R_I(x_{11},\ldots,x_{NN})$ *of all words of ribbon shape* I *over* $\{x_{11}<\ldots<x_{NN}\}$*. Moreover,* $\mathbf{N}_I$ *has a canonical filtration, whose levels are described by the quantum quasi-symmetric functions* $\hat{R}_I$ *of* [35].

Proof – As shown in [21], a linear basis of $\mathbf{N}_I$ is given by $\mathbf{n}_w = \mathbf{w}\square_{\alpha(\overline{I}^\sim)}$ for $w \in R(I)$. The action of the Chevalley generators e_i and f_i on $\mathbf{n}_w$ can only generate words w' such that $\mathrm{std}\,(w') \geq \mathrm{std}\,(w)$ for the weak order. This is because e_i can change an occurence of $i+1$ into i iff there is no i on its right, and similarly, f_i can change an occurence of i into $i+1$ iff there is no $i+1$ on its left. Moreover, reasoning as in the proof of Proposition 7.1, one can see that the action of e_i or f_i on $\mathbf{n}_w$ ($w \in R(I)$) can only generate terms of the same form $\mathbf{n}_{w'}$, $w' \in R(I)$. Indeed, the replacement of an occurence of i by $i+1$ can only create or suppress a descent by transforming it into an equality, in which case the resulting term is zero [21].

We know that the descent class

$$D_I = \{\sigma \in \mathfrak{S}_n | C(\sigma) = I\} = [\alpha(I)\,,\,\omega(I)]$$

is an interval for the weak order. Therefore, the maximal ribbon of shape I, *i.e.* the unique word v_I of shape I and weight $0^{N-\ell(I)}\overline{I}$, which has $\omega(I)$ as standardized, generates a submodule $\mathbf{N}'$ of $\mathbf{N}_I$. Moreover, the basis of this submodule obtained by applying monomials in the e_i's to $\mathbf{n}_{v_I}$ is constituted of those $\mathbf{n}_w$, for $w \in R(I)$ such that $\mathrm{std}\,(w) = \omega(I)$, which are in natural bijection with the quasi-ribbons of shape $\overline{I}$. Thus, $\mathbf{N}' \simeq \mathbf{D}_{\overline{I}}$.

It is easy to see that for any ribbon w of shape I, there is a monomial m in the generators e_i, f_i such that $m\mathbf{n}_w$ is a nonzero multiple of $\mathbf{n}_{v_I}$. Therfore, any submodule $\mathbf{M}$ of $\mathbf{N}_I$ contains $\mathbf{N}'$, so that $\mathbf{N}_I$ is indecomposable.

This also shows that as an $\mathcal{U}_0(gl_N)$-module, $\mathbf{N}_I$ is generated by the minimal ribbon of shape I, *i.e.* the unique word u_I of shape and weight I. For example, $u_{3121} = 1132143$. The standardized of this word is $\mathrm{std}\,(u_I) = \alpha(I)$.

The above description yields a canonical composition series for $\mathbf{N}_I$, whose simple factors M_σ are labelled by elements of the descent class $[\alpha(I), \omega(I)]$, and such that the basis of each M_σ is labelled by words $w \in R(I)$ such that $\mathrm{std}\,(w) = \sigma$. Clearly, $M_{\omega(I)} = \mathbf{N}'$ is the socle of $\mathbf{N}_I$. Now, the socle of $\mathbf{N}_I/M_{\omega(I)}$ is spanned by the classes of the $\mathbf{n}_w$ such that $\mathrm{std}\,(w)$ covers $\omega(I)$ in the pemutohedron, and the classes of the $\mathbf{n}_w$ such that $\mathrm{std}\,(w) = \sigma$ form the basis of a simple submodule isomorphic to $\mathbf{D}_{C(\sigma)}$. Iterating this procedure, we obtain the required composition series.

To see this, it is sufficient to observe that if $\mathbf{n}_{w'} = f_i(\mathbf{n}_w)$ or $e_i(\mathbf{n}_w)$, the only case where $\mathrm{std}\,(w') = \mathrm{std}\,(w)$ arises when f_i acts on the rightmost i, or when e_i acts on the leftmost $i+1$. Therefore, in $f_i\mathbf{n}_w$ or $e_i\mathbf{n}_w$, there is at most one term $\mathbf{n}_{w'}$ such that $\mathrm{std}\,(w') = \mathrm{std}\,(w)$.

Taking into account the inversion numbers of the $\mathrm{std}\,(w)$ labelling the composition factors of $\mathbf{N}_I$, one can define q-analogues of the quasi-symmetric expansions of the commutative images of the ribbons R_I. These are the same as the ones defined in [35] by means of quantum quasi-symmetric functions, which correspond to canonical filtrations of the indecomposable projective $H_n(0)$-modules. ∎

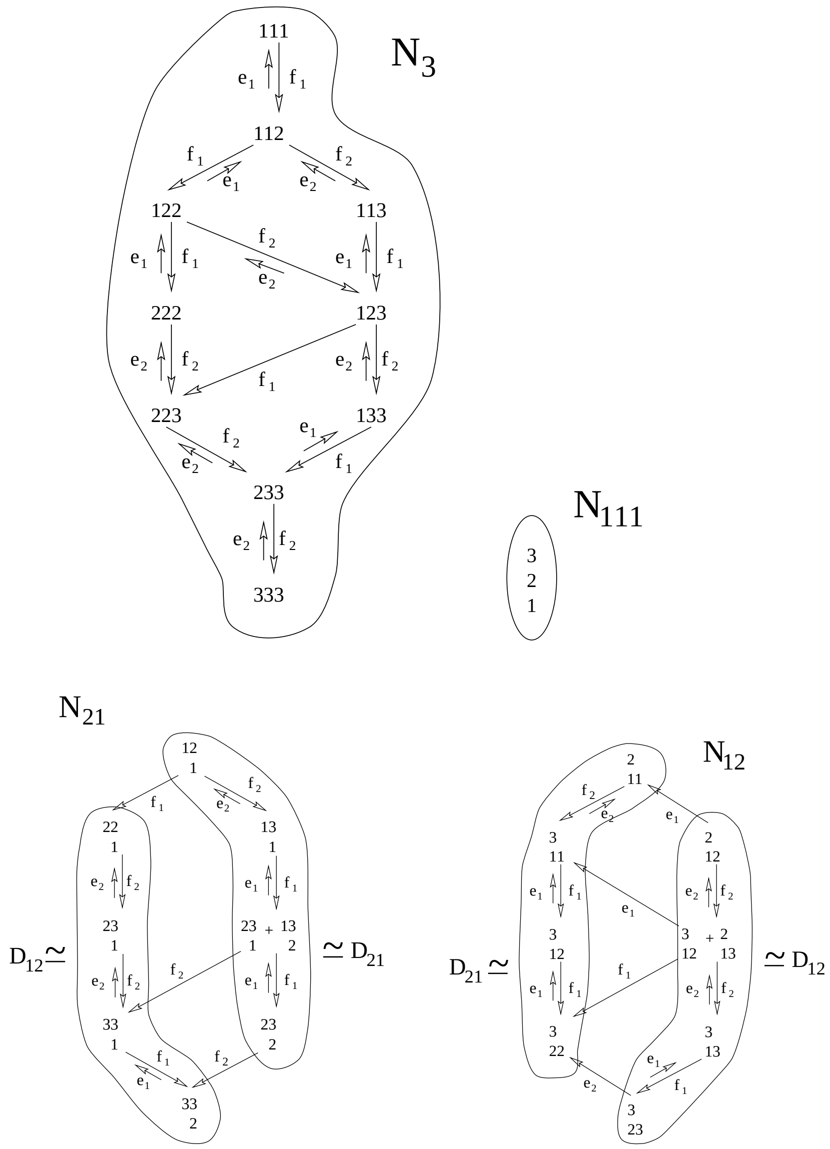


Figure 2: Decomposition of $V^{\otimes 3}$ under the action of $\mathcal{U}_0(gl_3)$

Using the decomposition [21]

$$V^{\otimes n} = \bigoplus_{I \vDash n} \mathbf{N}_I \tag{20}$$

as $\mathcal{U}_0(gl_N)$-modules, we can now give the complete classification of irreductible polynomial $\mathcal{U}_0(gl_N)$-modules and of the indecomposable direct summands of $V^{\otimes n}$.

Proposition 7.3 (i) *The* $\mathbf{D}_I$ *form a complete family of irreducible polynomial* $\mathcal{U}_0(gl_N)$-*modules.*
(ii) *The* $\mathbf{N}_I$ *form a complete family of indecomposable polynomial* $\mathcal{U}_0(gl_N)$-*modules which are direct summands of some* $V^{\otimes n}$.

The noncommutative analogues of quasi-ribbon functions arising in Proposition 7.2 form a linear basis of a commutative subalgebra of the hypoplactic algebra. This subalgebra is isomorphic to the algebra of quasi-symmetric functions and plays therefore the rôle of the character ring of polynomial $\mathcal{U}_0(gl_N)$-modules. Similarly, the algebra of noncommutative symmetric functions is a Grothendieck ring for "projective" polynomial $\mathcal{U}_0(gl_N)$-modules (*i.e.* direct summands of some $V^{\otimes n}$). The noncommutativity of this ring reflects the fact that at $q = 0$, the tensor products $M \otimes N$ and $N \otimes M$ are actually not isomorphic.

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