Branching functions of $A_{n-1}^{(1)}$ and Jantzen-Seitz problem for Ariki-Koike algebras

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Abstract

We study the restrictions of simple modules of Ariki-Koike algebras $\mathcal{H}_m(\boldsymbol{v})$ with set of parameters $\boldsymbol{v} = (\zeta; \zeta^{v_0}, \ldots, \zeta^{v_{l-1}})$, where ζ is an nth root of unity, to their subalgebras $\mathcal{H}_{m-j}(\boldsymbol{v})$. Using a theorem of Ariki and the crystal basis theory of Kashiwara, we relate this problem to the calculation of tensor product multiplicities of highest weight irreducible representations of the affine Lie algebra $A_{n-1}^{(1)}$. These multiplicities have a combinatorial description in terms of higher level paths or highest-lift multipartitions.

This enables us to solve the Jantzen-Seitz problem for Ariki-Koike algebras, that is, to determine which irreducible representations of $\mathcal{H}_m(\boldsymbol{v})$ restrict to irreducible representations of $\mathcal{H}_{m-1}(\boldsymbol{v})$. From a combinatorial point of view, this problem is identical to that of computing the tensor product of an $A_{n-1}^{(1)}$-module of level l and one of level 1.

We also consider natural generalisations of the Jantzen-Seitz problem corresponding to the product of a level l module by a level $l' > 1$ module, and from the commutativity of tensor products, we deduce a remarkable symmetry between the generalised Jantzen-Seitz conditions and the sets of parameters of the Ariki-Koike algebras.

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1 Introduction

In 1994, Ariki and Koike [4] introduced an analogue of the Iwahori-Hecke algebra for the complex reflection group $G(l,1,m)=(\mathbb{Z}/l\mathbb{Z})\wr\mathfrak{S}_m$. This algebra $\mathcal{H}_m(\boldsymbol{v})=\mathcal{H}_m(v;u_0,\dots,u_{l-1})$ depends on $l+1$ parameters $v,u_0,\dots,u_{l-1}$ and reduces to the group algebra of $G(l,1,m)$ when $v=1$ and $u_k=\exp(2ik\pi/l)$. Also, it generalises the Iwahori-Hecke algebras of the Coxeter groups of types A_{m-1} and B_m, which are obtained for $l=1$ and $l=2$ respectively. The algebra $\mathcal{H}_m(\boldsymbol{v})$ appeared independently in [5], where Hecke algebras of other types of complex reflection groups were also defined.

When the parameters are generic, $\mathcal{H}_m(\boldsymbol{v})$ is semisimple and its simple modules $S(\boldsymbol{\lambda})$ are labelled by l-tuples of partitions $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(l-1)})$ such that $|\boldsymbol{\lambda}|:=\sum_j|\lambda^{(j)}|=m$. In this paper, we are concerned with the following choice of parameters. Fix an integer $n\ge 2$, l integers $0\le v_0\le\ldots\le v_{l-1}<n$, and set

$$v=\zeta=\exp(2i\pi/n),\qquad u_k=\zeta^{v_k},\quad (0\le k<l).$$

The corresponding AK-algebra will be denoted by $\mathcal{H}_m(\mathbf{i})$, where $\mathbf{i}=(i_0,\dots,i_{n-1})$ and i_k is the number of v_j equal to k. For example, $\mathcal{H}_m(1,0,0)$ is the Hecke algebra of type A_{m-1} at a 3rd root of 1. It is known that $\mathcal{H}_m(\mathbf{i})$ is not semisimple [2]. In a recent paper [3] Ariki, inspired by a previous conjecture of Lascoux, Leclerc and Thibon for type A Hecke algebras [26, 27], proved the following theorem.

Let $G_0(\mathcal{H}_m(\mathbf{i}))$ be the Grothendieck group of finitely generated $\mathcal{H}_m(\mathbf{i})$-modules and let $\mathcal{G}(\mathbf{i})=\bigoplus_m G_0(\mathcal{H}_m(\mathbf{i}))$. Then the infinite-dimensional complex vector space $\mathcal{G}_{\mathbb{C}}(\mathbf{i})=\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{G}(\mathbf{i})$ is naturally endowed with the structure of an $A^{(1)}_{n-1}$-module, isomorphic to the level l irreducible representation $V(\Lambda_{\mathbf{i}})$ of highest weight $\Lambda_{\mathbf{i}}=\sum_k i_k\Lambda_k$. (Here, the Λ_k denote the fundamental weights of $A^{(1)}_{n-1}$.) Moreover, the basis of $\mathcal{G}(\mathbf{i})$ consisting of the classes of simple modules coincides with Kashiwara's upper global basis (or Lusztig's dual canonical basis) of $V(\Lambda_{\mathbf{i}})$. In particular, the simple $\mathcal{H}_m(\mathbf{i})$-modules are in one-to-one correspondence with the vertices of the crystal graph $B(\Lambda_{\mathbf{i}})$ of $V(\Lambda_{\mathbf{i}})$ that correspond to weight vectors of principal degree m.

It turns out that this canonical basis may be efficiently computed by embedding the $U_q(A^{(1)}_{n-1})$-module $V_q(\Lambda_{\mathbf{i}})$ in a suitable q-deformed Fock space. In the level 1 case, this was explained in [27] using the Fock space construction of Misra and Miwa [33]. The construction was generalised to arbitrary levels by Mathas [31], who thereby obtained the first complete classification of simple $\mathcal{H}_m(\mathbf{i})$-modules [32]. This Fock space construction yields a labelling of the vertices of the crystal graph by either a certain class of l-tuples of Young diagrams, or by a set of combinatorial objects called paths, which were introduced in the context of solvable lattice models by Andrews, Baxter and Forrester [1] and extensively studied by the Kyoto group (see *e.g.* [6]).

In this paper, we shall use Ariki's theorem to study the restriction of a simple $\mathcal{H}_m(\mathbf{i})$-module to the subalgebras $\mathcal{H}_{m-j}(\mathbf{i})$. In fact, we shall use some refined restriction operations defined by means of a natural central element $c_m\in\mathcal{H}_m(\mathbf{i})$ [3]. Indeed, if M is a $\mathcal{H}_m(\mathbf{i})$-module, the restricted module $M\!\downarrow_{\mathcal{H}_{m-1}(\mathbf{i})}$ splits into a direct sum of eigenspaces of c_{m-1}, which we denote by

$$M\!\downarrow_k,\qquad (0\le k<n).$$

These k-restriction operators were first defined for symmetric groups by Robinson (see [14]). More generally, we write $M\!\downarrow_k^j$ for the $\mathcal{H}_{m-j}(\mathbf{i})$-module obtained from M by j successive k-restrictions.

Let D be a simple $\mathcal{H}_m(\mathbf{i})$-module, and let $\mathbf{j} = (j_0, \ldots, j_{n-1})$ be another n-tuple of nonnegative integers. We say that D satisfies the generalised Jantzen-Seitz condition $\mathrm{JS}(\mathbf{j})$ if and only if

$$D{\downarrow_k}^{j_k+1} = 0\,, \qquad (k = 0, \ldots, n-1)\,.$$

In this case, we write $D \in \mathrm{JS}(\mathbf{j})$ (or also $b \in \mathrm{JS}(\mathbf{j})$ if b is the vertex of $B(\Lambda_{\mathbf{i}})$ corresponding to D). This is indeed a generalisation of the original Jantzen-Seitz condition, namely, we show that the (ordinary) restriction $D{\downarrow_{\mathcal{H}_{m-1}(\mathbf{i})}}$ is irreducible if and only if

$$D \in \mathrm{JS}(1, 0, \ldots, 0) \cup \mathrm{JS}(0, 1, 0, \ldots, 0) \cup \cdots \cup \mathrm{JS}(0, \ldots, 0, 1)\,.$$

Let b be the vertex of the crystal graph of $V(\Lambda_{\mathbf{i}})$ corresponding to D. It follows from Ariki's theorem that $D \in \mathrm{JS}(\mathbf{j})$ if and only if

$$\tilde{e}_k^{j_k+1}\, b = 0\,, \qquad (k = 0, \ldots, n-1)\,,$$

where $\tilde{e}_k$ denotes the Kashiwara operator acting on the crystal graph. This allows us to relate the generalised JS-problem to the calculation of tensor product multiplicities. Write $\Lambda_{\mathbf{j}} = \sum_k j_k \Lambda_k$, and denote by $\deg b$ the homogeneous degree of $b \in B(\Lambda_{\mathbf{i}})$. Then we obtain

$$\sum_{b \in B(\Lambda_{\mathbf{i}}) \cap \mathrm{JS}(\mathbf{j})} z^{\deg b} = \sum_{\Lambda} b^{\Lambda}_{\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}}}(z)\,,$$

where the $b^{\Lambda}_{\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}}}(z)$ are the branching functions of the tensor product $V(\Lambda_{\mathbf{j}}) \otimes V(\Lambda_{\mathbf{i}})$. Thus, the generating function of the number of simple $\mathcal{H}_m(\mathbf{i})$-modules satisfying the condition $\mathrm{JS}(\mathbf{j})$ is a sum of branching functions of $A^{(1)}_{n-1}$.

One striking fact about this formula is that $\mathbf{i}$ and $\mathbf{j}$ play symmetric roles, which means that the number of simple $\mathcal{H}_m(\mathbf{i})$-modules satisfying the condition $\mathrm{JS}(\mathbf{j})$ is the same as the number of simple $\mathcal{H}_m(\mathbf{j})$-modules satisfying the condition $\mathrm{JS}(\mathbf{i})$.

This result motivates a deeper study of the sets of multipartitions $\mathcal{Y}(\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}})$ which label the highest weight vectors of $V(\Lambda_{\mathbf{j}}) \otimes V(\Lambda_{\mathbf{i}})$. When both $\Lambda_{\mathbf{j}}$ and $\Lambda_{\mathbf{i}}$ are of level 1, *i.e.* when $\Lambda_{\mathbf{j}} = \Lambda_j$ and $\Lambda_{\mathbf{i}} = \Lambda_i$ are fundamental weights, the set $\mathcal{Y}(\Lambda_j, \Lambda_i)$ consists of a class of partitions described by Foda, Okado and Warnaar [12] in their work on RSOS solvable lattice models based on the coset $(\widehat{\mathfrak{sl}}_n)_1 \times (\widehat{\mathfrak{sl}}_n)_1 / (\widehat{\mathfrak{sl}}_n)_2$. The surprising observation that these partitions are the same as those occurring in the paper of Jantzen and Seitz [15] in the modular representation theory of symmetric groups was in fact the starting point of this work [10].

In this article we describe, much in the spirit of [12], the elements of $\mathcal{Y}(\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}})$ in the case when $\Lambda_{\mathbf{j}}$ is of level one and $\Lambda_{\mathbf{i}}$ is arbitrary. To this aim, we first consider the problem of relating the two sets $\mathcal{Y}(\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}})$ and $\mathcal{Y}(\Lambda_{\mathbf{i}}, \Lambda_{\mathbf{j}})$ for arbitrary dominant integral weights $\Lambda_{\mathbf{i}}$ and $\Lambda_{\mathbf{j}}$. Using the combinatorics of paths, we construct a simple bijection between $\mathcal{Y}(\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}})$ and $\mathcal{Y}(\sharp\Lambda_{\mathbf{i}}, \sharp\Lambda_{\mathbf{j}})$, where $\sharp$ denotes the root diagram automorphism exchanging Λ_i and Λ_{n-i}. We then apply this construction in the case of a fundamental weight $\Lambda_{\mathbf{j}} = \Lambda_j$ and thereby obtain the desired characterisation.

The paper is divided into two main parts. The first one, Section 2, is devoted to the calculation of branching functions using the combinatorics of paths and multipartitions. Since this language is probably not familiar to many readers interested in AK-algebras, we feel it appropriate to explain it in some detail. Thus, Sections 2.1 and 2.2 are almost entirely expository, our main references being [6] and [16]. In 2.1, we review the Fock space representations and their q-deformations, and describe how the vertices of the crystal graph of $V(\Lambda)$ may be labelled using either unrestricted paths or their corresponding highest-lift multipartitions. In 2.2, we use crystal basis theory to identify some specific classes of paths, called restricted paths, whose generating functions are the branching functions. We also introduce the corresponding restricted multipartitions. Section 2.3 deals with the involution $\sharp$ and Section 2.4 contains the description of $\mathcal{Y}(\Lambda_j, \Lambda_{\mathfrak{i}})$ obtained via $\sharp$.

The second part, Section 3, deals with AK-algebras. Section 3.1 introduces, following [4, 2, 3], the Ariki-Koike algebra and provides a realisation of it as a quotient of the affine Hecke algebra. Section 3.2 recalls the basic tools and results of the representation theory of AK-algebras, including the definition of the k-restriction operators and the statement of Ariki's theorem. Finally, in Sections 3.3 and 3.4, we present our results on the generalised JS-problem and interpret the combinatorial theorems of 2.3 and 2.4 in this context.

2 Combinatorics of branching functions of $A_{n-1}^{(1)}$

2.1 Highest weight modules, unrestricted paths and highest-lifts

We begin by recalling some standard notation (see *e.g.* [18] [16]).

Let $e_i, f_i, h_i, (0 \le i \le n-1)$ be the Chevalley generators of the affine Lie algebra $\widehat{\mathfrak{sl}}_n = A_{n-1}^{(1)}$. The degree generator and the canonical central element are denoted respectively by d and c. The derived algebra $\widehat{\mathfrak{sl}}'_n = [\widehat{\mathfrak{sl}}_n, \widehat{\mathfrak{sl}}_n]$ is the subalgebra obtained by omitting the degree generator d.

Let $\Lambda_0, \dots, \Lambda_{n-1}$ denote the fundamental weights and δ the null root. For $i \in \mathbb{Z}$, set $\Lambda_i = \Lambda_{(i \bmod n)}$, $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i0}^{(n)}\,\delta$ (where $\delta_{ij}^{(n)} = 1$ if $(i-j) \bmod n = 0$ and $\delta_{ij}^{(n)} = 0$ otherwise) and $\epsilon_i = \Lambda_{i+1} - \Lambda_i$. The α_i are the simple roots and the ϵ_i can be identified with the weights of the n-dimensional defining representation of $\mathfrak{sl}_n$ extended to an $\widehat{\mathfrak{sl}}'_n$-representation. The root lattice and the weight lattice are respectively $Q = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i$ and $P = \mathbb{Z}\delta \oplus (\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i)$. For $l \in \mathbb{N}$, let

$$P_l = \left\{ \sum_{i=0}^{n-1} a_i \Lambda_i \mid a_i \in \mathbb{Z},\ \sum_{i=0}^{n-1} a_i = l \right\}, \qquad P_l^+ = \left\{ \sum_{i=0}^{n-1} a_i \Lambda_i \in P_l \mid a_i \ge 0 \right\}$$

be the set of (classical) level l integral weights and the subset of dominant ones, respectively. Finally, let $P^+ = \mathbb{Z}\delta \oplus (\bigoplus_{i=0}^{n-1} \mathbb{N}\Lambda_i)$ be the set of dominant integral weights.

For each $\Lambda \in P^+$ there exists a unique integrable highest weight module $V(\Lambda)$ with highest weight Λ. It can be explicitly constructed as a submodule of a Fock space $\mathcal{F}(\Lambda)$ [6], as we now explain.

Throughout this section we fix $\Lambda \in P_l^+$ and we write

$$\Lambda = \Lambda_{v_0} + \Lambda_{v_1} + \cdots + \Lambda_{v_{l-1}}\,,$$

where we may assume that $0 \le v_0 \le v_1 \le \cdots \le v_{l-1} < n$. The level l Fock space $\mathcal{F}(\Lambda)$ is a tensor product of l level 1 Fock spaces

$$\mathcal{F}(\Lambda) = \mathcal{F}(\Lambda_{v_0}) \otimes \cdots \otimes \mathcal{F}(\Lambda_{v_{l-1}})\,,$$

and each level 1 Fock space representation is constructed as follows. As a vector space $\mathcal{F}(\Lambda_j) = \bigoplus_\lambda \mathbb{C}\,u_\lambda$, where λ runs through the set Π of all partitions. In other words, $\mathcal{F}(\Lambda_j)$ is an infinite dimensional $\mathbb{C}$-space with a distinguished basis u_λ labelled by Young diagrams. To define the action of $\widehat{\mathfrak{sl}}_n$ on $\mathcal{F}(\Lambda_j)$, one uses a colouring of the nodes of each

$\lambda \in \Pi$, namely the node in the ith row and the kth column of λ is filled with the colour $(k - i + j) \bmod n$. We write $\mu/\lambda = \boxed{r}$ to indicate that the Young diagram μ is obtained from λ by adding a node with colour r. Then $\widehat{\mathfrak{sl}}_n$ acts on $\mathcal{F}(\Lambda_j)$ by [6]

$$f_r u_\lambda = \sum_{\mu/\lambda=\boxed{r}} u_\mu \,, \tag{1}$$

$$e_r u_\mu = \sum_{\mu/\lambda=\boxed{r}} u_\lambda \,, \tag{2}$$

$$c\, u_\lambda = u_\lambda \,, \tag{3}$$

$$d\, u_\lambda = -N^0(\lambda) u_\lambda \,, \tag{4}$$

where $N^0(\lambda)$ is the number of 0-nodes of λ.

The level l action of $\widehat{\mathfrak{sl}}_n$ on $\mathcal{F}(\Lambda)$ is obtained by taking the tensor product of these level 1 actions in the standard way. The natural basis of $\mathcal{F}(\Lambda)$ consists of the monomial tensors

$$u_{\boldsymbol{\lambda}} = u_{\lambda^{(0)}} \otimes \cdots \otimes u_{\lambda^{(l-1)}}$$

indexed by l-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$.

Example 2.1 Take $n = 3$ and $\Lambda = 2\Lambda_1 + \Lambda_2$. Set $\boldsymbol{\lambda} = ((3,2),(1,1,1),(5,4,1))$. The coloured Young diagram of $\boldsymbol{\lambda}$ is

$$\begin{array}{|c|c|c|}\hline 1&2&0\\ \hline 0&1& \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|}\hline 1\\ \hline 0\\ \hline 2\\ \hline\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 2&0&1&2&0\\ \hline 1&2&0&1& \\ \cline{1-4} 0& & & & \\ \cline{1-1}\end{array}$$

The lowering operators of $\widehat{\mathfrak{sl}}_3$ act on $u_{\boldsymbol{\lambda}} \in \mathcal{F}(\Lambda)$ as follows:

$$\begin{aligned}
f_0 u_{\boldsymbol{\lambda}} &= 0, \\
f_1 u_{\boldsymbol{\lambda}} &= u_{((\mathbf{4},2),(1,1,1),(5,4,1))} + u_{((3,2),(1,1,1,\mathbf{1}),(5,4,1))} + u_{((3,2),(1,1,1),(\mathbf{6},4,1))} \\
&\quad + u_{((3,2),(1,1,1),(5,4,\mathbf{2}))}, \\
f_2 u_{\boldsymbol{\lambda}} &= u_{((3,\mathbf{3}),(1,1,1),(5,4,1))} + u_{((3,2,\mathbf{1}),(1,1,1),(5,4,1))} + u_{((3,2),(\mathbf{2},1,1),(5,4,1))} \\
&\quad + u_{((3,2),(1,1,1),(5,\mathbf{5},1))} + u_{((3,2),(1,1,1),(5,4,1,\mathbf{1}))}.
\end{aligned}$$

(The parts of $\boldsymbol{\lambda}$ which have been increased by 1 are printed in bold type). The degree operator acts by $d u_{\boldsymbol{\lambda}} = -7 u_{\boldsymbol{\lambda}}$. $\diamond$

It follows from (1), (2), (3) and (4), that for all $\boldsymbol{\lambda}$, $u_{\boldsymbol{\lambda}}$ is a weight vector of $\mathcal{F}(\Lambda)$ of weight $\mathrm{wt}\,(\boldsymbol{\lambda})$, where we define

$$\mathrm{wt}\,(\boldsymbol{\lambda}) = \Lambda - \sum_{i=0}^{n-1} N^i(\boldsymbol{\lambda}) \alpha_i \,, \tag{5}$$

and where $N^r(\boldsymbol{\lambda})$ is the total number of r-nodes of $\boldsymbol{\lambda}$.

Let $u_\emptyset$ denote the vacuum vector of $\mathcal{F}(\Lambda)$, that is, the vector labelled by the empty multipartition. It is clear that $u_\emptyset$ is a highest weight vector of $\mathcal{F}(\Lambda)$ of weight Λ. The submodule $U(\widehat{\mathfrak{sl}}_n)\, u_\emptyset$ is isomorphic to $V(\Lambda)$. In [6], an explicit basis of this submodule was constructed using combinatorial objects called *paths*. These we now describe.

Let $\mathcal{A}_l^+ = \{\sum_{i=0}^{n-1} a_i\epsilon_i \mid a_i \in \mathbb{N}, \sum a_i = l\}$ denote the set of weights of the lth symmetric power of $\mathbb{C}^n$ (considered as an $\widehat{\mathfrak{sl}}_n'$-representation). A *level l path* is an infinite sequence $p = (p_0, p_1, p_2, \ldots)$ of weights $p_k \in P_l$. A Λ-*path* is a level l path such that:

$$\text{for all } k \geq 0,\ p_{k+1} - p_k \in \mathcal{A}_l^+, \tag{6}$$

$$\text{for } k \text{ large enough, } p_k = \Lambda_{v_0+k} + \cdots + \Lambda_{v_{l-1}+k}. \tag{7}$$

Condition (7) means that, except for a finite number of indices, $p = (p_k)$ coincides with the *ground state path*

$$\overline{p} = (\overline{p}_k) = (\Lambda_{v_0+k} + \cdots + \Lambda_{v_{l-1}+k}) .$$

The smallest integer k_* such that $p_k = \overline{p}_k$ for $k \geq k_*$ is called the *length* of p and is denoted by $\ell(p)$. The set of Λ-paths is denoted by $\mathcal{P}(\Lambda)$.

Note that a Λ-path $p = (p_k)$ is completely determined by the associated sequence $\eta = (\eta_k)$ of elements $\eta_k = p_{k+1} - p_k$ of $\mathcal{A}_l^+$. It is sometimes more convenient to think of a Λ-path in this way as a sequence η of elements of $\mathcal{A}_l^+$ with the tail condition

$$\eta_k = \overline{\eta}_k := \epsilon_{v_0+k} + \cdots + \epsilon_{v_{l-1}+k} , \quad (k >> 0) .$$

We shall sometimes write $\eta_k = \sum_{j=0}^{l-1} \epsilon_{\gamma_j(k)}$, where $0 \leq \gamma_j(k) < n$ for $k = 0, 1, \ldots, l-1$.

The notion of path comes from the study of solvable lattice models, where, roughly speaking, a path corresponds to an eigenvector of the corner transfer matrix at the absolute temperature $q = 0$ (see *e.g.* [11]). It was found by Date, Jimbo, Kuniba, Miwa and Okado [6] that the set of Λ-paths is an appropriate labelling for a basis of weight vectors of $V(\Lambda)$. Namely, in the context of statistical mechanics, a path p has an *energy* $E(p)$ defined as follows. One introduces a function H on $\mathcal{A}_l^+ \times \mathcal{A}_l^+$ given, for $\alpha = \epsilon_{\mu_0} + \cdots + \epsilon_{\mu_{l-1}}$ and $\beta = \epsilon_{\nu_0} + \cdots + \epsilon_{\nu_{l-1}}$, by

$$H(\alpha, \beta) = \min_{\sigma \in \mathfrak{S}_l} \sum_{i=0}^{l-1} \theta(\mu_i - \nu_{\sigma(i)}) , \tag{8}$$

where $\theta(a) = 1$ if $a \geq 0$ and $\theta(a) = 0$ otherwise. Thus, for $n = 3 = l$, one has

$$H(\epsilon_0 + \epsilon_1 + \epsilon_2, \epsilon_0 + \epsilon_1 + \epsilon_2) = \theta(0-1) + \theta(1-2) + \theta(2-0) = 1 .$$

Then one puts

$$E(p) = \sum_{k=1}^{\infty} k \left(H(\eta_{k-1}, \eta_k) - H(\overline{\eta}_{k-1}, \overline{\eta}_k) \right) . \tag{9}$$

This is in fact a finite sum since, by condition (7) above, $\eta_k = \overline{\eta}_k$ for k large enough. Finally one defines the *weight* $\mathrm{wt}\,(p) \in P$ of p by:

$$\mathrm{wt}\,(p) = p_0 - E(p)\delta . \tag{10}$$

Theorem 2.2 [6] *Let $\Lambda \in P_l^+$. The formal character of the representation $V(\Lambda)$ of $\widehat{\mathfrak{sl}}_n$ is given by*

$$\mathrm{ch} V(\Lambda) = \sum_{p \in \mathcal{P}(\Lambda)} e^{\mathrm{wt}\,(p)} .$$

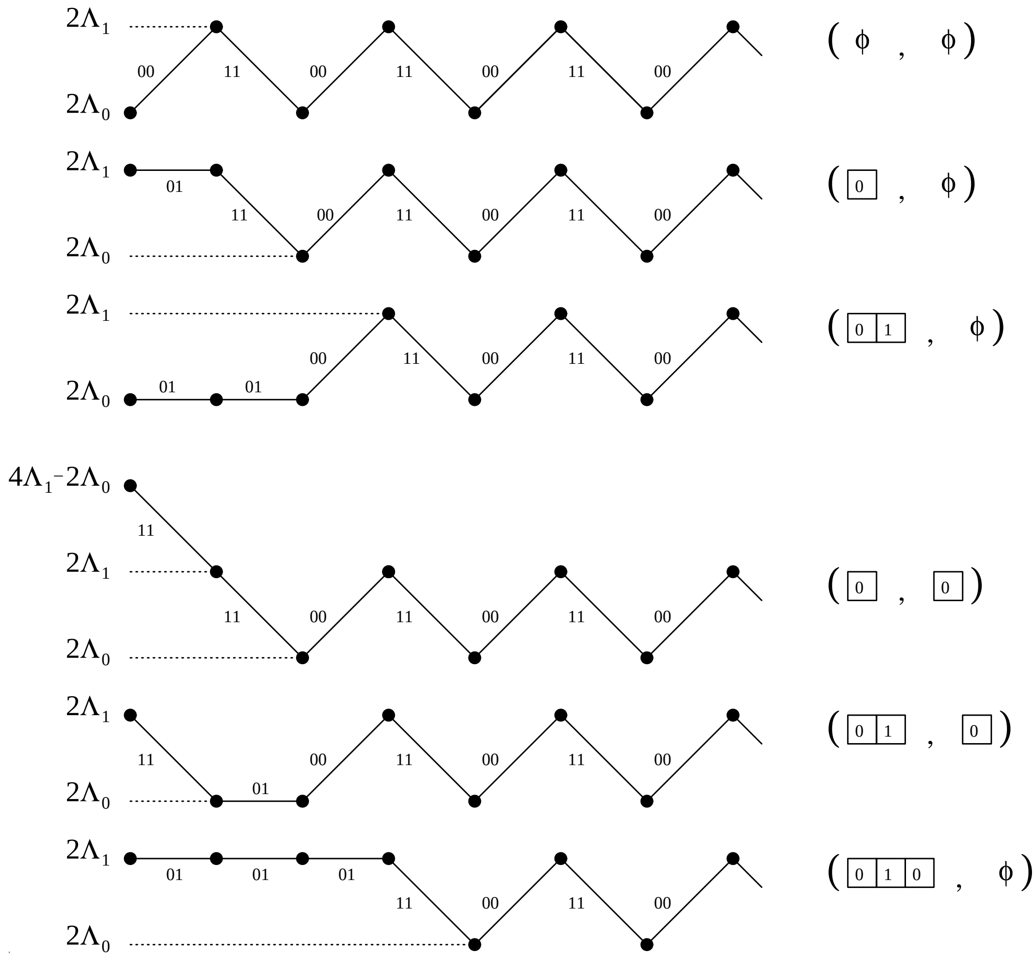


Figure 1: $2\Lambda_0$-paths for $\widehat{\mathfrak{sl}}_2$ and their highest-lift bipartitions

Example 2.3 We illustrate the theorem by enumerating a few paths in $\mathcal{P}(2\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$. We write for short 00, 01, 11, in place of $2\epsilon_0$, $\epsilon_0+\epsilon_1$, $2\epsilon_1$. The paths are given as sequences (η_k) of elements of $\mathcal{A}_2^+$.

path	energy	weight
$(00, 11, 00, 11, \ldots)$	0	$2\Lambda_0$
$(01, 11, 00, 11, \ldots)$	1	$2\Lambda_1 - \delta$
$(01, 01, 00, 11, \ldots)$	1	$2\Lambda_0 - \delta$
$(11, 11, 00, 11, \ldots)$	2	$4\Lambda_1 - 2\Lambda_0 - 2\delta$
$(11, 01, 00, 11, \ldots)$	2	$2\Lambda_1 - 2\delta$
$(01, 01, 01, 11, \ldots)$	2	$2\Lambda_1 - 2\delta$

These paths are shown in Figure 1, with the corresponding highest-lift multipartitions which will be defined below. Thus, one has

$$\mathrm{ch} V(2\Lambda_0) = e^{2\Lambda_0} + e^{2\Lambda_1-\delta} + e^{2\Lambda_0-\delta} + e^{4\Lambda_1-2\Lambda_0-2\delta} + 2e^{2\Lambda_1-2\delta} + \cdots$$

$\diamond$

The relationship between paths and highest weight representations of $\widehat{\mathfrak{sl}}_n$ was later clarified using the crystal basis theory of Kashiwara [33, 16, 20, 21]. This involves a q-deformation of $\widehat{\mathfrak{sl}}_n$.

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantized enveloping algebra of $\widehat{\mathfrak{sl}}_n$. We denote by $V_q(\Lambda)$ the irreducible $U_q(\widehat{\mathfrak{sl}}_n)$-module with highest weight Λ. We shall follow [16] and construct $V_q(\Lambda)$ as a submodule of a q-deformed level l Fock space $\mathcal{F}_q(\Lambda)$. As a vector space,

$$\mathcal{F}_q(\Lambda) = \sum_{\boldsymbol{\lambda} \in \Pi^l} \mathbb{Q}(q) v_{\boldsymbol{\lambda}} \, ,$$

that is, $\mathcal{F}_q(\Lambda)$ has a distinguished $\mathbb{Q}(q)$-basis $\{v_{\boldsymbol{\lambda}}\}$ labelled by the set of all l-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$.

To describe the action of $U_q(\widehat{\mathfrak{sl}}_n)$ requires some notation. First we colour the nodes of $\boldsymbol{\lambda}$ as before, filling the node of $\lambda^{(j)}$ which lies on the ith row and the kth column with $r = (k - i + v_j) \bmod n$. We write $\boldsymbol{\mu}/\boldsymbol{\lambda} = \boxed{r}$ to indicate that the multipartition $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding a node with colour r. In this case, we say that $\boldsymbol{\mu}/\boldsymbol{\lambda}$ is a *removable r-node* of $\boldsymbol{\mu}$ or an *addable r-node* of $\boldsymbol{\lambda}$. With each removable or addable r-node of $\boldsymbol{\lambda}$, there is associated a pair (d, j) of integers, where $d = k - i + v_j$ indicates the actual diagonal containing that node, and j indicates the component $\lambda^{(j)}$ of $\boldsymbol{\lambda}$ to which the node belongs. Then, define a total order on the set of removable and addable r-nodes of $\boldsymbol{\lambda}$ by:

$$(d, j) < (d', j') \qquad \Longleftrightarrow \qquad ((d < d') \text{ or } (d = d' \text{ and } j > j')) \, . \tag{11}$$

Example 2.4 Take $n = 3$, $l = 2$, $\Lambda = \Lambda_0 + \Lambda_1$ and consider the bipartition

$$\boldsymbol{\lambda} = ((9, 8, 7, 5, 4, 4, 1, 1), (9, 9, 7, 6, 5, 3, 3))$$

with coloured diagram

$$\begin{array}{ccccccccc}
0&1&2&0&1&2&0&1&2\\
2&0&1&2&0&1&2&0\\
1&2&0&1&2&0&1\\
0&1&2&0&1\\
2&0&1&2\\
1&2&0&1\\
0\\
2
\end{array}
\qquad
\begin{array}{ccccccccc}
1&2&0&1&2&0&1&2&0\\
0&1&2&0&1&2&0&1&2\\
2&0&1&2&0&1&2\\
1&2&0&1&2&0\\
0&1&2&0&1\\
2&0&1\\
1&2&0
\end{array}$$

The addable and removable 1-nodes of $\boldsymbol{\lambda}$ are ordered in the following way:

$$A_{-8,0} < A_{-5,0} < R_{-2,0} < R_{1,1} < R_{1,0} < A_{4,1} < R_{4,0} < A_{7,0} < A_{10,1} \, ,$$

where $A_{d,j}$ means an addable node in position (d, j) and $R_{d,j}$ a removable node in position (d, j). $\diamond$

Now suppose that $\boldsymbol{\mu}/\boldsymbol{\lambda}$ is the r-node (d, j). We put

$$\begin{aligned}
N_r^{<}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sharp\{ &\text{ addable } r\text{-nodes } (d', j') \text{ of } \boldsymbol{\lambda} \text{ such that } (d', j') < (d, j)\} \\
&- \sharp\{ \text{ removable } r\text{-nodes } (d', j') \text{ of } \boldsymbol{\lambda} \text{ such that } (d', j') < (d, j)\}, \\
N_r^{>}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sharp\{ &\text{ addable } r\text{-nodes } (d', j') \text{ of } \boldsymbol{\lambda} \text{ such that } (d', j') > (d, j)\} \\
&- \sharp\{ \text{ removable } r\text{-nodes } (d', j') \text{ of } \boldsymbol{\lambda} \text{ such that } (d', j') > (d, j)\}, \\
N_r(\boldsymbol{\lambda}) = \sharp\{ &\text{ addable } r\text{-nodes of } \boldsymbol{\lambda} \ \} - \ \sharp\{ \text{ removable } r\text{-nodes of } \boldsymbol{\lambda} \ \}.
\end{aligned}$$

Theorem 2.5 [16] *$U_q(\widehat{\mathfrak{sl}}_n)$ acts on $\mathcal{F}_q(\Lambda)$ by*

$f_r v_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu}} q^{N_r^>(\boldsymbol{\lambda},\boldsymbol{\mu})} v_{\boldsymbol{\mu}}$, *sum over all $\boldsymbol{\mu}$ such that* $\boldsymbol{\mu}/\boldsymbol{\lambda} = \boxed{r}$,

$e_r v_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\lambda}} q^{-N_r^<(\boldsymbol{\lambda},\boldsymbol{\mu})} v_{\boldsymbol{\lambda}}$, *sum over all $\boldsymbol{\lambda}$ such that* $\boldsymbol{\mu}/\boldsymbol{\lambda} = \boxed{r}$,

$q^{h_r} v_{\boldsymbol{\lambda}} = q^{N_r(\boldsymbol{\lambda})} v_{\boldsymbol{\lambda}}$,

$q^c v_{\boldsymbol{\lambda}} = q^l v_{\boldsymbol{\lambda}}$,

$q^d v_{\boldsymbol{\lambda}} = q^{-N^0(\boldsymbol{\lambda})} v_{\boldsymbol{\lambda}}$.

Example 2.6 We give the q-deformation of the calculation of Example 2.1. Take $n = 3$ and $\Lambda = 2\Lambda_1 + \Lambda_2$. Set $\boldsymbol{\lambda} = ((3,2),(1,1,1),(5,4,1))$. Then the lowering operators of $U_q(\widehat{\mathfrak{sl}}_3)$ act on $v_{\boldsymbol{\lambda}} \in \mathcal{F}_q(\Lambda)$ as follows:

$$\begin{aligned}
f_0 v_{\boldsymbol{\lambda}} &= 0 \\
f_1 v_{\boldsymbol{\lambda}} &= q\, v_{((4,2),(1,1,1),(5,4,1))} + q\, v_{((3,2),(1,1,1,1),(5,4,1))} \\
&\quad + v_{((3,2),(1,1,1),(6,4,1))} + v_{((3,2),(1,1,1),(5,4,2))} \\
f_2 v_{\boldsymbol{\lambda}} &= q\, v_{((3,3),(1,1,1),(5,4,1))} + q^3\, v_{((3,2,1),(1,1,1),(5,4,1))} \\
&\quad + q^2\, v_{((3,2),(2,1,1),(5,4,1))} + v_{((3,2),(1,1,1),(5,5,1))} \\
&\quad + q^3\, v_{((3,2),(1,1,1),(5,4,1,1))}
\end{aligned}$$

◇

Note 2.7 It is possible, and perhaps more natural, to define another level l q-deformed Fock space $\mathcal{F}'_q(\Lambda)$ by imitating the two steps of the $q = 1$ construction. First, one constructs the level 1 q-Fock spaces $\mathcal{F}_q(\Lambda_{v_j})$ as above. Then, one defines via the comultiplication of $U_q(\widehat{\mathfrak{sl}}_n)$

$$\mathcal{F}'_q(\Lambda) = \mathcal{F}_q(\Lambda_{v_0}) \otimes \cdots \otimes \mathcal{F}_q(\Lambda_{v_{l-1}})\,.$$

Clearly, the two Fock spaces $\mathcal{F}_q(\Lambda)$ and $\mathcal{F}'_q(\Lambda)$ coincide at $q = 1$. However, they are different in general. More precisely, $\mathcal{F}_q(\Lambda)$ and $\mathcal{F}'_q(\Lambda)$ are isomorphic as $U_q(\widehat{\mathfrak{sl}}_n)$-modules but the formulae giving the action of the Chevalley generators on multipartitions are not the same. This leads to two different ways of labelling the vertices of the same crystal graph. In this paper, we work exclusively with $\mathcal{F}_q(\Lambda)$ in contrast with Mathas [31] who uses $\mathcal{F}'_q(\Lambda)$. As a consequence, our labelling of the simple modules of the Ariki-Koike algebras differs from that of [31] (see below, Section 3.4).

Using this explicit description of the q-deformed Fock space, it is possible to compute its crystal basis [16]. To explain how the arrows of the crystal graph are obtained, we introduce the notion of a *good node*. Fix a residue r between 0 and $n-1$ and consider the sequence of removable and addable r-nodes of $\boldsymbol{\lambda}$, ordered as explained above. This sequence is called the *r-signature* of $\boldsymbol{\lambda}$. Thus for $n = 3$, $\Lambda = \Lambda_0 + \Lambda_1$ and the bipartition $\boldsymbol{\lambda}$ of Example 2.4, we get the following 1-signature:

$$A_{-8,0}A_{-5,0}R_{-2,0}R_{1,1}R_{1,0}A_{4,1}R_{4,0}A_{7,0}A_{10,1}\,.$$

Now recursively remove RA pairs (together with their subscripts) from this sequence until none remain. The sequence will then be of the form

$$A \cdots AR \cdots R.$$

The remaining nodes are called the *normal r-nodes* of $\boldsymbol{\lambda}$. The node corresponding to the leftmost R is termed a *good removable r-node* and that corresponding to the rightmost A is termed a *good addable r-node* (note that there is at most one of each). In the example above, the removal procedure first disposes of $R_{1,0}A_{4,1}$ and $R_{4,0}A_{7,0}$, and then disposes of $R_{1,1}A_{10,2}$ so that

$$A_{-8,0}A_{-5,0}R_{-2,0}$$

remains. Therefore $\boldsymbol{\lambda}$ has a good addable node on the -5 diagonal of $\lambda^{(0)}$ and a good removable node on the -2 diagonal of $\lambda^{(1)}$. In the case of 0-nodes and 2-nodes, we first obtain the sequences

$$A_{-6,1}R_{-3,1}A_{0,0}R_{3,1}A_{6,1}R_{6,0}A_{9,0}$$

and

$$R_{-7,0}A_{-1,1}A_{2,1}A_{2,0}R_{5,1}A_{5,0}R_{8,1}R_{8,0}$$

respectively which, after the removal procedure, produce

$$A_{-6,1}$$

and

$$A_{2,1}A_{2,0}R_{8,1}R_{8,0}$$

respectively. Thus $\boldsymbol{\lambda}$ has a good addable 0-node on the -6 diagonal of $\lambda^{(1)}$, but no good removable 0-node, and a good addable 2-node on the 2 diagonal of $\lambda^{(0)}$ and a good removable 2-node on the 8 diagonal of $\lambda^{(1)}$.

Theorem 2.8 [16] *Let $A \subset \mathbb{Q}(q)$ be the ring of rational functions without pole at $q = 0$. Let $L = \bigoplus_{\boldsymbol{\lambda}\in\Pi^l} A\, v_{\boldsymbol{\lambda}}$, and let B be the $\mathbb{Q}$-basis of L/qL given by $B = \{v_{\boldsymbol{\lambda}} \bmod qL \mid \boldsymbol{\lambda} \in \Pi^l\}$. Then (L, B) is a crystal basis of $\mathcal{F}_q(\Lambda)$. Moreover, the crystal graph contains the arrow*

$$v_{\boldsymbol{\lambda}} \xrightarrow{r} v_{\boldsymbol{\mu}}$$

if and only if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by adding a good r-node.

Let $v_\emptyset$ denote the vacuum vector of $\mathcal{F}_q(\Lambda)$. The submodule $U_q(\widehat{\mathfrak{sl}}_n)v_\emptyset$ is known to be isomorphic to $V_q(\Lambda)$. Therefore, using properties of crystal bases one can obtain the crystal graph of $V_q(\Lambda)$ as the connected component of that of $\mathcal{F}_q(\Lambda)$ which contains $v_\emptyset$. The multipartitions labelling the vertices of this subgraph have been described in [16] using the following map from multipartitions to paths.

Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$ and let $k_* = \max\{\lambda^{(0)}_1, \ldots, \lambda^{(l-1)}_1\}$ be the greatest part of the partitions $\lambda^{(j)}$. We define the path $p = \pi(\boldsymbol{\lambda}) \in \mathcal{P}(\Lambda)$ of length $\ell(p) = k_*$ by

$$\eta_k = \sum_{j=0}^{l-1} \epsilon_{v_j + k - \lambda^{(j)\prime}_{k+1}}\,, \tag{12}$$

where $\lambda^{(j)\prime}_k$ denotes the height of the kth column of the Young diagram $\lambda^{(j)}$. The sequence (η_k) is easily read from the coloured diagram of $\boldsymbol{\lambda}$ by taking the colours of the nodes immediately under the bottom end of each column. For instance, if $\boldsymbol{\lambda}$ is again the bipartition $\boldsymbol{\lambda}$ of Example 2.4 then, for convenience, writing ij instead of $\epsilon_i + \epsilon_j$, we have

$$\eta = (01, 11, 22, 02, 00, 22, 01, 02, 11, 01, 12, 02, 01, 12, \ldots)\,.$$

It was proved in [6, 16] that the map π is surjective and that each path p has a distinguished preimage $\boldsymbol{\lambda}$ called the *highest-lift* of p. It is characterised as follows.

We say that a multipartition $\boldsymbol{\lambda}$ is *cylindrical of highest weight* Λ if

$$\begin{aligned} &\lambda_i^{(j)} \geq \lambda_{i+v_{j+1}-v_j}^{(j+1)}, \qquad 0 \leq j \leq l-2, \quad i = 1, 2, \dots, \\ &\lambda_i^{(l-1)} \geq \lambda_{i+n+v_0-v_{l-1}}^{(0)}, \qquad i = 1, 2, \dots. \end{aligned}$$

If we make the convention to align the partitions so that the first rows beginning with a node of colour 0 are adjacent to one another, then the first condition means that the lengths of adjacent rows should not increase from left to right across the diagram. The second condition can be similarly checked by putting a copy of $\lambda^{(0)}$ on the right, but raised n rows.

Given two multipartitions $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, we say that $\boldsymbol{\lambda}$ is *higher than* $\boldsymbol{\mu}$ if

$$\lambda^{(j)\prime}_k \leq \mu^{(j)\prime}_k, \qquad 0 \leq j \leq l-1, \quad k = 1, 2, \dots,$$

that is, if the frontier of the Young diagram $\lambda^{(j)}$ is higher than that of $\mu^{(j)}$ for all j.

Now, for each path $p \in \mathcal{P}(\Lambda)$, there is a unique cylindrical preimage $\boldsymbol{\lambda} \in \pi^{-1}(p)$ which is higher than all other cylindrical preimages of p [6]. It is called the highest-lift of p and is denoted by $\boldsymbol{\lambda}(p)$. The following result highlights the correspondence between paths and their highest-lifts.

Lemma 2.9 [6] *If $p \in \mathcal{P}(\Lambda)$ then*

$$\mathrm{wt}(\boldsymbol{\lambda}(p)) = \mathrm{wt}(p)\,.$$

Proof: We outline the proof of this result in Appendix A. □

Let $\mathcal{Y}(\Lambda)$ denote the set of highest-lifts of the paths $p \in \mathcal{P}(\Lambda)$. For example, a few paths $p \in \mathcal{P}(2\Lambda_0)$ for $n = 2$, and their corresponding highest-lifts $\boldsymbol{\lambda} \in \mathcal{Y}(2\Lambda_0)$ are shown in Figure 1. Using the above result, we are now able to rewrite Theorem 2.2 in the following form (also due to [6]):

$$\mathrm{ch} V(\Lambda) = \sum_{\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)} e^{\mathrm{wt}(\boldsymbol{\lambda})}\,.$$

The significance of paths and highest-lifts here is in fact explained by the following theorem of Jimbo, Misra, Miwa and Okado.

Theorem 2.10 [16] *The crystal graph of $V_q(\Lambda)$ is the full subgraph of the crystal graph of $\mathcal{F}_q(\Lambda)$ with set of vertices $\mathcal{Y}(\Lambda)$.*

Thus, denoting by $(L(\Lambda), B(\Lambda))$ the crystal basis at $q = 0$ of $V_q(\Lambda)$, we can identify $B(\Lambda)$ with $\mathcal{P}(\Lambda)$ or $\mathcal{Y}(\Lambda)$. The crystal graph of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_q(2\Lambda_0)$ with vertices labelled by $\mathcal{Y}(2\Lambda_0)$ is shown in Figure 2, up to the principal degree 5.

We end this section by stating several useful results concerning Λ-paths and highest-lift multipartitions. We first give a direct characterisation of highest-lift multipartitions. It is shown in Appendix A that this characterisation is equivalent to that of [16].

Proposition 2.11 *Let $\boldsymbol{\lambda}$ be a cylindrical multipartition of highest weight Λ. Then $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ if and only if for all $k > 0$, among the colours appearing at the right ends of the length k rows of $\boldsymbol{\lambda}$, at least one element of $\{0, 1, \dots, n-1\}$ does not occur.*

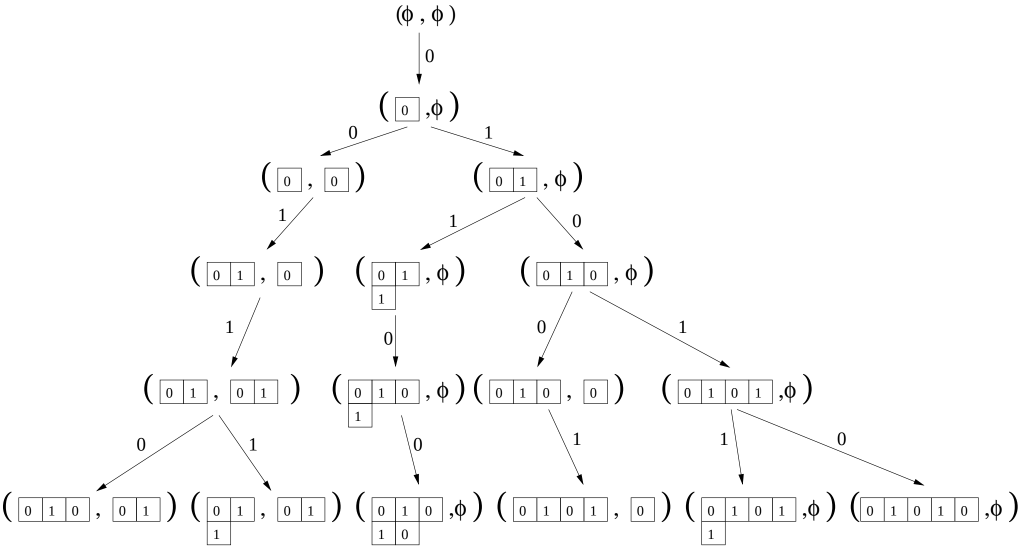
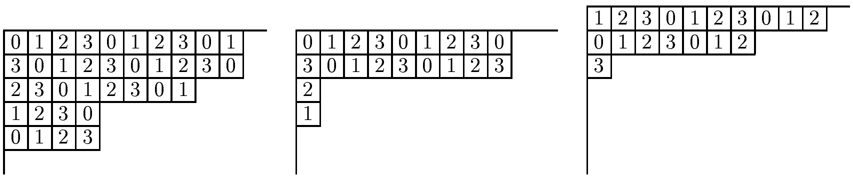


Figure 2: The crystal graph the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_q(2\Lambda_0)$ labelled by $\mathcal{Y}(2\Lambda_0)$

Note that when $\Lambda = \Lambda_i$ is a fundamental weight, we recover the usual characterisation of $\mathcal{Y}(\Lambda_i)$ as the set of *n-regular partitions*, *i.e.* partitions $\lambda = (\cdots 2^{m_2} 1^{m_1})$ with all $m_i < n$.

Example 2.12 Take $n = 4$, $\Lambda = 2\Lambda_0 + \Lambda_1$, and let $\boldsymbol{\lambda}$ be the multipartition with the following coloured diagram:



We immediately see that this is a cylindrical multipartition. To check that it is a highest-lift multipartition, we first note that 0 does not occur at the end of a row of length 1. There are no rows of length 2, 3, 5, 6, and k for $k > 10$ and so nothing to check in these cases. For rows of length 4, 7, 8, 9, and 10, we see that, for example, 1, 0, 0, 1 and 3 respectively do not occur. Thus our example is also highest-lift. $\diamond$

The following relations between a Λ-path and its highest-lift are also proved in Appendix A. We introduce some notation. Let σ denote the linear map defined by $\sigma(\Lambda_i) = \Lambda_{i-1}$. For $i \in \mathbb{Z}$, put $\alpha'_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$. Note that $\sum_{i=0}^{n-1} \alpha'_i = 0$.

Proposition 2.13 *Let $p \in \mathcal{P}(\Lambda)$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(p)$.*

1. *Let $\psi_r(k)$ be the number of r-nodes at the end of length k rows of $\boldsymbol{\lambda}$. Then,*

$$\eta_{k-1} - \sigma(\eta_k) = \sum_{j=0}^{l-1} \epsilon_{\gamma_j(k-1)} - \sum_{j=0}^{l-1} \epsilon_{\gamma_j(k)-1} = \sum_{r=0}^{n-1} \psi_r(k)\alpha'_r. \tag{13}$$

   Since $\boldsymbol{\lambda}$ is a highest-lift, one of the integers $\psi_0(k), \ldots, \psi_{n-1}(k)$ is 0, and hence these integers are unambiguously determined by Eq. (13).

2. *Let $m_r(k)$ be the number of r-nodes of $\boldsymbol{\lambda}$ belonging to columns to the right of the kth column (including the kth column). Then*

$$p_{k-1} - \overline{p}_{k-1} = -\sum_{r=0}^{n-1} m_r(k)\alpha'_r .$$

Example 2.14 We continue Example 2.12. Let $p = \pi(\boldsymbol{\lambda})$ and take $k = 9$. We have

$$\eta_8 - \sigma(\eta_9) = (\epsilon_0 + 2\epsilon_2) - (2\epsilon_0 + \epsilon_2) = -\epsilon_0 + \epsilon_2 = \alpha'_0 + \alpha'_3.$$

Correspondingly, there are two rows of length 9 having right end nodes of colours 0 and 3 respectively.

We have $p_8 = \Lambda_1 + 2\Lambda_2$ and $\overline{p}_8 = 2\Lambda_0 + \Lambda_1$, so that

$$p_8 - \overline{p}_8 = -2\Lambda_0 + 2\Lambda_2 = -\alpha'_0 + \alpha'_2 = -(3\alpha'_0 + 2\alpha'_1 + \alpha'_2 + 2\alpha'_3) .$$

On the other hand, we can check that the 9th and 10th columns of $\boldsymbol{\lambda}$ contain 8 nodes with colours $0, 0, 0, 1, 1, 2, 3, 3$. $\diamond$

Finally, we recall how to use paths or highest-lifts to compute the length of a string of the crystal graph of $V_q(\Lambda)$. Let $b \in B(\Lambda)$ and let $\varepsilon_r(b)$ (*resp.* $\varphi_r(b)$) denote the greatest integer k such that $\tilde{e}_r^k(b) \neq 0$ (*resp.* $\tilde{f}_r^k(b) \neq 0$). When b is identified with $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ or $p \in \mathcal{P}(\Lambda)$, we write as well $\varepsilon_r(b) = \varepsilon_r(\boldsymbol{\lambda}) = \varepsilon_r(p)$ and $\varphi_r(b) = \varphi_r(\boldsymbol{\lambda}) = \varphi_r(p)$. Given $\nu = \sum_j \nu_j \epsilon_j \in \mathcal{A}_l^+$, we put $\hat{\nu} = \sum_j \nu_j \Lambda_j \in P_l^+$. For $A = \sum_j a_j \Lambda_j$ we define

$$|A|_r^- = \begin{cases} -a_r & \text{if } a_r \leq 0, \\ 0 & \text{if } a_r > 0. \end{cases}$$

Proposition 2.15 [16]

1. *For $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$, $\varepsilon_r(\boldsymbol{\lambda})$ is equal to the number of normal removable r-nodes of $\boldsymbol{\lambda}$.*
2. *Let $p \in \mathcal{P}(\Lambda)$ and $\eta_k = p_{k+1} - p_k$ for all $k \geq 0$. Then*

$$\varepsilon_r(p) = \max_{k \geq 0} |p_k - \hat{\eta}_k|_r^- . \tag{14}$$

Example 2.16 Let $n = 3$, $\Lambda = 2\Lambda_1 + \Lambda_2$. Take $\boldsymbol{\lambda} = ((4, 2), (3, 1), (5))$, whose coloured diagram is

$$\begin{array}{|c|c|c|c|}\hline 1&2&0&1\\\hline 0&1&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&0\\\hline 0&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|c|c|c|}\hline 2&0&1&2&0\\\hline\end{array}$$

It is immediate from Proposition 2.11 that $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$. The r-signatures of $\boldsymbol{\lambda}$ are

$$\begin{array}{ll} r = 0 : & R_{0,1}R_{3,1}R_{6,2} \\ r = 1 : & A_{1,2}A_{1,1}(R_{1,0}A_{4,1})(R_{4,0}A_{7,2}) \\ r = 2 : & A_{-1,1}A_{-1,0}A_{2,0}A_{5,0} \end{array}$$

Hence $\varepsilon_0(\boldsymbol{\lambda}) = 3$ and $\varepsilon_1(\boldsymbol{\lambda}) = \varepsilon_2(\boldsymbol{\lambda}) = 0$. On the other hand, the path $p = \pi(\boldsymbol{\lambda})$ is given by

$$\begin{aligned}
&\eta = (122, 012, 022, 011, 222, 001, 112, 022, 001, \ldots),\\
&p = (-3\Lambda_0 + 2\Lambda_1 + 4\Lambda_2, -\Lambda_0 + \Lambda_1 + 3\Lambda_2, -\Lambda_0 + \Lambda_1 + 3\Lambda_2, 2\Lambda_1 + \Lambda_2,\\
&\quad -\Lambda_0 + \Lambda_1 + 3\Lambda_2, 2\Lambda_0 + \Lambda_1, 2\Lambda_1 + \Lambda_2, \Lambda_0 + 2\Lambda_2, 2\Lambda_0 + \Lambda_1, \ldots),\\
&p - \hat{\eta} = (-3\Lambda_0 + \Lambda_1 + 2\Lambda_2, -2\Lambda_0 + 2\Lambda_2, -2\Lambda_0 + \Lambda_1 + \Lambda_2, -\Lambda_0 + \Lambda_2,\\
&\quad -\Lambda_0 + \Lambda_1, 0, 0, 0, 0, \ldots).
\end{aligned}$$

Hence, Proposition 2.15(2) gives

$$\varepsilon_0(p) = \max(3, 2, 2, 1, 1, 0, 0, \ldots) = 3\,, \qquad \varepsilon_1(p) = \varepsilon_2(p) = 0\,.$$

$\diamond$

2.2 Tensor products, restricted paths and restricted highest-lifts

The tensor product of two $U_q(\widehat{\mathfrak{sl}}_n)$-modules carries the structure of a $U_q(\widehat{\mathfrak{sl}}_n)$-module defined via the coproduct

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i. \tag{15}$$

Let (L_1, B_1) and (L_2, B_2) be crystal bases of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$-modules M_1 and M_2. Let $B_1 \otimes B_2$ denote the basis $\{u \otimes v,\ u \in B_1,\ v \in B_2\}$ of $(L_1/qL_1) \otimes (L_2/qL_2)$. Then, Kashiwara [22, 23] has proved that $(L_1 \otimes L_2,\ B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$, with the action of $\tilde{e}_i$, $\tilde{f}_i$ on $B_1 \otimes B_2$ given by

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i u \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v), \\ u \otimes \tilde{f}_i v & \text{otherwise,} \end{cases} \tag{16}$$

$$\tilde{e}_i(u \otimes v) = \begin{cases} \tilde{e}_i u \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v), \\ u \otimes \tilde{e}_i v & \text{otherwise.} \end{cases} \tag{17}$$

This gives a powerful way of computing tensor product multiplicities for $U_q(\widehat{\mathfrak{sl}}_n)$ and $\widehat{\mathfrak{sl}}_n$. Let $\Lambda', \Lambda'', \Lambda \in P^+$. The multiplicity $c^{\Lambda}_{\Lambda'\,\Lambda''}$ of $V(\Lambda)$ in the tensor product $V(\Lambda') \otimes V(\Lambda'')$ is equal to the number of vertices $b_1 \otimes b_2$ of $B(\Lambda') \otimes B(\Lambda'')$ that satisfy

$$\mathrm{wt}\,(b_1 \otimes b_2) = \Lambda \quad \text{and} \quad \tilde{e}_i(b_1 \otimes b_2) = 0, \qquad (i = 0, \ldots, n-1).$$

By (17), this last condition is equivalent to the fact that $b_1 = b_{\Lambda'}$, the origin of the crystal graph of $V(\Lambda')$, and

$$\varepsilon_i(b_2) \leq \varphi_i(b_{\Lambda'}) = \langle \Lambda', h_i \rangle, \qquad (i = 0, 1, \ldots, n-1). \tag{18}$$

Hence we get,

Proposition 2.17 *The multiplicity $c^{\Lambda}_{\Lambda'\,\Lambda''}$ of $V(\Lambda)$ in $V(\Lambda') \otimes V(\Lambda'')$ is equal to the number of vertices b_2 of $B(\Lambda'')$ such that*

$$\mathrm{wt}\,(b_2) = \Lambda - \Lambda' \quad \textit{and} \quad \varepsilon_i(b_2) \leq \langle \Lambda', h_i \rangle, \qquad (i = 0, 1, \ldots, n-1). \tag{19}$$

From now on, we fix positive integers l', l'', $l = l' + l''$, and weights $\Lambda' \in P_{l'}^{+}$, $\Lambda'' \in P_{l''}^{+}$ and $\Lambda \in P_{l}^{+}$. We define the *branching function*

$$b^{\Lambda}_{\Lambda'\Lambda''}(z) = \sum_{k} z^k c^{\Lambda - k\delta}_{\Lambda'\Lambda''} \, . \tag{20}$$

Let us identify the crystal basis $B(\Lambda'')$ with the set $\mathcal{P}(\Lambda'')$ of Λ''-paths, as explained in Section 2.1. Using Proposition 2.17 and Eq. (14) of Proposition 2.15, it was proved in [16] that $b^{\Lambda}_{\Lambda'\Lambda''}(z)$ is equal to the generating function (with respect to the energy $E(p)$) of certain subsets of $\mathcal{P}(\Lambda'')$ coming from a class of restricted-solid-on-solid solvable models (RSOS models). We proceed to describe those paths in detail.

A level $l' + l''$ path $p = (p_k)$ is said to be (Λ', Λ'')-*restricted* if

1. $p - \Lambda' := (p_k - \Lambda') \in \mathcal{P}(\Lambda'')$, which implies that $p_{k+1} - p_k = \eta_k \in \mathcal{A}^{+}_{l''}$;
2. $p_k - \hat{\eta}_k \in P^{+}_{l'}$ for all $k \geq 0$.

The set of (Λ', Λ'')-restricted paths is denoted by $\mathcal{P}(\Lambda', \Lambda'')$. The ground state path $\overline{p} \in \mathcal{P}(\Lambda', \Lambda'')$ is defined to be $\overline{p} = \overline{p}'' + \Lambda'$ where $\overline{p}''$ is the ground state path of $\mathcal{P}(\Lambda'')$.

Example 2.18 We consider again the path $p \in \mathcal{P}(2\Lambda_1 + \Lambda_2)$ of Example 2.16. The path $p + \Lambda'$ is an element of $\mathcal{P}(\Lambda', 2\Lambda_1 + \Lambda_2)$ for all $\Lambda' = a_0\Lambda_0 + a_1\Lambda_1 + a_2\Lambda_2$ with $a_0 \geq 3, \ a_1 \geq 0, \ a_2 \geq 0$. ◇

Note that if $p \in \mathcal{P}(\Lambda', \Lambda'')$, then $p_k \in P^{+}_{l'+l''}$ for all $k \geq 0$. Note also that in the case where $\Lambda'' = \Lambda_0$, the previous definition is equivalent to Definition 2.12.1 of [11].

Since $\{p - \Lambda' \mid p \in \mathcal{P}(\Lambda', \Lambda'')\} \subset \mathcal{P}(\Lambda'')$, we can associate to the restricted path $p \in \mathcal{P}(\Lambda', \Lambda'')$ the highest-lift multipartition $\boldsymbol{\lambda}(p - \Lambda')$, that we also denote by $\boldsymbol{\lambda}(p)$, with a slight abuse of notation. Similarly, write $\ell(p) = \ell(p - \Lambda')$ and call it the *length* of $p \in \mathcal{P}(\Lambda', \Lambda'')$. Finally let

$$\mathcal{Y}(\Lambda', \Lambda'') = \{\boldsymbol{\lambda}(p) \mid p \in \mathcal{P}(\Lambda', \Lambda'')\}$$

denote the set of (Λ', Λ'')-*restricted highest-lift multipartitions.* Equivalently,

$$\mathcal{Y}(\Lambda', \Lambda'') = \{\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda'') \mid \varepsilon_i(\boldsymbol{\lambda}) \leq \langle \Lambda', h_i \rangle, \ 0 \leq i \leq n-1\} \, .$$

Theorem 2.19 [16] *The branching function is given by*

$$b^{\Lambda}_{\Lambda', \Lambda''}(z) = \sum_{\substack{p \in \mathcal{P}(\Lambda', \Lambda'') \\ p_0 = \Lambda}} z^{E(p)} = \sum_{\substack{\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda', \Lambda'') \\ \mathrm{wt}\,(\boldsymbol{\lambda}) = \Lambda - \Lambda' \,(\mathrm{mod}\, \delta)}} z^{N^0(\boldsymbol{\lambda})}.$$

Example 2.20 Take $n = 2$ and consider the branching function $b^{3\Lambda_0}_{\Lambda_0, 2\Lambda_0}(z)$. As can be seen from Figure 2, the first elements of $\mathcal{Y}(\Lambda_0, 2\Lambda_0)$ that contribute to this function are

$$(\emptyset \, , \emptyset) \, , \qquad ((3,1) \, , \emptyset) \, ,$$

whence $b^{3\Lambda_0}_{\Lambda_0, 2\Lambda_0}(z) = 1 + z^2 + \text{higher terms}$. Similarly, the first elements of $\mathcal{Y}(\Lambda_0, 2\Lambda_0)$ that contribute to $b^{\Lambda_0 + 2\Lambda_1}_{\Lambda_0, 2\Lambda_0}(z)$ are

$$((1) \, , \emptyset) \, , \qquad ((3) \, , \emptyset) \, , \qquad ((3) \, , (2)) \, , \qquad ((5) \, , \emptyset) \, ,$$

which gives $b^{\Lambda_0 + 2\Lambda_1}_{\Lambda_0, 2\Lambda_0}(z) = z + z^2 + 2z^3 + \text{higher terms}$.

Of course, one may use paths instead of multipartitions to obtain the same results. ◇

When Λ' or Λ'' is a fundamental weight, there is also a closed formula for $b^{\Lambda}_{\Lambda', \Lambda''}(z)$ in terms of theta-functions (see below, Section 3.3).

2.3 A bijection between $\mathcal{Y}(\Lambda', \Lambda'')$ and $\mathcal{Y}(\sharp\Lambda'', \sharp\Lambda')$

As indicated in Section 2.2, the tensor product $V(\Lambda') \otimes V(\Lambda'')$ may be calculated by enumerating certain elements of $\mathcal{P}(\Lambda'')$ or $\mathcal{Y}(\Lambda'')$. On the other hand, since $V(\Lambda') \otimes V(\Lambda'')$ is isomorphic to $V(\Lambda'') \otimes V(\Lambda')$, this tensor product may equally be calculated using $\mathcal{P}(\Lambda')$ or $\mathcal{Y}(\Lambda')$. Thus there exist weight-preserving bijections between $\mathcal{P}(\Lambda', \Lambda'')$ and $\mathcal{P}(\Lambda'', \Lambda')$ or between $\mathcal{Y}(\Lambda', \Lambda'')$ and $\mathcal{Y}(\Lambda'', \Lambda')$. It turns out that such bijections are not easy to describe in terms of the paths or the partitions involved. Moreover, one can check from small examples that such bijections cannot preserve the length of paths.

However, there exists an equally useful but much more easily described length-preserving bijection between $\mathcal{P}(\Lambda', \Lambda'')$ and $\mathcal{P}(\sharp\Lambda'', \sharp\Lambda')$, where $\sharp$ is the root diagram automorphism exchanging Λ_i and Λ_{-i}. This bijection is the manifestation of a simple correspondence between the tensor products of Fock spaces $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$ and $\mathcal{F}_q(\sharp\Lambda'') \otimes \mathcal{F}_q(\sharp\Lambda')$, which we discuss in Appendix B.

Definition 2.21 *Let $\Lambda = \sum_{i=0}^{n-1} a_i \Lambda_i \in P_l$, and p be a level l path, with $p_k = \sum_{i=0}^{n-1} a_i(k) \Lambda_i$. We put $\sharp\Lambda = \sum_{i=0}^{n-1} a_i \Lambda_{-i}$, and we define the path $\sharp p$ by $(\sharp p)_k = \sum_{i=0}^{n-1} a_i(k) \Lambda_{k-i}$.*

Note that $(\sharp p)_k$ differs from $\sharp(p_k)$ by a shift of indices. Hereafter, $\sharp p_k$ will always mean $(\sharp p)_k$.

Proposition 2.22 *Let $\Lambda' \in P_{l'}^+$ and $\Lambda'' \in P_{l''}$. The map $p \mapsto \sharp p$ defines a bijection between $\mathcal{P}(\Lambda', \Lambda'')$ and $\mathcal{P}(\sharp\Lambda'', \sharp\Lambda')$. Furthermore, the paths p and $\sharp p$ have the same length.*

Proof: For fixed k, we have

$$p_k = \Lambda_{w_0(k)} + \Lambda_{w_1(k)} + \cdots + \Lambda_{w_{l'+l''-1}(k)},$$

where $0 \leq w_i(k) < n$. Since p is a (Λ', Λ'')-restricted path, we may assume that $w_j(k) = \gamma_{j-l'}(k)$ for $l' \leq j < l' + l''$. Then

$$p_{k+1} = \Lambda_{w_0(k)} + \cdots + \Lambda_{w_{l'-1}(k)} + \Lambda_{w_{l'}(k)+1} + \cdots + \Lambda_{w_{l'+l''-1}(k)+1},$$

so that

$$\begin{aligned}
\sharp\eta_k &= (\sharp p_{k+1} - \sharp\Lambda'') - (\sharp p_k - \sharp\Lambda'') = \sharp p_{k+1} - \sharp p_k \\
&= (\Lambda_{k+1-w_0(k)} + \cdots + \Lambda_{k+1-w_{l'-1}(k)} + \Lambda_{k-w_{l'}(k)} + \cdots + \Lambda_{k-w_{l'+l''-1}(k)}) \\
&\quad -(\Lambda_{k-w_0(k)} + \cdots + \Lambda_{k-w_{l'-1}(k)} + \Lambda_{k-w_{l'}(k)} + \cdots + \Lambda_{k-w_{l'+l''-1}(k)}) \\
&= (\Lambda_{k+1-w_0(k)} + \cdots + \Lambda_{k+1-w_{l'-1}(k)}) - (\Lambda_{k-w_0(k)} + \cdots + \Lambda_{k-w_{l'-1}(k)}) \\
&= \epsilon_{k-w_0(k)} + \cdots + \epsilon_{k-w_{l'-1}(k)}\,.
\end{aligned}$$

Thus $\sharp\eta_k \in \mathcal{A}_{l'}^+$. Furthermore, on setting $\Lambda' = \Lambda_{u_0} + \cdots + \Lambda_{u_{l'-1}}$ and $\Lambda'' = \Lambda_{v_0} + \cdots + \Lambda_{v_{l''-1}}$, we have for each $k \geq \ell(p)$,

$$p_k = \Lambda_{u_0} + \cdots + \Lambda_{u_{l'-1}} + \Lambda_{v_0+k} + \cdots + \Lambda_{v_{l''-1}+k},$$

whence

$$\sharp p_k = \Lambda_{k-u_0} + \cdots + \Lambda_{k-u_{l'-1}} + \Lambda_{-v_0} + \cdots + \Lambda_{-v_{l''-1}},$$

which shows that $\sharp p - \sharp\Lambda''$ is an element of $\mathcal{P}(\sharp\Lambda')$. It also shows that $\ell(\sharp p - \sharp\Lambda'') \leq \ell(p - \Lambda')$. Exchanging the roles of p and $\sharp p$ yields $\ell(p - \Lambda') \leq \ell(\sharp p - \sharp\Lambda'')$, and therefore $\ell(p - \Lambda') = \ell(\sharp p - \sharp\Lambda'')$.

Now, comparing the expression

$$\sharp p_k = \Lambda_{k-w_0(k)} + \cdots + \Lambda_{k-w_{l'+l''-1}(k)}$$

with the above expression for $\sharp\eta_k$, we see that $\sharp p$ is $(\sharp\Lambda'', \sharp\Lambda')$-restricted. Finally the bijectivity of $p \mapsto \sharp p$ follows from the fact that it is an involution between two finite sets with the same cardinality. □

We now map the paths p and $\sharp p$ to their corresponding multipartitions, thereby establishing a bijection between $\mathcal{Y}(\Lambda', \Lambda'')$ and $\mathcal{Y}(\sharp\Lambda'', \sharp\Lambda')$. Let $\boldsymbol{\lambda} = \boldsymbol{\lambda}(p)$ be the highest-lift of $p \in \mathcal{P}(\Lambda', \Lambda'')$ and define $\sharp\boldsymbol{\lambda} = \boldsymbol{\lambda}(\sharp p)$ to be the highest-lift of $\sharp p$.

Theorem 2.23 *$\sharp\boldsymbol{\lambda}$ is the unique element of $\mathcal{Y}(\sharp\Lambda'', \sharp\Lambda')$ for which, for all $k > 0$ and $0 \le r < n$, the number of length k rows with right end of colour r is equal to the number of length k rows of $\boldsymbol{\lambda}$ with left end of colour $-r$. In particular, $\boldsymbol{\lambda}$ and $\sharp\boldsymbol{\lambda}$ have the same number of rows of length k, for all $k > 0$.*

Proof: Let $k > 0$ and write p_{k-1} in the form

$$p_{k-1} = \sum_{j=0}^{l'+l''-1} \Lambda_{w_j(k-1)}.$$

Then

$$\begin{aligned} p_k &= \sum_{j=0}^{l'+l''-1} \Lambda_{w_j(k-1)} + \sum_{j=0}^{l''-1} \epsilon_{\gamma_j(k-1)}, \\ p_{k+1} &= \sum_{j=0}^{l'+l''-1} \Lambda_{w_j(k-1)} + \sum_{j=0}^{l''-1} \epsilon_{\gamma_j(k-1)} + \sum_{j=0}^{l''-1} \epsilon_{\gamma_j(k)}. \end{aligned}$$

Applying the map $p \mapsto \sharp p$ yields (taking care to transform the ϵ terms correctly)

$$\begin{aligned} \sharp p_{k-1} &= \sum_{j=0}^{l'+l''-1} \Lambda_{k-1-w_j(k-1)}, \\ \sharp p_k &= \sum_{j=0}^{l'+l''-1} \Lambda_{k-w_j(k-1)} - \sum_{j=0}^{l''-1} \epsilon_{k-1-\gamma_j(k-1)}, \\ \sharp p_{k+1} &= \sum_{j=0}^{l'+l''-1} \Lambda_{k+1-w_j(k-1)} - \sum_{j=0}^{l''-1} \epsilon_{k-\gamma_j(k-1)} - \sum_{j=0}^{l''-1} \epsilon_{k-\gamma_j(k)}. \end{aligned}$$

Therefore

$$\sharp\eta_k = \sharp p_{k+1} - \sharp p_k = \sum_{j=0}^{l'+l''-1} \epsilon_{k-w_j(k-1)} + \sum_{j=0}^{l''-1} \left(\epsilon_{k-1-\gamma_j(k-1)} - \epsilon_{k-\gamma_j(k-1)} - \epsilon_{k-\gamma_j(k)}\right)$$

and

$$\sharp\eta_{k-1} = \sharp p_k - \sharp p_{k-1} = \sum_{j=0}^{l'+l''-1} \epsilon_{k-1-w_j(k-1)} - \sum_{j=0}^{l''-1} \epsilon_{k-1-\gamma_j(k-1)},$$

whereupon, if $\psi_r^\sharp(k)$ is the number of r-nodes at the right ends of the length k rows of $\sharp\boldsymbol{\lambda}$, on using Proposition 2.13 Eq. (13), we obtain:

$$\sum_{r=0}^{n-1} \psi_r^\sharp(k)\alpha'_r = \sharp\eta_{k-1} - \sigma(\sharp\eta_k) = -\sum_{j=0}^{l''-1} \epsilon_{k-2-\gamma_j(k-1)} + \sum_{j=0}^{l''-1} \epsilon_{k-1-\gamma_j(k)}.$$

In the case of $\boldsymbol{\lambda}$, expanding (13) gives:

$$\sum_{j=0}^{l''-1} \epsilon_{\gamma_j(k-1)} - \sum_{j=0}^{l''-1} \epsilon_{\gamma_j(k)-1} = \sum_{r=0}^{n-1} \psi_r(k)\alpha'_r.$$

On applying the linear map $\Lambda_i \mapsto \Lambda_{k-1-i}$, so that $\epsilon_i \mapsto -\epsilon_{k-2-i}$ and $\alpha_i \mapsto \alpha_{k-1-i}$, to both sides of this formula, we obtain:

$$-\sum_{j=0}^{l''-1} \epsilon_{k-2-\gamma_j(k-1)} + \sum_{j=0}^{l''-1} \epsilon_{k-1-\gamma_j(k)} = \sum_{r=0}^{n-1} \psi_r(k)\alpha'_{k-1-r}.$$

Thus $\psi^\sharp_r(k) = \psi_{k-1-r}(k)$ for $0 \le r < n$. Since $\psi_{k-1-r}(k)$ is the multiplicity of the colour $-r$ at the left ends of the length k rows of $\boldsymbol{\lambda}$, this multiplicity is equal to the multiplicity of the colour r at the right ends of the length k rows of $\sharp\boldsymbol{\lambda}$.

That $\sharp\boldsymbol{\lambda}$ is determined uniquely follows since the colour of the nodes at the right ends of the length k rows determine, via Eq. (13), the path $\pi(\sharp\boldsymbol{\lambda})$, which in turn determines $\sharp\boldsymbol{\lambda}$ since it is a highest-lift multipartition. □

Example 2.24 To illustrate Theorem 2.23, consider for $n = 3$ the highest-lift multipartition $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_1 + 2\Lambda_2, \Lambda_0 + \Lambda_1)$ whose coloured diagram is

$$\begin{array}{l} 0\,1\,2\,0\,1\,2\,0\,1\,2 \\ 2\,0\,1\,2\,0\,1\,2\,0 \\ 1\,2\,0\,1\,2\,0\,1 \\ 0\,1\,2\,0\,1 \\ 2\,0\,1\,2 \\ 1\,2\,0\,1 \\ 0 \\ 2 \end{array} \qquad \begin{array}{l} 1\,2\,0\,1\,2\,0\,1\,2\,0 \\ 0\,1\,2\,0\,1\,2\,0\,1\,2 \\ 2\,0\,1\,2\,0\,1\,2 \\ 1\,2\,0\,1\,2\,0 \\ 0\,1\,2\,0\,1 \\ 2\,0\,1 \\ 1\,2\,0 \end{array}$$

By definition, $\sharp\boldsymbol{\lambda}$ is a multipartition of highest weight $2\Lambda_1 + \Lambda_2$ comprising 3 partitions. From Theorem 2.23, this multipartition must have 3 rows of length 9, with left end of colour 0, 1 and 1. These can only be placed as the first rows of the 3 partitions in one way so as to obtain a cylindrical multipartition: two in the first partition and one in the second. Next, a row of length 8, beginning with colour 0 can only be placed in the second partition to obtain a cylindrical multipartition. Proceeding in this way, distributing the rows in order of decreasing length, we obtain the following multipartition

$$\begin{array}{l} 1\,2\,0\,1\,2\,0\,1\,2\,0 \\ 0\,1\,2\,0\,1\,2\,0\,1\,2 \\ 2\,0\,1\,2\,0 \\ 1\,2\,0\,1 \\ 0\,1\,2 \\ 2\,0\,1 \end{array} \qquad \begin{array}{l} 1\,2\,0\,1\,2\,0\,1\,2\,0 \\ 0\,1\,2\,0\,1\,2\,0\,1 \\ 2\,0\,1\,2\,0 \\ 1 \\ 0 \end{array} \qquad \begin{array}{l} 2\,0\,1\,2\,0\,1\,2 \\ 1\,2\,0\,1\,2\,0\,1 \\ 0\,1\,2\,0\,1\,2 \\ 2\,0\,1\,2 \end{array}$$

◇

In fact, when $l' = 1$ a simple algorithm determines $\sharp\boldsymbol{\lambda}$. Indeed, $\boldsymbol{\lambda}$ is an l''-tuple multipartition, and from the second part of Theorem 2.23, $\sharp\boldsymbol{\lambda}$ is a single partition obtained by collecting together all the rows of the constituent partitions of $\boldsymbol{\lambda}$.

Example 2.25 Let $\boldsymbol{\lambda}$ be the multipartition of Example 2.12. We can check that $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_0, 2\Lambda_0 + \Lambda_1)$. Then $\sharp\boldsymbol{\lambda} \in \mathcal{Y}(2\Lambda_0 + \Lambda_3, \Lambda_0)$ is the following single partition

0	1	2	3	0	1	2	3	0	1
3	0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1	
0	1	2	3	0	1	2	3	0	
3	0	1	2	3	0	1	2		
2	3	0	1	2	3	0			
1	2	3	0						
0	1	2	3						
3									
2									
1									

One can check that, for each k, the colours of the leftmost nodes of the rows of length k are the n-complements of those of the rightmost nodes of the rows of length k in $\boldsymbol{\lambda}$. $\diamond$

2.4 Another description of $\mathcal{Y}(\Lambda_u, \Lambda)$

Let Λ_u be a fixed fundamental weight and let $\Lambda \in P_l^+$. By means of Proposition 2.15, the multipartitions $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ which belong to $\mathcal{Y}(\Lambda_u, \Lambda)$, may be obtained by calculating the integers $\varepsilon_r(\boldsymbol{\lambda})$ using the normal nodes of $\boldsymbol{\lambda}$, and checking that $\varepsilon_u(\boldsymbol{\lambda}) \leq 1$ and $\varepsilon_j(\boldsymbol{\lambda}) = 0$ for $j \neq u$. In this section, we apply Theorem 2.23 to obtain an alternative description of $\mathcal{Y}(\Lambda_u, \Lambda)$, similar to that obtained in [12] in the case where Λ has level $l = 1$.

Theorem 2.26 *Let $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ and let $\underline{\lambda}$ be the partition which, for each $k > 0$, has as many rows of length k as $\boldsymbol{\lambda}$. Then $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_u, \Lambda)$ if and only if $\underline{\lambda} \in \mathcal{Y}(\Lambda_{-u})$ and, for each $k > 0$, the number of length k rows of $\underline{\lambda}$ with left end of colour r is equal to the number of length k rows of $\boldsymbol{\lambda}$ with right end of colour $-r$.*

Proof: The 'only if' part follows immediately from Theorem 2.23. Let us prove the converse. Suppose that $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ is such that $\underline{\lambda} \in \mathcal{Y}(\Lambda_{-u})$ and, for each $k > 0$, the number of length k rows of $\underline{\lambda}$ with left end of colour r is equal to the number of length k rows of $\boldsymbol{\lambda}$ with right end of colour $-r$. Put $p = \pi(\boldsymbol{\lambda}) \in \mathcal{P}(\Lambda)$ and write

$$p_k = \sum_{j=0}^{n-1} \beta_j(k) \Lambda_j.$$

We first prove the following

Lemma 2.27 *The path $\underline{p} := \pi(\underline{\lambda}) \in \mathcal{P}(\Lambda_{-u})$ is given by*

$$\underline{p} = \overline{p}_{\Lambda_{-u}} - \sharp\Lambda + \sharp p\,,$$

where $\overline{p}_{\Lambda_{-u}}$ denotes the ground state path of $\mathcal{P}(\Lambda_{-u})$.

Proof of Lemma 2.27: Since $p_k - \hat{\eta}_k$ is not dominant for only a finite number of k (it is always dominant if $k \geq k_*$, the length of p), we can find a dominant integral weight χ such that $p_k + \chi - \hat{\eta}_k$ is always dominant. Then $p + \chi \in \mathcal{P}(\chi, \Lambda)$ with the actual points given by

$$(p + \chi)_k = \sum_{j=0}^{n-1} (\beta_j(k) + x_j) \Lambda_j,$$

where $\chi = \sum_{j=0}^{n-1} x_j \Lambda_j$. Thence, by Proposition 2.22, $\sharp(p+\chi) \in \mathcal{P}(\sharp\Lambda, \sharp\chi)$. Theorem 2.23 shows that the multipartition $\widehat{\boldsymbol{\lambda}} \in \mathcal{Y}(\sharp\Lambda, \sharp\chi)$ corresponding to this path has, for each r and k, as many length k rows with right end of colour r as does $\boldsymbol{\lambda}$ with left end of colour $-r$. The same multipartition corresponds to the $\sharp\chi$-path $\sharp(p+\chi) - \sharp\Lambda$.

Let $m_j(k)$ be the number of j-nodes in and to the right of the kth column of the partitions composing $\widehat{\boldsymbol{\lambda}}$. By Proposition 2.13, we have

$$(\sharp(p+\chi) - \sharp\Lambda)_k = \sum_{j=0}^{n-1} x_j \Lambda_{k-j} - \sum_{j=0}^{n-1} m_j(k+1)\alpha'_j.$$

From the assumption on $\underline{\lambda}$, the rows of $\widehat{\boldsymbol{\lambda}}$ and $\underline{\lambda}$ are in bijection, and the colours of the nodes are preserved. Therefore,

$$\begin{aligned}
\underline{p}_k &= \Lambda_{k-u} - \sum_{j=0}^{n-1} m_j(k+1)\alpha'_j \\
&= \Lambda_{k-u} + (\sharp(p+\chi) - \sharp\Lambda)_k - \sum_{j=0}^{n-1} x_j \Lambda_{k-j} \\
&= \Lambda_{k-u} - \sharp\Lambda + \sum_{j=0}^{n-1} \beta_j(k)\Lambda_{k-j},
\end{aligned}$$

as claimed. ∎

We now prove that $\underline{\lambda} \in \mathcal{Y}(\sharp\Lambda, \Lambda_{-u})$. This amounts to showing that for each $k \geq 0$,

$$p^\flat_k := \Lambda_{k-u} + \sum_{j=0}^{n-1} \beta_j(k)\Lambda_{k-j} \tag{21}$$

is dominant. Assume the contrary. Then for some i, $p^\flat_i$ is dominant and $p^\flat_{i-1}$ is not. On using (12), we have

$$\begin{aligned}
p^\flat_{i-1} = p^\flat_i - \epsilon_{i-1-u-\underline{\lambda}'_i} &= \sum_{j=0}^{n-1} \beta_j(i)\Lambda_{i-j} + \Lambda_{i-u} - \epsilon_{i-1-u-\underline{\lambda}'_i} \\
&= \sum_{j=0}^{n-1} \beta_j(i)\Lambda_{i-j} + \Lambda_{i-u} - \Lambda_{i-u-\underline{\lambda}'_i} + \Lambda_{i-1-u-\underline{\lambda}'_i}.
\end{aligned}$$

Since $p^\flat_{i-1}$ is not dominant whereas $p^\flat_i$ is, it follows that $\beta_{u+\underline{\lambda}'_i}(i) = -\delta^{(n)}_{0,\underline{\lambda}'_i}$. Comparing this expression with (21) at $k = i-1$ gives

$$\sum_{j=0}^{n-1} \beta_j(i-1)\Lambda_{i-1-j} = \sum_{j=0}^{n-1} \beta_j(i)\Lambda_{i-j} + \Lambda_{i-u} - \Lambda_{i-1-u} - \Lambda_{i-u-\underline{\lambda}'_i} + \Lambda_{i-1-u-\underline{\lambda}'_i},$$

whereby, on applying the linear transformation $\Lambda_j \mapsto \Lambda_{i-1-j}$ for all j, we obtain:

$$\begin{aligned}
p_{i-1} = \sum_{j=0}^{n-1} \beta_j(i-1)\Lambda_j &= \sum_{j=0}^{n-1} \beta_j(i)\Lambda_{j-1} + \Lambda_{u-1} - \Lambda_u - \Lambda_{u+\underline{\lambda}'_i-1} + \Lambda_{u+\underline{\lambda}'_i} \\
&= \sum_{j=0}^{n-1} \beta_j(i)\Lambda_{j-1} - \epsilon_{u-1} + \epsilon_{u+\underline{\lambda}'_i-1},
\end{aligned}$$

so that

$$p_i - p_{i-1} = \sum_{j=0}^{n-1} \beta_j(i)\epsilon_{j-1} + \epsilon_{u-1} - \epsilon_{u+\underline{\lambda}'_i-1}.$$

Since $\beta_{u+\underline{\lambda}'_i}(i) = -\delta^{(n)}_{0,\underline{\lambda}'_i}$, the coefficient of $\epsilon_{u+\underline{\lambda}'_i-1}$ on the right side of this expression is negative, which contradicts the fact that p is a Λ-path. Hence $\underline{\lambda} \in \mathcal{Y}(\sharp\Lambda, \Lambda_{-u})$.

Finally, it follows from Theorem 2.23 that $\sharp\underline{\lambda} \in \mathcal{Y}(\Lambda_u, \Lambda)$ and that $\sharp\underline{\lambda} = \boldsymbol{\lambda}$. $\square$

Example 2.28 To illustrate this theorem, let $n = 4$ and $\Lambda = \Lambda_1 + 2\Lambda_3$, and consider the following highest-lift multipartition $\boldsymbol{\lambda}$ which has highest weight Λ

$$\begin{array}{l}
1\,2\,3\,0\,1\,2\,3\,0\,1\\
0\,1\,2\,3\,0\,1\,2\,3\\
3\,0\,1\,2\,3\,0\\
2\,3\,0\\
1\,2\,3
\end{array}
\qquad
\begin{array}{l}
3\,0\,1\,2\,3\,0\,1\,2\,3\,0\\
2\,3\,0\,1\,2\,3\,0\,1\,2\,3\\
1\,2\,3\,0\,1\,2\\
0\,1\,2\,3\,0\,1\\
3\,0\,1\,2\,3\\
2\\
1
\end{array}
\qquad
\begin{array}{l}
3\,0\,1\,2\,3\,0\,1\,2\\
2\,3\,0\,1\\
1\,2\,3\,0\\
0\,1\,2
\end{array}$$

Now construct the partition $\underline{\lambda}$ having, for each $k = 1, 2, \ldots$, the same number of rows of length k as does $\boldsymbol{\lambda}$. On colouring $\underline{\lambda}$ with $-u = 1$, we obtain

$$\begin{array}{l}
1\,2\,3\,0\,1\,2\,3\,0\,1\,2\\
0\,1\,2\,3\,0\,1\,2\,3\,0\,1\\
3\,0\,1\,2\,3\,0\,1\,2\,3\\
2\,3\,0\,1\,2\,3\,0\,1\\
1\,2\,3\,0\,1\,2\,3\,0\\
0\,1\,2\,3\,0\,1\\
3\,0\,1\,2\,3\,0\\
2\,3\,0\,1\,2\,3\\
1\,2\,3\,0\,1\\
0\,1\,2\,3\\
3\,0\,1\,2\\
2\,3\,0\\
1\,2\,3\\
0\,1\,2\\
3\\
2
\end{array}$$

We see that for each k, the colours at the ends of the length k rows of $\boldsymbol{\lambda}$ are the 4-complements of the colours at the beginnings of the length k rows of $\underline{\lambda}$. Using Theorem 2.26, we therefore conclude that $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_3, \Lambda_1 + 2\Lambda_3)$.

If we colour $\underline{\lambda}$ instead with $-u \neq 1$, we immediately see that the colours at the ends of the length k rows of $\boldsymbol{\lambda}$ are not the 4-complements of the colours at the beginnings of the length k rows of $\underline{\lambda}$. Therefore, $\boldsymbol{\lambda} \notin \mathcal{Y}(\Lambda_u, \Lambda_1 + 2\Lambda_3)$ if $u \neq 3$. $\diamond$

3 Ariki-Koike algebras

3.1 Ariki-Koike algebras and affine Hecke algebras

The Ariki-Koike algebra $\mathcal{H}_m(v; u_0, \ldots, u_{l-1})$ is a deformation of the group algebra of the complex reflection group $G(l, 1, m) = (\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_m$, the wreath product of a cyclic group of order l with the symmetric group $\mathfrak{S}_m$. The group $G(l, 1, m)$ can be realised as the group of monomial $m \times m$ matrices whose entries are lth roots of unity. It is generated by the permutation matrices of the elementary transpositions

$$\sigma_i = (i, i+1), \qquad i = 1, \dots, m-1,$$

together with the matrix $\sigma_0 = \mathrm{diag}\,(\omega, 1, \dots, 1)$, where ω is a primitive lth root of unity. These generators satisfy

$$\sigma_0\sigma_1\sigma_0\sigma_1 = \sigma_1\sigma_0\sigma_1\sigma_0, \tag{22}$$
$$\sigma_0^l = 1, \tag{23}$$
$$\sigma_0\sigma_i = \sigma_i\sigma_0, \qquad \text{for } i \geq 2, \tag{24}$$

in addition to the usual Moore-Coxeter relations for the generators of $\mathfrak{S}_m$.

Definition 3.1 [4, 5] *The Ariki-Koike algebra $\mathcal{H}_m(v; u_0, u_1, \dots, u_{l-1})$ is the unital associative $\mathbb{C}$-algebra generated by $T_0, T_1, \dots, T_{m-1}$ subject to the relations:*

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \leq i \leq m-2, \tag{25}$$
$$T_iT_j = T_jT_i, \qquad |i-j| > 1, \tag{26}$$
$$(T_i - v)(T_i + 1) = 0, \quad 1 \leq i \leq m-1, \tag{27}$$
$$T_0T_1T_0T_1 = T_1T_0T_1T_0, \tag{28}$$
$$(T_0 - u_0)(T_0 - u_1)\cdots(T_0 - u_{l-1}) = 0, \tag{29}$$

where $v, u_0, u_1, \dots, u_{l-1}$ are complex parameters.

For convenience, we use $\mathcal{H}_m(\boldsymbol{v})$ to denote $\mathcal{H}_m(v; u_0, u_1, \dots, u_{l-1})$. In the case where $v = 1$ and $u_j = \omega^j$, $\mathcal{H}_m(\boldsymbol{v})$ is isomorphic to the group algebra of $G(l, 1, m)$. For $l = 1$ and $l = 2$, $\mathcal{H}_m(\boldsymbol{v})$ is the Hecke algebra of types A_{m-1} and B_m respectively.

The group $G(l, 1, m)$ is a quotient of the affine Weyl group $\hat{W}_m = \mathbb{Z} \wr \mathfrak{S}_m$. Similarly, $\mathcal{H}_m(\boldsymbol{v})$ is a quotient of the affine Hecke algebra $\hat{H}_m(v)$, which is generated by invertible elements $T_1, \dots, T_{m-1}$, $y_1, \dots, y_m$ subject to the relations (25) (26) (27) of the Hecke algebra of type A_{m-1} and to

$$y_iy_j = y_jy_i, \qquad 1 \leq i, j \leq m, \tag{30}$$
$$y_jT_i = T_iy_j, \qquad \text{for } j \neq i, i+1, \tag{31}$$
$$T_iy_iT_i = v\, y_{i+1}, \qquad 1 \leq i \leq m-1. \tag{32}$$

Note that by (32), the generators $y_2, \dots, y_m$ are redundant, since

$$y_k = v^{-k+1}T_{k-1}T_{k-2}\dots T_1y_1T_1\dots T_{k-2}T_{k-1}, \qquad 2 \leq k \leq m.$$

Also, it follows from Eqs. (30) (32) that

$$T_1y_1T_1y_1 = y_1T_1y_1T_1. \tag{33}$$

In fact, $\hat{H}_m(v)$ may equally be defined as the associative algebra generated by $T_1, \dots, T_{m-1}$ and y_1, subject to the relations (25) (26) (27) (33) plus relation (31) with $j = 1$. This shows that $\mathcal{H}_m(\boldsymbol{v})$ is the quotient of $\hat{H}_m(v)$ by the single relation

$$(y_1 - u_0)(y_1 - u_1)\cdots(y_1 - u_{l-1}) = 0, \tag{34}$$

T_0 being taken equal to y_1. The images of the y_k's under the natural projection of $\hat{H}_m(v)$ onto $\mathcal{H}_m(\boldsymbol{v})$ will still be denoted by y_k. They generate a commutative subalgebra of $\mathcal{H}_m(\boldsymbol{v})$. In the case where $l = 1 = u_0$, where $\mathcal{H}_m(\boldsymbol{v})$ reduces to the Hecke algebra $H_m(v)$ of type A_{m-1}, we have $y_1 = T_0 = 1$, and the elements

$$L_k = \frac{y_k - 1}{v - 1}, \qquad 2 \le k \le m \tag{35}$$

are called the Jucys-Murphy elements of $H_m(v)$.

The affine Hecke algebra $\hat{H}_m(v)$ admits a faithful representation by symmetrisation operators acting on the ring of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ [30, 28]. In this representation, the generators of $\hat{H}_m(v)$ act by

$$y_i f = x_i^{-1} f\,, \qquad T_i f = (v-1)\pi_i f + \sigma_i f\,, \qquad f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]\,, \tag{36}$$

where σ_i is the transposition exchanging x_i and x_{i+1} and π_i is the isobaric divided difference

$$\pi_i f = \frac{x_i f - x_{i+1}\sigma_i(f)}{x_i - x_{i+1}}\,. \tag{37}$$

Since symmetric expressions in the x_i are scalars for these operators, it is clear that the power-sums $y_1^k + \cdots + y_m^k$ are in the center of $\hat{H}_m(v)$ (they are in fact generators of the center). In particular,

$$c_m = y_1 + \cdots + y_m \tag{38}$$

acts as a scalar on every irreducible representation of $\hat{H}_m(v)$ or of its quotient $\mathcal{H}_m(\boldsymbol{v})$.

3.2 Representation theory

The representation theory of $\mathcal{H}_m(\boldsymbol{v})$ has been studied in [4, 2]. We first have the following criterion of semi-simplicity.

Theorem 3.2 [2] *The algebra $\mathcal{H}_m(\boldsymbol{v})$ is semisimple if and only if the parameters satisfy*

$$v^d u_i \neq u_j \quad (\text{ for } i \neq j, d \in \mathbb{Z}, |d| < m) \qquad \text{and} \qquad [m]! \neq 0,$$

where $[m]! = \prod_{j=1}^m [j]$ and $[j] = 1 + v + \cdots + v^{j-1}$.

In the semisimple case, the irreducible representations have been constructed, and we have:

Theorem 3.3 [4] *In the case where $\mathcal{H}_m(\boldsymbol{v})$ is semisimple, the full set of non-equivalent irreducible representations of $\mathcal{H}_m(\boldsymbol{v})$ is labelled by the set of l-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$ such that $|\boldsymbol{\lambda}| = \sum_j |\lambda^{(j)}| = m$.*

The irreducible $\mathcal{H}_m(\boldsymbol{v})$-module indexed by $\boldsymbol{\lambda}$ will be denoted by $S(\boldsymbol{\lambda})$. In [4], a basis $\{b_{\boldsymbol{\mathcal{T}}}\}$ of $S(\boldsymbol{\lambda})$ is constructed. It is labelled by standard multitableaux of shape $\boldsymbol{\lambda}$, that is, l-tuples $\boldsymbol{\tau} = (\tau_0, \ldots, \tau_{l-1})$ of Young tableaux such that τ_i is of shape $\lambda^{(i)}$ and each integer from 1 to m occurs exactly once. The vectors $b_{\boldsymbol{\mathcal{T}}}$ are simultaneous eigenvectors of the y_k, and there holds

$$y_k\, b_{\boldsymbol{\mathcal{T}}} = u_{s(k)} v^{c(k)} b_{\boldsymbol{\mathcal{T}}}\,, \tag{39}$$

where $s(k)$ is the index of the component of $\boldsymbol{\tau}$ in which k occurs, and $c(k)$ is the content of the box k of $\lambda^{(s(k))}$. It follows that c_m acts on $S(\boldsymbol{\lambda})$ as the scalar

$$c_{\boldsymbol{\lambda}} = \sum_{j=0}^{l-1} u_j \left(\sum_{x \in \lambda^{(j)}} v^{c(x)} \right) . \tag{40}$$

Suppose now that the parameters are integral powers of v:

$$u_j = v^{v_j}, \qquad 0 \le j \le l-1. \tag{41}$$

Then, by Theorem 3.2, $\mathcal{H}_m(\boldsymbol{v})$ is not semisimple. Let M be any $\mathcal{H}_m(\boldsymbol{v})$-module on which c_m has a unique eigenvalue c, and let $M\!\downarrow$ denote the restriction of M to $\mathcal{H}_{m-1}(\boldsymbol{v})$. Then $M\!\downarrow$ splits as the direct sum of the generalised eigenspaces of c_{m-1}. Since all u_j are powers of v, the eigenvalues of c_{m-1} differ from those of c_m by just a power of v.

Definition 3.4 [3] *For $i \in \mathbb{Z}$, the i-restriction $M\!\downarrow_i$ of M is defined as the generalised eigenspace of c_{m-1} in $M\!\downarrow$ corresponding to the eigenvalue $c - v^i$. Similarly, the i-induced module $M\!\uparrow_i$ is the c_{m+1}-generalised eigenspace of eigenvalue $c + v^i$ in the induced module $M\!\uparrow$.*

This definition applies in particular to indecomposable $\mathcal{H}_m(\boldsymbol{v})$-modules M, and therefore can be extended to any module M. Let $\mathcal{G}_m(\boldsymbol{v}) = G_0(\mathcal{H}_m(\boldsymbol{v}))$ be the Grothendieck group of the category of finite-dimensional $\mathcal{H}_m(\boldsymbol{v})$-modules, and

$$\mathcal{G}(\boldsymbol{v}) = \bigoplus_{m \ge 0} \mathcal{G}_m(\boldsymbol{v}) .$$

As shown in [27] for $l = 1$, and in [3] for the general case, one can define on $\mathcal{G}_{\mathbb{C}}(\boldsymbol{v}) = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{G}(\boldsymbol{v})$ an action of the affine Kac-Moody algebra $\widehat{\mathfrak{gl}}_\infty$ if v is not a root of unity, and of $\widehat{\mathfrak{sl}}_n$ if v is a primitive nth root of unity, by setting

$$e_i[M] = [M\!\downarrow_i], \qquad f_i[M] = [M\!\uparrow_i] . \tag{42}$$

In both cases, $\mathcal{G}_{\mathbb{C}}(\boldsymbol{v})$ is the level l irreducible representation $V(\Lambda)$ with highest weight

$$\Lambda = \Lambda_{v_0} + \cdots + \Lambda_{v_{l-1}} .$$

Ariki's theorem can now be stated as follows:

Theorem 3.5 [3] *Under the identification of $\mathcal{G}_{\mathbb{C}}(\boldsymbol{v})$ with $V(\Lambda)$, the basis of $\mathcal{G}_{\mathbb{C}}(\boldsymbol{v})$ consisting of the classes of the irreducible modules of the various $\mathcal{H}_m(\boldsymbol{v})$ is mapped to the canonical basis of $V(\Lambda)$.*

Here, the canonical basis is understood in the sense of Lusztig. It is the same as the global upper crystal basis of Kashiwara. As a consequence, there is a one-to-one correspondence between the simple $\mathcal{H}_m(\boldsymbol{v})$-modules and the vertices of the crystal graph of $V(\Lambda)$ whose associated weight vectors have principal degree m.

Moreover, in the $l = 1$ case and when v is a primitive root of unity of prime order p, it was shown in [27] that the crystal graph coincides with the p-good lattice of Kleshchev describing socles of restricted simple modules of symmetric groups in characteristic p [25].

3.3 The generalised Jantzen-Seitz problem

Henceforth, we assume that v is a primitive nth root of unity, and that the relations (41) are satisfied. Clearly, we may also assume that $0 \le v_0 \le \ldots \le v_{l-1} < n$. We put $\mathbf{i} = (i_0, \ldots, i_{n-1})$ where i_k is the number of v_j equal to k and we write $\mathcal{H}_m(\mathbf{i})$ for the Ariki-Koike algebra with this choice of parameters. We denote by $\Lambda_{\mathbf{i}} = \sum_k i_k \Lambda_k$ the corresponding $\widehat{\mathfrak{sl}}_n$-weight.

The Jantzen-Seitz problem consists in the determination of those simple $\mathcal{H}_m(\mathbf{i})$-modules which remain irreducible after restriction to $\mathcal{H}_{m-1}(\mathbf{i})$. We shall translate this problem in the language of crystal bases.

Proposition 3.6 *Let D be a simple $\mathcal{H}_m(\mathbf{i})$-module, and let $b \in B(\Lambda_{\mathbf{i}})$ be the corresponding vertex of the crystal graph of $V(\Lambda_{\mathbf{i}})$. Fix k in $\{0, \ldots, l-1\}$. The following three conditions on $j > 0$ are equivalent:*

(i) *j is the smallest integer such that $D\!\downarrow_k^{j+1} = 0$.*

(ii) *$\varepsilon_k(b) = j$.*

(iii) *$[D\!\downarrow_k^{j}] = j!\,[D']$ for some simple $\mathcal{H}_{m-j}(\mathbf{i})$-module D'.*

The equivalence between (i) *and* (ii) *also holds for $j = 0$.*

Proof: Let $\{G(b) \mid b \in B(\Lambda_{\mathbf{i}})\}$ denote the global upper crystal basis of $V(\Lambda_{\mathbf{i}})$ (at $q = 1$). The fact that (i) and (ii) are equivalent follows from Lemma 5.1.1 (i), p. 470 of [24] and Theorem 3.5. Also, by Lemma 5.1.1 (ii) of [24], if $\varepsilon_k(b) = j$ then

$$[D\!\downarrow_k^{j}] = e_k^j[D] = e_k^j\, G(b) = j!\, G(\tilde{e}_k^j b) = j!\,[D'],$$

where D' is simple by Theorem 3.5. Conversely assume (iii), *i.e.*

$$e_k^{(j)} G(b) = G(b'), \tag{43}$$

where $e_k^{(j)} = e_k^j/j!$ is the jth divided power of e_k and b' is the vertex corresponding to D'. By repeated use of Eq. (5.3.8) of [24], we see that

$$e_k^{(j)} G(b) = \binom{\varepsilon_k(b)}{j} G(\tilde{e}_k^j b) + \text{ other terms,}$$

so that, if $j \neq 0$, (43) implies that $j = \varepsilon_k(b)$. □

Definition 3.7 *Let $\mathbf{j} = (j_0, \ldots, j_{n-1}) \in \mathbb{N}^n$. We say that a simple $\mathcal{H}_m(\mathbf{i})$-module D satisfies the generalised Jantzen-Seitz condition* $\mathrm{JS}(\mathbf{j})$ *if and only if*

$$D\!\downarrow_k^{j_k+1} = 0 \quad \text{for } k = 0, \ldots, n-1.$$

In this case, we write $D \in \mathrm{JS}(\mathbf{j})$.

It follows from Proposition 3.6 that the set of irreducible representations of $\mathcal{H}_m(\mathbf{i})$ which restrict to irreducible representations of $\mathcal{H}_{m-1}(\mathbf{i})$ is

$$\mathrm{JS}(1, 0, \ldots, 0) \cup \mathrm{JS}(0, 1, 0, \ldots, 0) \cup \cdots \cup \mathrm{JS}(0, \ldots, 0, 1)\,.$$

Proposition 3.6 also implies that, in general, the vertices b of $B(\Lambda_{\mathbf{i}})$ corresponding to simple $\mathcal{H}_m(\mathbf{i})$-modules D satisfying $\mathrm{JS}(\mathbf{j})$ are characterised by

$$\varepsilon_k(b) \leq j_k \quad \text{for } k = 0, \ldots, n-1. \tag{44}$$

For $\Lambda \in P^+$, the number of those b satisfying also $\mathrm{wt}(b) = \Lambda - \Lambda_{\mathbf{j}}$ is equal by Proposition 2.17 to the tensor product multiplicity $c^{\Lambda}_{\Lambda_{\mathbf{j}}\,\Lambda_{\mathbf{i}}}$ of $V(\Lambda)$ in $V(\Lambda_{\mathbf{j}}) \otimes V(\Lambda_{\mathbf{i}})$. Therefore, the generating function of the number of $\mathcal{H}_m(\mathbf{i})$-modules D satisfying $\mathrm{JS}(\mathbf{j})$ is a sum of branching functions of $\widehat{\mathfrak{sl}}_n$.

To state this result precisely, let us introduce some notation. For $b \in B(\Lambda_{\mathbf{i}})$, let $\deg b$ denote the homogeneous degree of b, that is, $\deg b := -\langle \mathrm{wt}\,(b)\,,\, d\rangle$. If b is labelled by a multipartition $\boldsymbol{\lambda}$, then $\deg b = N^0(\boldsymbol{\lambda})$. We write $b \in \mathrm{JS}(\mathbf{j})$ if $G(b) = [D]$ for a module $D \in \mathrm{JS}(\mathbf{j})$.

Theorem 3.8 *Let $l_{\mathbf{i}}$, $l_{\mathbf{j}}$ denote the levels of $\Lambda_{\mathbf{i}}$ and $\Lambda_{\mathbf{j}}$. We have*

$$\sum_{b \in B(\Lambda_{\mathbf{i}}) \cap \mathrm{JS}(\mathbf{j})} z^{\deg b} = \sum_{\Lambda} b^{\Lambda}_{\Lambda_{\mathbf{j}},\Lambda_{\mathbf{i}}}(z)\,,$$

where Λ runs through the weights of $P^+_{l_{\mathbf{i}}+l_{\mathbf{j}}}$ congruent to $\Lambda_{\mathbf{i}}+\Lambda_{\mathbf{j}}$ modulo the $\mathbb{Z}$-lattice spanned by the α'_i. □

When one of the factors of the tensor product is of level one, the branching functions $b^{\Lambda}_{\Lambda'\,\Lambda''}(z)$ can be explicitly evaluated in terms of theta functions [19, 17], as we shall now recall.

Let $\mathfrak{h} = (\bigoplus_{i=0}^{n-1} \mathbb{C}\, h_i) \oplus \mathbb{C}d$ and $\bar{\mathfrak{h}} = \bigoplus_{i=1}^{n-1} \mathbb{C}\, h_i$ denote the Cartan subalgebras of $\widehat{\mathfrak{sl}}_n$ and $\mathfrak{sl}_n$, respectively. One identifies $\bar{\mathfrak{h}}^*$ with the subspace $\{\Lambda \in \mathfrak{h}^* \mid \Lambda(c) = \Lambda(d) = 0\}$. The linear map $\Lambda \mapsto \bar{\Lambda}$ defined by

$$\bar{\Lambda}_i = \Lambda_i - \Lambda_0, \qquad \bar{\delta} = 0,$$

is a natural projection of $\mathfrak{h}^*$ onto $\bar{\mathfrak{h}}^*$. Note that $\bar{\alpha}_i = \alpha'_i$. The root lattice of $\mathfrak{sl}_n$ gets identified with $\bar{Q} = \bigoplus_{i=1}^{n-1} \mathbb{Z}\bar{\alpha}_i$. Let $(\cdot\,,\,\cdot)$ denote the usual bilinear form on $\mathfrak{h}^*$ defined by

$$(\Lambda_i\,,\,\Lambda_j) = \min(i,j) - \frac{ij}{n}\,, \qquad (\delta\,,\,\Lambda_i) = 1, \qquad (\delta\,,\,\delta) = 0.$$

We write $|\Lambda|^2 = (\Lambda\,,\,\Lambda)$. Recall that the Weyl group of $\mathfrak{sl}_n$ is isomorphic to $\mathfrak{S}_n$, the elementary transposition σ_i being represented by the orthogonal reflection of $\bar{\mathfrak{h}}^*$ with respect to $\bar{\alpha}_i$.

For $\mu \in \bar{\mathfrak{h}}$ and $m \in \mathbb{Z}_{>0}$, define the *theta function*

$$\Theta_{\mu,m}(z) = \sum_{\alpha \in \bar{Q}} z^{\frac{m}{2}\left|\alpha + \frac{1}{m}\mu\right|^2}\,, \tag{45}$$

and let $\eta(z) = z^{1/24}\prod_{k\geq 1}(1-z^k)$ be the Dedekind modular function. Then, for a fundamental weight Λ'', and for $\Lambda \equiv \Lambda' + \Lambda'' \bmod \bar{Q}$, Eq. (4.17) of [17] reads

$$\boldsymbol{b}^{\Lambda}_{\Lambda'\,\Lambda''}(z) = \eta(z)^{-(n-1)} \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w)\Theta_{-L(\bar{\Lambda}'+\bar{\rho})+(L-1)w(\bar{\Lambda}+\bar{\rho})\,,\,L(L-1)}(z)\;, \tag{46}$$

where $L = n+(\delta,\Lambda)$ and $\rho = \sum_{i=0}^{n-1}\Lambda_i$. The branching function $b^{\Lambda}_{\Lambda'\,\Lambda''}(z)$ of Eq. (20) differs from $\boldsymbol{b}^{\Lambda}_{\Lambda'\,\Lambda''}(z)$ by a power of z, namely

$$\boldsymbol{b}^{\Lambda}_{\Lambda'\,\Lambda''}(z) = z^{\gamma(\Lambda',\Lambda'';\Lambda)} b^{\Lambda}_{\Lambda'\,\Lambda''}(z)$$

where

$$\gamma(\Lambda',\Lambda'';\Lambda) = \frac{|\Lambda'+\rho|^2}{2(l'+n)} + \frac{|\Lambda''+\rho|^2}{2(l''+n)} - \frac{|\Lambda+\rho|^2}{2(l+n)} - \frac{|\rho|^2}{2n} \tag{47}$$

and l, l', l'' denote the respective levels of $\Lambda, \Lambda', \Lambda''$.

Example 3.9 Take $n=2$, $\Lambda' = \Lambda_0+\Lambda_1$ and $\Lambda'' = \Lambda_0$. Then $L=5$, and the nonzero branching functions correspond to $\Lambda = 2\Lambda_0+\Lambda_1$ and $\Lambda = 3\Lambda_1$. Equation (46) gives

$$\boldsymbol{b}^{2\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) = \eta(z)^{-1}\left[\Theta_{-2\bar\Lambda_1,20}(z) - \Theta_{-18\bar\Lambda_1,20}(z)\right] .$$

Here,

$$\Theta_{-2\bar\Lambda_1,20}(z) = \sum_{k\in\mathbb{Z}} z^{10\left|k\alpha_1 - \frac{1}{20}\alpha_1\right|^2} = \sum_{k\in\mathbb{Z}} z^{20\left(k-\frac{1}{20}\right)^2} ,$$

$$\Theta_{-18\bar\Lambda_1,20}(z) = \sum_{k\in\mathbb{Z}} z^{10\left|k\alpha_1 - \frac{9}{20}\alpha_1\right|^2} = \sum_{k\in\mathbb{Z}} z^{20\left(k-\frac{9}{20}\right)^2} ,$$

so that

$$\begin{aligned}\boldsymbol{b}^{2\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) &= z^{1/20}\eta(z)^{-1}\sum_{k\in\mathbb{Z}}\left(z^{20k^2-2k} - z^{20k^2-18k+4}\right)\\ &= z^{1/120}(1+z+2z^2+3z^3+4z^4+6z^5+8z^6+11z^7+\cdots) ,\end{aligned}$$

and since (47) gives $\gamma = \frac{1}{120}$, we have finally

$$b^{2\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) = 1+z+2z^2+3z^3+4z^4+6z^5+8z^6+11z^7+\cdots .$$

Likewise,

$$\begin{aligned}\boldsymbol{b}^{3\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) &= \eta(z)^{-1}\left[\Theta_{6\bar\Lambda_1,20}(z) - \Theta_{-26\bar\Lambda_1,20}(z)\right]\\ &= z^{9/20}\eta(z)^{-1}\sum_{k\in\mathbb{Z}}\left(z^{20k^2+6k} - z^{20k^2-26k+8}\right)\\ &= z^{49/120}(1+z+z^2+2z^3+3z^4+4z^5+6z^6+8z^7+\cdots) .\end{aligned}$$

Here, (47) yields $\gamma = -\frac{71}{120}$ so that

$$b^{3\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) = z+z^2+z^3+2z^4+3z^5+4z^6+6z^7+\cdots .$$

In terms of Ariki-Koike algebras, using Theorem 3.8, we have found that the generating function for the number of $\mathcal{H}_m(1,0)$-modules satisfying JS(1,1), (or equivalently for the number of $\mathcal{H}_m(1,1)$-modules satisfying JS(1,0)) is equal to

$$\begin{aligned}&b^{2\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z) + b^{3\Lambda_1}_{\Lambda_0+\Lambda_1,\Lambda_0}(z)\\ &\qquad = \prod_{i>0}\frac{1}{1-z^i}\sum_{k\in\mathbb{Z}}(z^{20k^2-2k} + z^{20k^2+6k+1} - z^{20k^2-18k+4} - z^{20k^2-26k+9})\\ &\qquad = 1+2z+3z^2+4z^3+6z^4+9z^5+12z^6+17z^7\cdots .\end{aligned}$$

◇

3.4 Labelling of modules

Let $\Lambda_{\mathbf{i}} \in P_l^+$. As explained in Section 2.1, there are (at least) three ways of labelling the vertices of the crystal graph of $V(\Lambda_{\mathbf{i}})$. One may use paths $p \in \mathcal{P}(\Lambda_{\mathbf{i}})$, their highest-lifts $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_{\mathbf{i}})$, or the set $\mathcal{M}(\Lambda_{\mathbf{i}})$ of multipartitions coming from the second q-deformation $\mathcal{F}'_q(\Lambda_{\mathbf{i}})$ of the level l Fock space $\mathcal{F}(\Lambda_{\mathbf{i}})$ (see Note 2.7).

From the point of view of Ariki-Koike algebras, there is a canonical way of labelling the irreducible representations of $\mathcal{H}_m(\mathbf{i})$. This is as follows. Under the specialisation of parameters of Section 3.3, the modules $S(\boldsymbol{\lambda})$ are in general reducible. An important property of $S(\boldsymbol{\lambda})$ is that it is endowed with a natural bilinear form compatible with the action of $\mathcal{H}_m(\mathbf{i})$. Let $\operatorname{rad} S(\boldsymbol{\lambda})$ denote the radical of this form. It was proved by Graham and Lehrer that $S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda})$ is either a simple module or 0, and that all simple modules arise this way [13]. (This generalises a similar result of Dipper and James for type A Hecke algebras [7] and Dipper, James and Murphy for type B [9].) Then Mathas, building on Ariki's theorem, characterised the set of multipartitions $\boldsymbol{\lambda}$ such that $S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda}) \neq 0$ and proved that it coincides with $\mathcal{M}(\Lambda_{\mathbf{i}})$ [31]. (Strictly speaking, Mathas' labels are obtained by conjugating and switching the components of the $\boldsymbol{\lambda} \in \mathcal{M}(\Lambda_{\mathbf{i}})$. This arises from a different convention for labelling the $S(\boldsymbol{\lambda})$. We shall ignore here such minor modifications.) This makes it natural to define

$$D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda}), \qquad (\boldsymbol{\lambda} \in \mathcal{M}(\Lambda_{\mathbf{i}})).$$

Unfortunately, it appears difficult to give a non-recursive description of $\mathcal{M}(\Lambda_{\mathbf{i}})$, whereas Proposition 2.11 provides such a simple characterisation of $\mathcal{Y}(\Lambda_{\mathbf{i}})$. This prompts us to use $\mathcal{Y}(\Lambda_{\mathbf{i}})$ as an alternative set of labels for irreducible representations of $\mathcal{H}_m(\mathbf{i})$. The bijection between the two labellings may be obtained by following a sequence of arrows back to the highest weight vertex in the crystal graph labelled by $\mathcal{Y}(\Lambda_{\mathbf{i}})$, and then applying the reversed sequence to the highest weight vertex of the crystal graph labelled by $\mathcal{M}(\Lambda_{\mathbf{i}})$. We denote the image of $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_{\mathbf{i}})$ under this bijection by $\tilde{\boldsymbol{\lambda}} \in \mathcal{M}(\Lambda_{\mathbf{i}})$. Then, for $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_{\mathbf{i}})$, define $\tilde{D}(\boldsymbol{\lambda}) = D(\tilde{\boldsymbol{\lambda}})$.

Example 3.10 The crystal graph of the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_q(2\Lambda_0)$ labelled by $\mathcal{M}(2\Lambda_0)$ is shown in Figure 3. By comparing with Figure 2, we get the following relations between the two labellings of $\mathcal{H}_m(2,0)$-modules:

$$\tilde{D}((2),(2)) = D((2,1),(1)), \qquad \tilde{D}((3),(2)) = D((3,1),(1)).$$

The other modules for $m \leq 5$ have the same label in both systems of indexation. $\diamond$

We can now apply the results of Section 2.3 and Section 2.4 and state:

Corollary 3.11

(i) *The $\mathcal{H}_m(\mathbf{i})$-module $\tilde{D}(\boldsymbol{\lambda})$ satisfies the condition* JS($\mathbf{j}$) *if and only if $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda_{\mathbf{j}}, \Lambda_{\mathbf{i}})$.*

(ii) *The $\mathcal{H}_m(\mathbf{i})$-module $\tilde{D}(\boldsymbol{\lambda})$ restricts to a simple $\mathcal{H}_{m-1}(\mathbf{i})$-module if and only if $\boldsymbol{\lambda}$ satisfies the conditions of Theorem* 2.26 *for some integer u between* 0 *and* $n-1$.

(iii) *The map $\tilde{D}(\boldsymbol{\lambda}) \mapsto \tilde{D}(\sharp\boldsymbol{\lambda})$ establishes a one-to-one correspondence between the simple $\mathcal{H}_m(\mathbf{i})$-modules satisfying* JS($\mathbf{j}$) *and the simple $\mathcal{H}_m(\sharp\mathbf{j})$-modules satisfying* JS($\sharp\mathbf{i}$), *where $\sharp\mathbf{i} = (i_0, i_{n-1}, \ldots, i_1)$.*

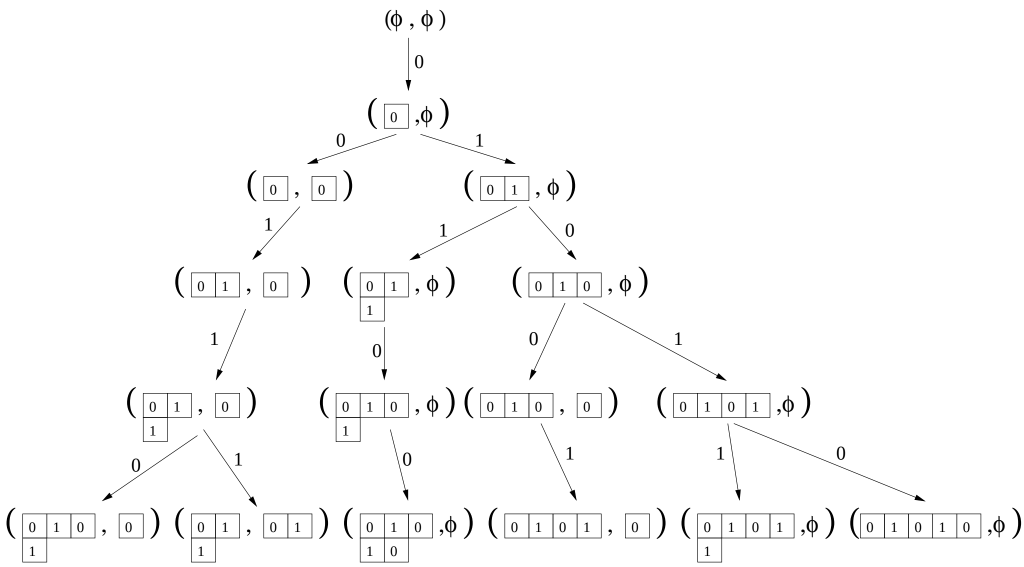


Figure 3: The crystal graph the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_q(2\Lambda_0)$ labelled by $\mathcal{M}(2\Lambda_0)$

Example 3.12 Let us determine the simple $\mathcal{H}_5(2,0)$-modules $D \in \mathrm{JS}(1,0) \cup \mathrm{JS}(0,1)$. Recall that the Hecke algebra H_m of type B_m depends on two parameters q and Q in the notation of [9], and coincides with the AK-algebra $\mathcal{H}_m(v; u_0, u_1)$ where $v = q$, $u_0 = -1$ and $u_1 = Q$. Therefore, if $q = Q = -1$, H_m coincides with $\mathcal{H}_m(0,2)$, which is isomorphic to $\mathcal{H}_m(2,0)$ by sending T_0 to $-T_0$. Thus, this question is equivalent to finding the irreducible representations of the Hecke algebra of type B_5 with both parameters q and Q equal to -1, which restrict to irreducible representations of the Hecke algebra of type B_4 (with the same parameters).

By Proposition 2.11, or by inspecting the graph of Figure 2, one determines the following list of simple $\mathcal{H}_5(2,0)$-modules:

$$\tilde{D}((3),(2)),\quad \tilde{D}((2,1),(2)),\quad \tilde{D}((3,2),\emptyset),\quad \tilde{D}((4),(1)),\quad \tilde{D}((4,1),\emptyset),\quad \tilde{D}((5),\emptyset).$$

Among them, by Corollary 3.11 (i), the modules $\tilde{D}((3),(2))$ and $\tilde{D}((5),\emptyset)$ satisfy $\mathrm{JS}(1,0)$ and the module $\tilde{D}((4),(1))$ satisfies $\mathrm{JS}(0,1)$. This can also be checked using Figure 2.

Moreover, by Corollary 3.11 (iii), $\tilde{D}((3),(2))$ and $\tilde{D}((5),\emptyset)$ are in correspondence with the $\mathcal{H}_5(1,0)$-modules $\tilde{D}(3,2)$ and $\tilde{D}(5)$, respectively, and $\tilde{D}((4),(1))$ corresponds to the $\mathcal{H}_5(0,1)$-module $\tilde{D}(4,1)$. These are the modules of each algebra satisfying the condition $\mathrm{JS}(2,0)$. Note that $\mathcal{H}_5(1,0)$ and $\mathcal{H}_5(0,1)$ are both isomorphic to the Hecke algebra of type A_4 at $q = -1$. $\diamond$

A Proofs of Propositions 2.11 and 2.13 and Lemma 2.9

Proof of 2.11. We shall prove that 2.11 is equivalent to Proposition 3.4 of [16]. Let $\boldsymbol{\lambda} = (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$. We put

$$t_{jk} = v_j - \lambda^{(j)\prime}_{k+1}, \qquad (k \geq 0,\ 0 \leq j < l), \tag{48}$$

and we extend this definition to all $j \in \mathbb{Z}$ by setting

$$t_{j+l,k} = t_{j,k} + n. \tag{49}$$

Then $\boldsymbol{\lambda}$ is a cylindrical multipartition of highest weight Λ if and only if

$$t_{j,k} \leq t_{j+1,k}, \qquad (k \geq 0,\ j \in \mathbb{Z}). \tag{50}$$

By Proposition 3.4 of [16], $\boldsymbol{\lambda} \in \mathcal{Y}(\Lambda)$ if and only if for each $k \geq 0$, there exists an integer $j_* = j_*(k)$ such that

$$t_{j_*+1,k} > t_{j_*,k+1}. \tag{51}$$

Let us show that (50) and (51) imply the characterisation given in Proposition 2.11. For $j_* < j \leq j_* + l$,

$$t_{j,k} \geq t_{j_*+1,k} > t_{j_*,k+1} = t_{j_*+l,k+1} - n \geq t_{j,k+1} - n,$$

whence $t_{j,k} > t_{j,k+1} - n$ holds for all j by (49). Thus $\lambda^{(j)\prime}_{k+1} > \lambda^{(j)\prime}_{k} - n$, so that each partition $\lambda^{(j)}$ is n-regular. Therefore, $\lambda^{(j)}$ has $t_{j,k} - t_{j,k-1} < n$ rows of length k. Using (48), we find that the colours at the end of these rows are the $\bmod n$ residues of

$$t_{j,k-1} + k, t_{j,k-1} + k + 1, t_{j,k-1} + k + 2, \ldots, t_{j,k} + k - 1, \tag{52}$$

whereupon those at the beginning of these rows are the $\bmod n$ residues of

$$t_{j,k-1} + 1, t_{j,k-1} + 2, t_{j,k-1} + 3, \ldots, t_{j,k}. \tag{53}$$

It follows that the colours appearing at the beginnings of all the rows of length k in $\boldsymbol{\lambda}$ are the $\bmod n$ residues of (53) as j varies over $j_*(k-1) < j \leq j_*(k-1) + l$. Set $j_* = j_*(k-1)$. For $0 \leq r < n$, let $r' \in \mathbb{Z}$ be the unique value such that $r' \bmod n = r$ and such that $t_{j_*,k} < r' \leq t_{j_*+l,k}$. Since $t_{j_*+l,k} = t_{j_*,k} + n < t_{j_*+1,k-1} + n$, this r' is the only possible value appearing in (53) for $j_* < j \leq j_* + l$ that gives rise to r. Let $\theta_{r',k}$ be the smallest value of θ such that $t_{\theta,k} \geq r'$. Then the number of times that r' appears in (53) for $j_* < j \leq j_* + l$, is equal to

$$\psi_{r'+k-1} = \theta_{r',k-1} - \theta_{r',k}, \tag{54}$$

for all $r' \in \mathbb{Z}$. (Note that $\theta_{r',k-1} \geq \theta_{r',k}$ for all r'.) With this definition, the property (49) of t_{jk} implies that this number is also given by $\psi_{r+k-1} = \psi_{r'+k-1}$.

In the case $r' = t_{j_*+1,k-1}$, note that $\theta_{r',k-1} = \theta_{r',k}$ so that $\psi_{r'+k-1} = 0$ implying that $t_{j_*+1,k-1} \bmod n$ does not appear at the beginning of a row of length k. Furthermore, on comparing (52) and (53), we find that the number of r-nodes at the end of length k rows of $\boldsymbol{\lambda}$ is given by ψ_r. In particular, for $r' = t_{j_*+1,k-1}$, the colour $(r' + k - 1) \bmod n$ does not appear at the end of a row of length k.

Hence (50) and (51) imply the characterisation given in Proposition 2.11. This reasoning may be reversed, thus showing the converse. □

Proof of 2.13.1. Let $p \in \mathcal{P}(\Lambda)$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(p)$. Let $\psi_r(k)$ be the number of r-nodes at the right ends of length k rows of $\boldsymbol{\lambda}$. From (12) and (48), we obtain

$$\eta_k = \sum_{j=0}^{l-1} \epsilon_{t_{j,k}+k}, \tag{55}$$

so that

$$
\begin{aligned}
\eta_{k-1} - \sigma(\eta_k) &= \sum_{j=0}^{l-1} \left(\epsilon_{t_{j,k-1}+k-1} - \epsilon_{t_{j,k}+k-1} \right) \\
&= \sum_{j=0}^{l-1} \left(\alpha'_{t_{j,k-1}+k} + \alpha'_{t_{j,k-1}+k-1} + \cdots + \alpha'_{t_{j,k}+k-1} \right).
\end{aligned}
$$

Since the subscripts here are the values in (52), we immediately obtain

$$
\eta_{k-1} - \sigma(\eta_k) = \sum_{r=0}^{n-1} \psi_r(k) \alpha'_r. \tag{56}
$$

□

Proof of 2.13.2. For $K \geq 0$, define a path $p^{(K)}$ by $p_K^{(K)} = p_K$ and

$$
p_{k+1}^{(K)} - p_k^{(K)} = \sum_{j=0}^{l-1} \epsilon_{t_{j,K+k}}, \qquad (k \geq 0). \tag{57}
$$

Then $p^{(K)}$ is the ground state of $\mathcal{P}(\Lambda^{(K)})$ where

$$
\Lambda^{(K)} = \sum_{j=0}^{l-1} \Lambda_{t_{j,K}}.
$$

This also implies that $p_{K+1}^{(K)} = p_{K+1}$, so that $p^{(K)}$ has its Kth and $(K+1)$th points in common with p. For $k, K > 0$, it follows from (57) that,

$$
p_{k-1}^{(K-1)} - p_{k-1}^{(K)} = p_k^{(K-1)} - p_k^{(K)} + \sum_{j=0}^{l-1} \epsilon_{t_{j,K}+k-1} - \sum_{j=0}^{l-1} \epsilon_{t_{j,K-1}+k-1}. \tag{58}
$$

Together, (55) and (56) imply that

$$
\sum_{j=0}^{l-1} \epsilon_{t_{j,K}+K-1} - \sum_{j=0}^{l-1} \epsilon_{t_{j,K-1}+K-1} = \sigma(\eta_K) - \eta_{K-1} = -\sum_{r=0}^{n-1} \psi_r(K) \alpha'_r, \tag{59}
$$

where $\psi_r(K)$ is the number of r-nodes appearing at the end of a length K row of $\boldsymbol{\lambda}$. Shifting the subscripts in (59) and combining with (58) yields:

$$
p_{k-1}^{(K-1)} - p_{k-1}^{(K)} = p_k^{(K-1)} - p_k^{(K)} - \sum_{r=0}^{n-1} \psi_r(K) \alpha'_{r-K+k}, \qquad (k, K > 0). \tag{60}
$$

Then, for $k \leq K$, repeated use of (60) yields:

$$
\begin{aligned}
p_{k-1}^{(K-1)} - p_{k-1}^{(K)} &= p_K^{(K-1)} - p_K^{(K)} - \sum_{j=k}^{K} \sum_{r=0}^{n-1} \psi_r(K) \alpha'_{r-K+j} \\
&= -\sum_{r=0}^{n-1} m_r(K, k) \alpha'_r,
\end{aligned}
$$

(using $p_K^{(K-1)} = p_K = p_K^{(K)}$) where $m_r(K,k)$ is the multiplicity of the colour r in the length K rows of $\boldsymbol{\lambda}$ in or to the right of the kth column. Hence, if K_* is the length of p so that $p^{(K_*)}$ is the ground state of $\mathcal{P}(\Lambda)$, then for $k \le K_*$,

$$p_{k-1} - p_{k-1}^{(K_*)} = p_{k-1}^{(k-1)} - p_{k-1}^{(K_*)} = -\sum_{i=k}^{K_*}\sum_{r=0}^{n-1} m_r(i,k)\alpha'_r = -\sum_{r=0}^{n-1} m_r(k)\alpha'_r,$$

where $m_r(k)$ is the number of r-nodes in $\boldsymbol{\lambda}$ in or to the right of the kth column. For $k > K_*$, both sides of this expression are clearly 0. □

Proof of 2.9. Let $p \in \mathcal{P}(\Lambda)$ and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(p)$. Proposition 2.13(2) implies that

$$p_0 = \Lambda + N^0(\boldsymbol{\lambda})\delta - \sum_{r=0}^{n-1} N^r(\boldsymbol{\lambda})\alpha_r.$$

Once it is established that $N^0(\boldsymbol{\lambda}) = E(p)$, we immediately obtain the desired result from the definitions (5) and (10):

$$\mathrm{wt}\,(p) = p_0 - E(p)\delta = \Lambda - \sum_{r=0}^{n-1} N^r(\boldsymbol{\lambda})\alpha_r = \mathrm{wt}\,(\boldsymbol{\lambda}).$$

The fact that $N^0(\boldsymbol{\lambda}) = E(p)$ is not immediate. The details of its verification may be extracted from Prop. 5.6 of [6]. □

B Tensor products and the $\sharp$ involution

The $\sharp$ involution allowed us to map the highest weight vertices of the crystal graph of $V_q(\Lambda')\otimes V_q(\Lambda'')$ to those of $V_q(\sharp\Lambda'')\otimes V_q(\sharp\Lambda')$. In this appendix, we sketch the construction of a similar map at the level of the $U_q(\widehat{\mathfrak{sl}}_n)$-modules.

The symmetry $i \leftrightarrow n-i$ of the Dynkin diagram of type $A_{n-1}^{(1)}$ induces an involutive automorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$e_i^\sharp = e_{-i}\,, \qquad f_i^\sharp = f_{-i}\,, \qquad (q^{h_i})^\sharp = q^{h_{-i}}\,, \qquad (q^d)^\sharp = q^d\,.$$

Given a multipartition $\boldsymbol{\lambda} = (\lambda^{(0)},\ldots,\lambda^{(l-1)})$ we define $\boldsymbol{\lambda}' = (\lambda^{(l-1)\prime},\ldots,\lambda^{(0)\prime})$ where $'$ is the conjugation of partitions. If $\boldsymbol{\lambda}$ labels a vector $v_{\boldsymbol{\lambda}}$ of $\mathcal{F}_q(\Lambda)$, so that the nodes of the main diagonal of $\lambda^{(j)}$ are coloured with $v_j \bmod n$, then we regard $\boldsymbol{\lambda}'$ as labelling a vector $v_{\boldsymbol{\lambda}'}$ of $\mathcal{F}_q(\sharp\Lambda)$ and we colour the nodes of the main diagonal of $\lambda'^{(j)}$ with $-v_{l-1-j} \bmod n$. The following lemma is a generalisation to level $l > 1$ of Lemma 7.5 of [27].

Lemma B.1 *Let $x \mapsto x'$ be the semi-linear map from $\mathcal{F}_q(\Lambda)$ to $\mathcal{F}_q(\sharp\Lambda)$ defined by*

$$\left(\sum_{\boldsymbol{\lambda}} \phi_{\boldsymbol{\lambda}}(q)\, v_{\boldsymbol{\lambda}}\right)' = \sum_{\boldsymbol{\lambda}} \phi_{\boldsymbol{\lambda}}(q^{-1})\, v_{\boldsymbol{\lambda}'}\,.$$

Then, one has

$$f_i^\sharp(u') = (q^{-1-h_i} f_i u)', \quad e_i^\sharp(u') = (q^{h_i-1} e_i u)', \qquad (u \in \mathcal{F}_q(\Lambda),\ 0 \le i \le n-1).$$

Proof: Let $\langle \cdot\,,\,\cdot\rangle$ be the scalar product on $\mathcal{F}_q(\Lambda)$ for which the natural basis $\{v_{\boldsymbol{\lambda}}\}$ is orthonormal. Using the formulae of Theorem 2.5, one can check that, for $u, v \in \mathcal{F}_q(\Lambda)$,

$$\begin{aligned}
\langle q^h u\,,\, v\rangle &= \langle u\,,\, q^h v\rangle, && (h \in \mathfrak{h}),\\
\langle f_i u\,,\, v\rangle &= \langle u\,,\, q^{h_i-1} e_i v\rangle, && (0 \le i \le n-1),\\
\langle e_i u\,,\, v\rangle &= \langle u\,,\, q^{-1-h_i} f_i v\rangle, && (0 \le i \le n-1).
\end{aligned}$$

Let us prove that

$$f_i^\sharp(u') = (q^{-1-h_i} f_i u)', \qquad (u \in \mathcal{F}_q(\Lambda),\ 0 \le i \le n-1).$$

By linearity, it is sufficient to show that

$$\langle f_i^\sharp v_{\boldsymbol{\lambda}'}\,,\, v_{\boldsymbol{\mu}'}\rangle = \langle (q^{-1-h_i} f_i v_{\boldsymbol{\lambda}})'\,,\, v_{\boldsymbol{\mu}'}\rangle \tag{61}$$

for all basis vectors $v_{\boldsymbol{\lambda}}$ of $\mathcal{F}_q(\Lambda)$ and $v_{\boldsymbol{\mu}'}$ of $\mathcal{F}_q(\sharp\Lambda)$. Using again Theorem 2.5 for $\mathcal{F}_q(\Lambda)$ and $\mathcal{F}_q(\sharp\Lambda)$, we see that

$$\langle f_i^\sharp v_{\boldsymbol{\lambda}'}\,,\, v_{\boldsymbol{\mu}'}\rangle = \overline{\langle v_{\boldsymbol{\lambda}}\,,\, e_i v_{\boldsymbol{\mu}}\rangle}\,,$$

where, for $\phi(q) \in \mathbb{Q}(q)$, we set $\overline{\phi(q)} = \phi(q^{-1})$. Then,

$$\overline{\langle v_{\boldsymbol{\lambda}}\,,\, e_i v_{\boldsymbol{\mu}}\rangle} = \overline{\langle q^{-1-h_i} f_i v_{\boldsymbol{\lambda}}\,,\, v_{\boldsymbol{\mu}}\rangle} = \langle (q^{-1-h_i} f_i v_{\boldsymbol{\lambda}})'\,,\, v_{\boldsymbol{\mu}'}\rangle$$

whence the result. The formula for $e_i^\sharp(u')$ is proved similarly. $\square$

Consider now the tensor product $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$ with the $U_q(\widehat{\mathfrak{sl}}_n)$-module structure determined by the comultiplication (15). Define a semi-linear map $'$ from $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$ to $\mathcal{F}_q(\sharp\Lambda'') \otimes \mathcal{F}_q(\sharp\Lambda')$ by

$$u \otimes v \mapsto (u \otimes v)' = v' \otimes u'\ . \tag{62}$$

Introducing on $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$ the scalar product $\langle \cdot\,,\,\cdot\rangle$ for which the tensors $v_{\boldsymbol{\lambda}} \otimes v_{\boldsymbol{\mu}}$ form an orthonormal basis, and arguing as in the proof of Lemma B.1, we obtain:

Lemma B.2 *Let $w \in \mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$. Then,*

$$f_i^\sharp(w') = (q^{-1-h_i} f_i w)'\,, \quad \text{and} \quad e_i^\sharp(w') = (q^{h_i-1} e_i w)'\ .$$

In particular, if w is a highest weight vector of weight Λ in $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$, then w' is a highest weight vector of weight $\sharp\Lambda$ in $\mathcal{F}_q(\sharp\Lambda'') \otimes \mathcal{F}_q(\sharp\Lambda')$.

Let $V_q(\Lambda) \subset \mathcal{F}_q(\Lambda)$ be the irreducible component generated by the vacuum vector $\emptyset_\Lambda$ of $\mathcal{F}_q(\Lambda)$. The tensor product $V_q(\Lambda') \otimes V_q(\Lambda'')$ is the submodule of $\mathcal{F}_q(\Lambda') \otimes \mathcal{F}_q(\Lambda'')$ generated by the action of $U_q(\widehat{\mathfrak{sl}}_n) \otimes U_q(\widehat{\mathfrak{sl}}_n)$ on $\emptyset_{\Lambda'} \otimes \emptyset_{\Lambda''}$. From Lemma B.1, we see that $V_q(\Lambda') \otimes V_q(\Lambda'')$ is mapped by $'$ onto $V_q(\sharp\Lambda'') \otimes V_q(\sharp\Lambda')$, and from Lemma B.2, we deduce that the map $'$ respects the decomposition into $U_q(\widehat{\mathfrak{sl}}_n)$-modules.

In particular, highest weight vectors of weight Λ of $V_q(\Lambda') \otimes V_q(\Lambda'')$ are mapped by $'$ to highest weight vectors of weight $\sharp\Lambda$ of $V_q(\sharp\Lambda'') \otimes V_q(\sharp\Lambda')$.

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