Algebraic Combinatorics with MAPLE and ACE

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We illustrate the use of *Maple* in algebraic combinatorics with three examples: double Schubert polynomials, ribbon tableaux and noncommutative symmetric functions. Most of the calculations presented in this paper were done by means of **ACE**, a collection of *Maple* packages developed at the University of Marne-la-Vallée.

Introduction

Experimentation in Algebraic Combinatorics, as in any other branch of Discrete Mathematics, requires large samples of data obtained by means of computer programs. Due to the algebraic aspects of the field, these data can then be processed in many different ways, e.g. one can compute generating polynomials for different statistics on a given set of combinatorial objects, try to factorize these polynomials, use them as coefficients of symmetric functions, specialize the variables, and so on. A computer algebra system like *Maple* is well suited for this kind of manipulations. Specialized *Maple* packages for handling the usual objects of Algebraic Combinatorics (permutations, words, Young tableaux, symmetric functions, Schubert polynomials etc.) have been written in recent years and regrouped in a coherent environment called **ACE** (Algebraic Combinatorics Environment), developed at the University of Marne-la-Vallée.

Here, we illustrate the use of *Maple* and **ACE** on three problems rather representative of the field. The first one is the search for a "Pieri formula" for double Schubert polynomials. Experimental data were easily obtained by means of the **SP** package of **ACE**, and we explain how it was analyzed and eventually led to a correct conjecture. The second one is a problem on symmetric functions, formulated in terms of generalizations of Young tableaux called ribbon (or rim-hook) tableaux. We explain how the generating functions (spin polynomials) of these objects have been implemented. Taking the spin polynomials as coefficients of symmetric functions and specializing the parameter at a root of unity, we arrived at an interesting conjecture in representation theory (which is still unproved at the time of writing). Finally, we explain how the same kind of manipulations with noncommutative symmetric functions (using **NCSF**) led to the discovery of a one-parameter family of idempotents in the group algebra of the symmetric group, interpolating between all known examples of Lie idempotents, i.e. idempotents projecting a homogeneous component of the free associative algebra onto the free Lie algebra.

The choice of *Maple* for developing **ACE** was motivated by the fact that this system, which is widely used among combinatorists, provides a good interface and can be used on all classical computer systems (*Unix*, *MsDos* or *Macintosh*).

Schubert polynomials

Computations in the ring $\mathbb{Z}[\mathbf{x}]$ of polynomials in several variables $\mathbf{x} := \{x_1, \ldots, x_n\}$ become rapidly intractable since dimensions grow very quickly with the number of variables and with the degree. For certain specific problems, appropriate linear bases allow to compact the data, but we need to keep track of the multiplicative structure. Geometry motivates the definition of such bases ([3],[1]).

Lascoux and Schützenberger [14] defined *the simple Schubert polynomials*, which form a basis of $\mathbb{Z}[\mathbf{x}]$ as a free module over the ring of symmetric functions. This basis is indexed by permutations of $\mathfrak{S}_n$ (and thus is finite) and algebraic manipulations of simple Schubert polynomials mostly reduce to combinatorial constructions on permutations.

Computations can be simplified by introducing a second set of parameters $\mathbf{y} := \{y_1, \ldots, y_n\}$ and defining the corresponding (double) *Schubert polynomials* [11] as follows. The maximal Schubert polynomial $\mathbb{X}_{w_0}$ with $w_0 := (n, \ldots, 2, 1)$ is:

$$\mathbb{X}_{w_0} := \prod_{i+j \leq n} (x_i - y_j) \quad ,$$

and the general Schubert polynomials are all its images by divided differences [15] acting only on $\mathbf{x}$. More precisely:

$$\mathbb{X}_w := \partial_{w^{-1}w_0}(\mathbb{X}_{w_0}) \quad ,$$

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where $\partial_{w^{-1}w_0}$ denotes the divided difference associated with the permutation $w^{-1}w_0$ (*cf.* [15]).

Simple Schubert polynomials $\mathbf{X}_w$ are obtained by specializing all y_i to 0 in $\mathbb{X}_w$. They contain as a subfamily the Schur functions which form a fundamental linear basis of the ring of symmetric functions. Combinatorics on symmetric functions mostly involves partitions and Young tableaux and extends to a combinatorics on permutations to deal with Schubert polynomials.

For example, the multiplicative structure of $\mathbb{Z}[\mathbf{y}][\mathbf{x}]$ with basis Schubert polynomials relies on the Monk formula [16], [8]. This formula gives the decomposition on the Schubert basis of the product of any Schubert polynomial by a single variable. The remarkable fact is that there is no multiplicity in this multiplication, *e.g.* $x_4\,\mathbb{X}_{5,2,7,3,6,4,1} = -\mathbb{X}_{5,3,7,2,6,4,1} + \mathbb{X}_{5,2,7,6,3,4,1} + \mathbb{X}_{5,2,7,4,6,3,1} + y_3\mathbb{X}_{5,2,7,3,6,4,1}$.

On the other hand, the multiplicative structure of the ring of symmetric functions is given by Pieri's formula that gives the decomposition on the Schur basis of the product of a Schur function by a complete or elementary symmetric function. Lascoux and Schützenberger have given the corresponding Pieri formula [14] for simple Schubert polynomials, *i.e.* the product of a simple Schubert polynomial by a complete or elementary symmetric function. This Pieri formula still has no multiplicity.

We were naturally led to investigate a Pieri formula for Schubert polynomials in $\mathbf{x}$ and $\mathbf{y}$. For that purpose, *Maple* was used to experiment, with **ACE** and more precisely the **SP** package devoted to computations with Schubert polynomials.

Elementary symmetric functions are defined via the recursion: $e_i(x_1,\ldots,x_n) = x_n\,e_{i-1}(x_1,\ldots,x_{n-1}) + e_i(x_1,\ldots,x_{n-1})$, implemented as follows:

```
> e2x:=proc(i, n) local j;
>   if (i=0) then
>     1
>   elif (i>n) then
>     0
>   else
>     x.n*e2x(i-1, n-1) + e2x(i, n-1)
>   fi
> end;
```

The **SP** package provides the `ToXX` function that transforms any algebraic expression in $\mathbf{x}$, $\mathbf{y}$ and Schubert polynomials into a linear combination of Schubert polynomials by iterating Monk's formula. The following *Maple* procedure computes then the decomposition on the general Schubert basis of the product of $e_i(x_1,\ldots,x_n)$ by $\mathbb{X}_w$.

```
> Pieri:=proc(i, n, w)
>   ToXX(e2x(i, n)*XX[op(w)], 'collect');
> end;
```

Now, a sample computation such as:

```
> w:=[4,7,1,5,6,2,3]:
> Pieri(3, 5, w);
```

$(y_5y_4y_7 + y_6y_5y_1 + y_6y_1y_7 + y_6y_5y_7 + y_4y_1y_7 + y_6y_5y_4 + y_5y_1y_7 + y_6y_1y_4 + y_6y_4y_7 + y_5y_1y_4)\mathbb{X}_{4,7,1,5,6,2,3} + (y_5y_7 + y_1y_7 + y_5y_4 + y_4y_7 + y_1y_4 + y_5y_1)\mathbb{X}_{4,7,1,5,8,2,3,6} + (y_1 + y_4 + y_7)\mathbb{X}_{4,7,1,6,8,2,3,5} + (y_4y_7 + y_5y_7 + y_6y_5 + y_5y_4 + y_6y_7 + y_6y_4)\mathbb{X}_{4,7,2,5,6,1,3} + \mathbb{X}_{4,8,1,6,7,2,3,5} + (y_6y_1 + y_5y_1 + y_6y_5 + y_1y_4 + y_5y_4 + y_6y_4)\mathbb{X}_{4,8,1,5,6,2,3,7} + (y_4 + y_5 + y_7)\mathbb{X}_{4,7,2,5,8,1,3,6} + \mathbb{X}_{5,7,1,6,8,2,3,4} + \mathbb{X}_{4,7,2,6,8,1,3,5} + (y_4 + y_5 + y_1)\mathbb{X}_{4,8,1,5,7,2,3,6} + (y_6 + y_4 + y_5)\mathbb{X}_{4,8,2,5,6,1,3,7} + \mathbb{X}_{4,8,2,5,7,1,3,6}$

shows that coefficients in the expansion are symmetric polynomials (and in fact elementary symmetric functions) in some subsets of $\mathbf{y}$. Pieri's formula for simple Schubert polynomials is obtained by specializing all y_k to 0. All coefficients e_i are sent to 0 except $e_0 = 1$. Thus, the coefficients "1" in Pieri's formula for simple Schubert polynomials should be interpreted as elementary symmetric functions of degree 0.

To further describe the previous output $\sum c_\nu\,\mathbb{X}_\nu$ of the computation of $e_i(x_1,\ldots,x_n)\,\mathbb{X}_w$, we shall rather write $\sum c_\nu\,\mathrm{cycle}(\nu w^{-1})$ where $\mathrm{cycle}(\nu w^{-1})$ is the cycle decomposition of νw^{-1}:

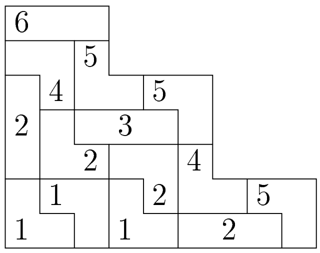
```
> XX2Cycle:=proc(expr, w)
>   if (type(expr, `+`) or type(expr, `*`)) then
>     map(XX2Cycle, expr, w)
>   elif (type(expr, indexed)) then
>     Perm2Cycle(MultPerm([op(expr)],
>                InvPerm(w)))
>   else
>     expr
>   fi
> end:
> XX2Cycle(Pieri(3, 5, w), w);
```

$(y_5y_4y_7 + y_6y_5y_1 + y_6y_1y_7 + y_6y_5y_7 + y_4y_1y_7 + y_6y_5y_4 + y_5y_1y_7 + y_6y_1y_4 + y_6y_4y_7 + y_5y_1y_4)[] + (y_5y_7 + y_1y_7 + y_5y_4 + y_4y_7 + y_1y_4 + y_5y_1)[[6,8]] + (y_1 + y_4 + y_7)[[5,6,8]] + [[1,2],[5,6,8]] + (y_4 + y_5 + y_7)[[1,2],[6,8]] + (y_4y_7 + y_5y_7 + y_6y_5 + y_5y_4 + y_6y_7 + y_6y_4)[[1,2]] + [[5,6,7,8]] + (y_4 + y_5 + y_1)[[6,7,8]] + (y_6y_1 + y_5y_1 + y_6y_5 + y_1y_4 + y_5y_4 + y_6y_4)[[7,8]] + [[4,5,6,8]] + [[1,2],[6,7,8]] + (y_6 + y_4 + y_5)[[1,2],[7,8]]$

The combinatorial description of the cycles appearing in this formula is very simple and is obtained from the case of simple Schubert polynomials (see [14]). This type of experiments leads us to the obtention of an explicit Pieri formula for (double) Schubert polynomials.

Generating functions of ribbon tableaux

The investigation of certain problems in representation theory leads to the consideration of *k-ribbon tableaux*, which are generalizations of Young tableaux obtained by tiling a Ferrers diagram with k-ribbons (or rim hooks) instead of boxes, and filling them with numbers weakly increasing along rows and strictly increasing along columns [20, 12]. The precise condition for columns is that the lowest and rightmost cell (called the root) of a ribbon labelled i should not be on the top of a ribbon labelled j if $j \geq i$. A typical 3-ribbon tableau is shown below:



With a ribbon tableau T, one associates a monomial x^T by associating with each ribbon labelled i the variable x_i and taking the product. For example, the monomial corresponding to the above example is $x_1^3 x_2^4 x_3^1 x_4^2 x_5^3 x_6$. The sequence $\mu = (3, 4, 1, 2, 3, 1)$ of exponents of x^T is called the *weight* of T, while the sequence $\lambda = (9, 9, 6, 6, 6, 3, 3)$ of lengths of rows of T is the *shape* of T.

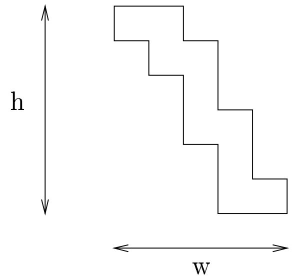
Let $Tab_k(\lambda, \mu)$ denote the set of all k-ribbon tableaux of shape λ and weight μ. The 1-ribbon tableaux are just ordinary Young tableaux, and the combinatorial expansion of Schur polynomials:

$$s_\lambda(\mathbf{x}) = \sum_{T \in Tab_1(\lambda,.)} x^T ,$$

can be generalized to:

$$G_\lambda^{(k)}(\mathbf{x}) = \sum_{T \in Tab_k(\lambda,.)} x^T .$$

It is not difficult to show that when this polynomial is not zero (*i.e.* when λ has empty k-core), $G_\lambda^{(k)}(\mathbf{x})$ is equal to a product of Schur functions $s_{\lambda^{(0)}}(\mathbf{x}) \ldots s_{\lambda^{(k-1)}}(\mathbf{x})$, where $(\lambda^{(0)}, \ldots, \lambda^{(k-1)})$ is the k-quotient of λ. However, for $k \geq 2$, one can read on ribbon tableaux an information which is not available on ordinary Young tableaux, namely the *heights* of the ribbons:



Define the spin of a ribbon tableau as:

$$s(T) = \frac{1}{2} \sum_{R \in T} (h(R) - 1) ,$$

and introduce the q-generating function:

$$G_\lambda^{(k)}(\mathbf{x}; q) = \sum_{T \in Tab_k(\lambda,.)} q^{s(T)}\, x^T .$$

It is no longer obvious that $G_\lambda^{(k)}(\mathbf{x}; q)$ is actually symmetric in the x_i's. This was proved quite recently using techniques from the theory of quantum affine algebras [12]. Assuming that this is true we could write:

$$G_\lambda^{(k)}(\mathbf{x}; q) = \sum_\mu \Big(\sum_{T \in Tab_k(\lambda,\mu)} q^{s(T)} \Big) m_\mu(\mathbf{x}) ,$$

where m_μ is the monomial symmetric polynomial indexed by the partition μ.

It appeared on small examples that the symmetric functions $H_\lambda^{(k)}(\mathbf{x}; q) = G_{k\lambda}^{(k)}(\mathbf{x}; q)$ form for each $k \geq 1$ a basis of the ring of symmetric functions interpolating between the Schur functions $s_\lambda = H_\lambda^{(1)}$ and the Hall-Littlewood functions $Q'_\lambda = H_\lambda^{(N)}(\mathbf{x})$ (N sufficiently large). By analogy with the results of [13] and [2], we were led to conjecture that in the case $\lambda = \nu^k := (\nu_1, \ldots, \nu_1, \nu_2, \ldots \nu_2, \ldots)$ (each part repeated k times), $H_\lambda^{(k)}(\mathbf{x}; q)$ has interesting properties when q is specialized to a primitive k-th root of unity. To get some confidence in this conjecture we needed some large examples. The first really significant ones we were interested in being $H_{222111}^{(3)}(\mathbf{x}, e^{2\pi i/3})$ and $H_{22221111}^{(4)}(\mathbf{x}, i)$.

The resulting conjecture in representation theory can be stated as follows. Let V_ν be a polynomial irreducible representation of $GL(n, \mathbb{C})$ with highest weight ν, and let γ be the endomorphism of $V_\nu^{\otimes k}$ such that $\gamma(v_1 \otimes \cdots \otimes v_k) = v_2 \otimes \cdots \otimes v_k \otimes v_1$. Let $E^{(r)}$ be the eigenspace of γ with eigenvalue ω^r, where $\omega = e^{2ri\pi/k}$. $E^{(r)}$ is $GL(n)$-stable, and the multiplicity of the irreducible V_μ in $E^{(r)}$ is equal to the coefficient of s_μ in the reduction modulo $1 - q^k$ of $H_\lambda^{(k)}(\mathbf{x}; q)$ with $\lambda = \nu^k$. In terms of symmetric functions this amounts to say that:

$$H_\lambda^{(k)}(\mathbf{x}, \omega) = (-1)^{(k-1)|\nu|} s_\nu(x_1^k, x_2^k, \ldots) .$$

To compute the spin polynomials:

$$\sigma_{\lambda\mu}^{(k)} = \sum_{T \in Tab_k(\lambda,\mu)} q^{s(T)} \ ,$$

we used a *Maple* program based on the following properties. First, $\sigma_{\lambda\mu}^{(k)} = q^{n(k-1)/2} F_{\alpha\mu}^{(k)}(q^{-1/2})$, where $\alpha = \lambda' + \rho_n$, $\rho_n = (n-1, n-2, \ldots, 1, 0)$, $n \geq \lambda_1$, and the polynomials $F_{\alpha\mu}^{(k)}$ are defined by:

$$F_{\alpha\mu}^{(k)}(q) = \sum_{T \in Tab_k(\lambda,\mu)} \sum_{R \in T} q^{w(R)-1} \ ,$$

$w(R)$ being the width of R. These polynomials satisfy the recurrence relation:

$$F_{\alpha\mu}^{(k)}(q) = \sum_p q^{I_p} F_{\alpha_p,\mu^*}^{(k)}(q) \ ,$$

where $\mu = (\mu_1, \ldots, \mu_r)$, $\mu^* = (\mu_1, \ldots, \mu_{r-1})$ and the sequences α_p are obtained in the following way. Subtract to α all the distinct permutations of the vector $[0, \ldots, 0, k, \ldots, k]$ (n components, k repeated μ_r times). Among the resulting vectors, discard all those having at least two identical components, and sort the other ones in decreasing order, storing the inversion number of the sorting permutation. The α_p are the resulting sorted vectors, and the I_p are the corresponding inversion numbers.

The F polynomials are computed by the *Maple* program `F` given below, which is the core of the H-function program. The conversion to the Schur basis can be performed either by Stembridge's package **SF** (in the share library) or by the **SYMF** package of **ACE**. Note the use of the `option remember` which allows the simple implementation of a complicated recursion. The auxiliary function:

```
> q_inv := proc(perm) ... end
```

returns q^I where I is the inversion number of a permutation `perm`.
For example, the two domino tableaux of shape $(4,4)$ and weight $(3,1)$ and the five standard domino tableaux of the same shape have respective generating functions given by:

```
> F(2, [2,3,4,5], [3,1]);
```

$$q^2 + 1$$

```
> F(2, [2,3,4,5], [1,1,1,1]);
```

$$2\,q^4 + 3\,q^2 + 1$$

Similarly, a call to `F(4,[4,5,6,7,12,13,14,15], [1$12])` gives the generating function of the 5913600 standard 4-ribbon tableaux of shape (88884444) needed to compute the coefficient of $m_{(1^{12})}$ in $H^{(4)}_{22221111}$, that is:

$$\begin{array}{llllllll}
275\,q^{36} & + & 2596\,q^{34} & + & 12881\,q^{32} & + \\
44648\,q^{30} & + & 119438\,q^{28} & + & 258259\,q^{26} & + \\
462332\,q^{24} & + & 694552\,q^{22} & + & 883474\,q^{20} & + \\
956690\,q^{18} & + & 883474\,q^{16} & + & 694552\,q^{14} & + \\
462332\,q^{12} & + & 258259\,q^{10} & + & 119438\,q^{8} & + \\
44648\,q^{6} & + & 12881\,q^{4} & + & 2596\,q^{2} & + \\
275 & & & & &
\end{array}$$

```
> ## computes the polynomial F^(k)_{alpha,mu}
> ## alpha is entered as an increasing sequence
>
> F := proc(k, alpha, mu)
>   local
>     r,weight,i,length,j,perm,fact,res,ll;
>   option remember;
>   r:=nops(mu); length:=nops(alpha);
>   for i to length while alpha[i]=i-1 do od;
>   if i=length+1 then
>     res:=1
>   elif r=0 then
>     res:=0
>   else
>     weight:=[seq(0, i=1..length-mu[r]),
>              seq(k, i=1..mu[r])];
>     ll:=combinat['permute'](weight);
>     res:=0;
>     for i in ll do
>       perm:=[seq(alpha[j]-i[j], j=1..length)];
>       fact:=invers(perm);
>       if fact=0 then
>         next
>       else
>         res:=expand(res+fact*
>                     F(k,sort(perm),
>                       [op(1..r-1,mu)]))
>       fi
>     od
>   fi;
>   res
> end ;
```

Noncommutative symmetric functions

We give a brief introduction to the theory of noncommutative symmetric functions (*cf.* [7, 9, 4, 10] for more details). The algebra of *noncommutative symmetric functions* is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \ldots\rangle$ generated by an infinite sequence of noncommutative indeterminates $(S_n)_{n\geq 1}$, called *complete* symmetric functions. One defines then two noncommutative analogues Ψ_n and Φ_n of commutative power sums symmetric functions by setting:

$$\Psi_n = \sum_{i_1+\cdots+i_r=n} (-1)^{r-1}\ i_r\ S_{i_1}\ \ldots\ S_{i_r}\ , \qquad (1)$$

$$\Phi_n = \sum_{i_1+\cdots+i_r=n} (-1)^{r-1}\, \frac{n}{r}\, S_{i_1}\, \ldots\, S_{i_r}\ . \qquad (2)$$

A composition of n is a sequence I of strictly positive integers whose sum $|I|$ is equal to n. For any composition $I = (i_1, \ldots, i_r)$, one defines $S^I = S_{i_1} \ldots S_{i_r}$. The homogeneous component $\mathbf{Sym}_n$ of order n of $\mathbf{Sym}$ is then the subspace of $\mathbf{Sym}$ generated by all S^I where I is a composition of n. Note that the family (S^I) forms an homogeneous basis of $\mathbf{Sym}$. The set of all compositions of an integer n can be equipped with the *reverse refinement order*, denoted $\preceq$. One has for example $(5,2,3) \preceq (1,4,2,1,2)$. The noncommutative *ribbon Schur* functions (R_I) are then defined by:

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)}\, S^J\ , \qquad (3)$$

where $\ell(I)$ denotes the *length* of I. The family (R_I) is also an homogeneous basis of $\mathbf{Sym}$.

We recall that an integer $i \in [1, n-1]$ is said to be a *descent* of $w \in \mathfrak{S}_n$ iff $w(i) > w(i+1)$. The set $\mathrm{Des}(w)$ of these integers is called the *descent set* of w. We also associate with any composition $I = (i_1, \ldots, i_r)$ of n the subset $D(I) = \{i_1, i_1+i_2, \ldots, i_1+\cdots+i_r\}$ of $[1, n-1]$. Solomon showed that the subspace of $\mathbb{Q}[\mathfrak{S}_n]$ generated by the elements:

$$D_I = \sum_{\mathrm{Des}(w)=D(I)} w\ ,$$

is a subalgebra of $\mathbb{Q}[\mathfrak{S}_n]$ (see [19]). This subalgebra is called the *descent algebra* of $\mathfrak{S}_n$ and is denoted by Σ_n. One can now define a linear isomorphism:

$$\alpha :\ \mathbf{Sym} = \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n \ \longrightarrow\ \Sigma = \bigoplus_{n=0}^{\infty} \Sigma_n\ ,$$

by setting $\alpha(R_I) = D_I$ for every composition I. It is interesting to notice that the image of S^I by α is equal to:

$$D_{\subset I} = \sum_{\mathrm{Des}(w)\subset D(I)} w\ .$$

A *Lie idempotent* is an idempotent of the algebra of some symmetric group of fixed order that projects an homogeneous component of the free associative algebra onto the free Lie algebra (see *e.g.* [6, 18]). It turns out that several classical Lie idempotents belong to Solomon's descent algebra, which allows us to interpret them as noncommutative symmetric functions (using α^{-1}). It can for instance be proved that the image by α^{-1} of Dynkin's Lie idempotent $\frac{1}{n}\,\theta_n$, where:

$$\begin{aligned}\theta_n &= [[\ldots[[1,2],3],\ldots],n] \\ &= \sum_{k=0}^{n-1} (-1)^k\, D_{=\{1,2,\ldots,k\}}\ ,\end{aligned}$$

(and $[x,y]$ stands for the Lie bracketing $xy - yx$), is exactly $\frac{1}{n}\,\Psi_n$. The noncommutative symmetric function $\frac{1}{n}\,\Phi_n$ can be interpreted in the same way as another classical Lie idempotent, *i.e.* Solomon's Eulerian idempotent (see [5, 7, 17]).

The complete symmetric functions of the alphabet $A/\,(1-q)$ are the noncommutative symmetric functions $S_n(\frac{A}{1-q})$ defined by:

$$\begin{aligned}\sigma(t; \frac{A}{1-q}) &= 1 + \sum_{n=1}^{+\infty} S_n(\frac{A}{1-q})\, t^n \\ &= \overleftarrow{\prod_{k\geq 0}}\ \sigma(q^k\, t; A) \\ &= \ldots\ \sigma(q^2\, t; A)\ \sigma(q\, t; A)\ \sigma(t; A)\ ,\end{aligned}$$

where $\sigma(x; A)$ denotes the generating series (in the variable x) of noncommutative complete symmetric functions. Define

$$\varphi_n(q) = \frac{1-q^n}{n}\ \Psi_n\left(\frac{A}{1-q}\right)\ ,$$

where $\Psi_n\left(\frac{A}{1-q}\right)$ is obtained by substituting $S_n(\frac{A}{1-q})$ in place of S_n in (1). This element encodes an interesting Lie idempotent of Σ_n that interpolates among several classical Lie idempotents. It can be indeed shown that $\varphi_n(0) = \Psi_n/n$ which encodes Dynkin's idempotent and that $\varphi_n(1) = \Phi_n/n$ which corresponds to Solomon's Eulerian idempotent. Moreover, when q is a primitive nth root of unity, $\varphi_n(q)$ specializes to Klyachko's idempotent (*cf.* [18, 9]). This motivated the search of an explicit formula for $\varphi_n(q)$.

The following *Maple* program was used to compute the decomposition of $\varphi_n(q)$ on the basis of ribbon functions:

```
> varphi := proc(n)
>   SpDirect(Ps[n], q) ;
>   ToR((1-q^n)*"/n, collect) ;
>   map(normal, ") ;
> end ;
```

It is based on the **NCSF** functions:

`ToR`	#	change to basis `R`
`SpDirect`	#	transformation of alphabet A/(1-q)→A

NCSF is a package of **ACE**, devoted to computation with noncommutative symmetric functions. For $n = 3, 4$, the procedure `varphi` gives:

$$\varphi_2(q) = \frac{R_2}{2} - \frac{R_{1,1}}{2}$$

$$\begin{aligned}
\varphi_3(q) &= \frac{R_3}{3} - \frac{qR_{2,1}}{3\,q+3} - \frac{R_{1,2}}{3\,q+3} + \frac{R_{1,1,1}}{3} \\
\varphi_4(q) &= \frac{R_4}{4} - \frac{q^2R_{3,1}}{4\,q^2+4\,q+4} - \frac{qR_{2,2}}{4\,q^2+4\,q+4} \\
&\quad + \frac{q^2R_{2,1,1}}{4\,q^2+4\,q+4} - \frac{R_{1,3}}{4\,q^2+4\,q+4} \\
&\quad + \frac{qR_{1,2,1}}{4\,q^2+4\,q+4} + \frac{R_{1,1,2}}{4\,q^2+4\,q+4} - \frac{R_{1,1,1,1}}{4}
\end{aligned}$$

We were thus lead to look for a formula of the form:

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} (-1)^{\ell(I)-1} \frac{q^{f(I)}}{g(I;q)} R_I$$

where $f(I)$ denotes an integer and $g(I;q)$ stands for a polynomial in q. When $|I| = 5$, the first values of the polynomials $g(I;q)$ are:

$$\begin{aligned}
g([5];q) &= 1\ , \\
g([4,1];q) &= q^3+q^2+q+1 = \begin{bmatrix}4\\1\end{bmatrix}_q\ , \\
g([3,2];q) &= q^3+q^2+q+1 = \begin{bmatrix}4\\1\end{bmatrix}_q\ , \\
g([3,1,1];q) &= (q^2+1)(q^2+q+1) = \begin{bmatrix}4\\2\end{bmatrix}_q\ , \\
g([2,3];q) &= q^3+q^2+q+1 = \begin{bmatrix}4\\1\end{bmatrix}_q\ , \\
g([2,2,1];q) &= (q^2+1)(q^2+q+1) = \begin{bmatrix}4\\2\end{bmatrix}_q\ , \\
g([2,1,2];q) &= (q^2+1)(q^2+q+1) = \begin{bmatrix}4\\2\end{bmatrix}_q\ , \\
g([2,1,1,1];q) &= q^3+q^2+q+1 = \begin{bmatrix}4\\3\end{bmatrix}_q\ .
\end{aligned}$$

Here, $\begin{bmatrix}n\\q\end{bmatrix}_q$ stands for the q-binomial coefficient (or Gaussian polynomial) defined by:

$$\begin{bmatrix}n\\p\end{bmatrix}_q = \frac{(q)_n}{(q)_p(q)_{n-p}}\ ,$$

with $(q)_n = (1-q)\dots(1-q^n)$. From other similar observations, it was easy to conjecture that:

$$g(I;q) = \begin{bmatrix}n-1\\ \ell(I)-1\end{bmatrix}_q\ .$$

Before stating the result for $f(I)$, we need a definition. Let I be a composition of n and let $D(I) = \{d_1, d_2, \dots, d_{r-1}\}$ be the associated subset of $[1, n-1]$. The major index of I is defined by:

$$\mathrm{maj}\,(I) = d_1 + \cdots + d_{r-1}\ .$$

For $n = 5$, explicit computations lead to the table:

I	[5]	[4,1]	[3,2]	[3,1,1]
$f(I)$	0	3	2	4
$\mathrm{maj}\,(I)$	0	4	3	7
$\mathrm{maj}\,(I) - f(I)$	0	1	1	3

I	[2,2,1]	[2,1,2]	[2,1,1,1]	[1,4]
$f(I)$	3	2	3	0
$\mathrm{maj}\,(I)$	6	5	9	1
$\mathrm{maj}\,(I) - f(I)$	3	3	6	1

We see that $\mathrm{maj}\,(I) - f(I)$ only depends on the length of I. Furthermore, the corresponding values are here $\{0, 1, 3, 6\}$ which are the four first triangular numbers. Similar experimental results showed that:

$$f(I) = \mathrm{maj}\,(I) - \binom{\ell(I)}{2}\ ,$$

up to the order 8. Hence we were led to the experimental formula

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{\ell(I)-1}}{\begin{bmatrix}n-1\\ \ell(I)-1\end{bmatrix}_q} q^{\mathrm{maj}\,(I)-\binom{\ell(I)}{2}} R_I\ .$$

which was proved three months later [9].

References

[1] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, Schubert cells and cohomology of the spaces G/P, *Uspekhi.Mat.Nauk*, **28**, pp. 3–26, (1973).

[2] C. Carré and B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, *J. Alg. Comb.*, **4**, pp. 201–231, (1995).

[3] M. Demazure, Désingularisation des variétés de Schubert, *Annales E.N.S.*, **6**, pp. 163–172, (1974).

[4] G. Duchamp, A. Klyachko, D. Krob and J.-Y. Thibon, Noncommutative symmetric functions III: Deformations of Cauchy and convolution algebras, *Disc. Math. and Theor. Comput. Sci.* (to appear).

[5] A.M. Garsia, Combinatorics of the free Lie algebra and the symmetric group, in Analysis, et cetera..., *Jurgen Moser Festschrift, Academic Press*, New York, pp. 309–262, (1989).

[6] A.M. Garsia and C. Reutenauer, A decomposition of Solomon's descent algebra, *Adv. in Math*, **77**, pp. 189–262, (1989).

[7] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh and J.-Y. Thibon, Noncommutative symmetric functions, *Adv. in Math.*, **112**, pp. 218–348, (1995).

[8] A. Kohnert and S. Veigneau, Using Schubert toolkit to compute with polynomials in several variables, *Proc. of the 8-th Conference "Formal Power Series and Algebraic Combinatorics"*, Minneapolis, pp. 283–293, (1996).

[9] D. Krob, B. Leclerc and J.-Y. Thibon, Noncommutative symmetric functions II: Transformations of alphabets, *Int. J. of Alg. and Comput.* (to appear).

[10] D. Krob and J.-Y. Thibon, Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$, *J. Alg. Comb.*, (to appear).

[11] A. Lascoux, Classes de Chern des variétés de drapeaux, *C.R. Acad. Sci. Paris*, **295**, pp. 393–398, (1982).

[12] A. Lascoux, B. Leclerc and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties, Preprint, (1996).

[13] A. Lascoux, B. Leclerc and J.-Y. Thibon, Green polynomials and Hall-Littlewood functions at roots of unity, *Europ. J. Combinatorics*, **15**, pp. 173–180, (1994).

[14] A. Lascoux and M.P. Schützenberger, Polynômes de Schubert, *C.R. Acad. Sci. Paris*, **294**, pp. 447–450, (1982).

[15] I.G. Macdonald, Notes on Schubert polynomials, *Publ. LACIM*, **6**, UQAM, Montréal, (1991).

[16] D. Monk, The geometry of flags manifolds, *Proc. London M.S. (3)*, **9**, pp. 253–286, (1959).

[17] C. Reutenauer, Theorem of Poincaré-Birkhoff-Witt, logarithm and representations of the symmetric group whose order are the Stirling numbers, *in Combinatoire énumérative, Proceedings*, Montréal 1985 (G. Labelle and P. Leroux Eds.), *Lecture Notes in Math.*, **1234**, Springer, pp. 267–284, (1986).

[18] C. Reutenauer, Free Lie algebras, *Oxford University Press*, (1993).

[19] L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra*, **41**, pp. 255–268, (1976).

[20] D. Stanton and D. White, A Schensted algorithm for rim-hook tableaux, *J. Comb. Theory A.*, **40**, pp. 211–247, (1985).

[21] J.-Y. Thibon and S. Veigneau, Using ACE, an algebraic Combinatorics Environment for Maple, *Proceedings of the 8-th International Conference on "Formal Power Series And Algebraic Combinatorics"*, Minneapolis, pp. 461–468, (1996).