LECTURES ON NONCOMMUTATIVE SYMMETRIC FUNCTIONS

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Abstract

This is the text of lectures delivered at the RIMS (Kyoto University) in July 1998. It presents the basic structures of the theory of noncommutative symmetric functions, with emphasis on the parallel with the commutative theory and on the representation theoretical interpretations. Some examples involving descent algebras and characters of symmetric groups are discussed in detail.

Introduction

The theory of noncommutative symmetric functions is the outgrowth of a program initiated in 1993. The starting point was the theory of quasi-determinants of Gelfand and Retakh [33, 34], which is the analogue of the theory of determinants for matrices with entries in a noncommutative ring. Many classical determinantal identities can be lifted to the level of quasi-determinants [58]. Such determinantal identities are widely used in the theory of symmetric functions, and most of them can be translated into formulas involving Schur functions. The original idea was then to look for some noncommutative analogue of the theory of symmetric functions, in which quasi-determinants would replace determinants.

Such a theory does exist, and the quasi-determinants arise in applications to enveloping algebras, roots of noncommutative polynomials, noncommutative continued fractions, Padé aprroximants or orthogonal polynomials [32, 85, 35, 36]. These calculations usually take place in a skew field, which is the field of quotients of a free associative algebra **Sym**, which appears as the proper analogue of the classical algebra of symmetric functions.

It is well known that symmetric functions have several interpretations in representation theory. It turns out that most of these interpretations have an analogue in the noncommutative case. It is this aspect of the theory which will be the main subject of these lectures.

After reviewing briefly the relevant features of the classical theory (Section 1), we describe in detail the algebraic structure of the Hopf algebra of integral noncommutative symmetric functions (Section 2), including the duality with quasi-symmetric functions, and the connection with descent algebras. The representation theoretical interpretations are discussed in Section 3. The first one, involving Hecke algebras at $q = 0$, leads us to the definition of quantum quasi-symmetric functions. The second one, a quantum matrix algebra at $q = 0$, provides us with the relevant analogues of the Robinson-Schensted correspondence and of the plactic algebra. Finally, a quantized enveloping algebra at $q = 0$, for which one has a natural notion of Demazure module, and the character formula for these modules, leads to a new action of the Hecke algebra on polynomials, from which one can define quasi-symmetric and noncommutative analogues of Hall-Littlewood functions. Section 4 presents a choice of examples. First, we analyze three idempotents of the group algebra of the symmetric groups involved in the combinatorics of the Hausdorff series, and exhibit a natural one-parameter family interpolating between them. Next, we show that similar calculations can give the decomposition of the tensor products of certain representations of symmetric groups. We conclude by the diagonalization of the iterated left q-bracketing operator of the free associative algebra.

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Notations. — We essentially use the notations of Macdonald's book [78] for commutative symmetric functions. A minor change is that the algebra of symmetric functions is denoted by Sym, the coefficients being taken in some field $\mathbb{K}$ rather than in $\mathbb{Z}$. The symmetric group is denoted by $\mathfrak{S}_n$. The Coxteter generators $(i, i+1)$ are denoted by s_i.

1 Some highlights of the commutative theory

Modern textbooks in Algebra usually close their account of symmetric functions with the so-called "Fundamental Theorem" of the theory, stating that the ring of symmetric polynomials in n variables

$$Sym(X) = \mathbb{K}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$$

($\mathbb{K}$ is some field of characteristic 0) is freely generated by the *elementary symmetric polynomials*

$$e_k = \sum_{i_1 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad (k = 1, 2, \ldots, n)\,.$$

Thus, $Sym(X)$ is just a polynomial algebra $\mathbb{K}[e_1, \ldots, e_n]$, with a particular grading defined by $\deg(e_k) = k$.

This is not, however, the end of the story, and the 475 pages of the second edition of Macdonald's book [78] do not suffice to exhaust the subject, which is still an active area of research.

The algebra Sym of *symmetric functions* is obtained by letting $n \to \infty$. The classical theory of symmetric functions can therefore be regarded as the study of the polynomial algebra $\mathbb{K}[e_k \,:\, k \geq 1]$ over an infinite sequence of indeterminates, graded by $\deg(e_k) = k$. Note that the original variables x_i can be eliminated from this definition.

What makes this algebra interesting is the existence of many natural bases (labelled by *partitions*), of a canonical scalar product, of a Hopf algebra structure, and of several other algebraic operations, and also, its many interpretations in representation theory, algebraic geometry, or mathematical physics.

First of all, Sym, as an algebra, has several distinguished sets of generators. Let $\lambda(t) = \sum_{k\geq 0} e_k t^k$ be the generating series of elementary symmetric functions. Then, the *complete homogeneous functions* h_n can be defined by their generating series $\sigma(t) = \lambda(-t)^{-1} = \sum_{k\geq 0} h_k t^k$, and the power-sums p_n by $\psi(t) = \sum_{k\geq 1} p_k t^{k-1} = \frac{d}{dt} \log \sigma(t)$ (the Newton formulas). One has as well $Sym = \mathbb{K}[h_1, h_2, \ldots] = \mathbb{K}[p_1, p_2, \ldots]$.

A Hopf algebra structure is defined by means of the comultiplication $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$, and the antipode $\tilde{\omega}(p_k) = -p_k$. It can be shown that Sym is isomorphic to its graded dual.

The dimension of the homogeneous component of degree n of Sym is $\dim Sym_n = |\mathrm{Part}\,(n)| = p(n)$, the number of partitions of n. Linear bases of Sym are therefore naturally labelled by partitions. The simplest ones are m_λ (monomial symmetric functions), $e_\lambda, h_\lambda, p_\lambda$ (products $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$ etc.), and the Schur functions s_λ, which are the fundamental ones in representation theory.

There is a canonical scalar product, which can be defined by either of the following equivalent formulas:

$$\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \langle p_\lambda, p^*_\mu \rangle = \delta_{\lambda\mu}$$

(where $p^*_\mu = p_\mu / z_\mu$). This scalar product materializes the self duality of Sym, in that Δ is the adjoint of the multiplication map $f \otimes g \mapsto fg$. The equivalence of these definitions, as well as the self duality, is a consequence of the following property

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \Leftrightarrow \prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(X) v_\lambda(Y)$$

and of the classical Cauchy identity for Schur functions

$$\sum_\lambda s_\lambda(X)s_\lambda(Y) = \prod_{i,j}(1-x_iy_j)^{-1}.$$

There is a second comultiplication defined by $\delta(p_k) = p_k \otimes p_k$. This comultiplication is dual to a multiplication known as the *internal product* $*$, *i.e.* $\langle f * g, h\rangle = \langle f \otimes g, \delta(h)\rangle$. The internal product corresponds to the (pointwise) product of central functions on the symmetric group, via the Frobenius characteristic map $\chi^\lambda \mapsto \mathrm{ch}\,(\chi^\lambda) = s_\lambda$, where the χ^λ are the irreducible characters. This is an isometric isomorphism $R(\mathfrak{S}_n) \to Sym_n$, where $R(\mathfrak{S}_n)$ is the vector space spanned by the irreducible characters. One has then $\mathrm{ch}\,(\phi\psi) = \mathrm{ch}\,(\phi) * \mathrm{ch}\,(\psi)$.

The Frobenius character formula expresses the value $\chi(\sigma)$ of a character χ on a permutation σ of cycle type μ as $\chi^\lambda(\mu) = \langle s_\lambda, p_\mu\rangle$.

The cycle index is the map $Z : \mathbb{K}\mathfrak{S}_n \to Sym_n$, mapping a permutation of cycle type μ onto p_μ, induces a canonical linear isomorphism between the center of the group algebra and symmetric functions of degree n. Then, the character formula can be rewritten as

$$\chi(\sigma) = \langle \mathrm{ch}\,(\chi)\,,\, Z(\sigma)\rangle\,.$$

Another important point is the induction formula, interpreting the ordinary multiplication of symmetric functions in terms of induced representations. If $f = \mathrm{ch}\,(\xi)$ and $g = \mathrm{ch}\,(\eta)$ are the characteristics of two representations of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, then $fg = \mathrm{ch}\,(\chi)$ where χ is the character of $\mathfrak{S}_{m+n}$ induced by the character $\xi \times \eta$ of its subgroup $\mathfrak{S}_m \times \mathfrak{S}_n$.

Schur functions can also be interpreted as characters of $GL(n, \mathbf{C})$. The polynomial representations V_λ of $GL(n, \mathbf{C})$ are parametrized by partitions of length at most n, and $s_\lambda(x_1, \ldots, x_n) = \mathrm{trace}_{V_\lambda}(X)$ where X is the diagonal matrix $X = \mathrm{diag}\,(x_1, \ldots, x_n)$.

Schur functions correspond in a similar way to the irreducible representations of the q-deformed structures $H_n(q)$, $U_q(gl_n)$ or $F_q(GL_n)$, for generic q. For example, in the case of the Hecke algebra $H_n(q)$, which is the algebra generated by elements $T_1, \ldots, T_{n-1}$ satisfying the braid relations plus the quadratic ones $(T_i + 1)(T_i - q) = 0$, the character formula can be written as [101, 9, 91]

$$\chi^\lambda_\mu(q) = \langle s_\lambda(X)\,,\, (q-1)^{-\ell(\mu)} h_\mu((q-1)X)\rangle\,, \tag{1}$$

where the "λ-ring notation" $h_\mu((q-1)X)$ means the following. In general, if X and Y are two (multi-) sets of variables, identified with the formal sum of their elements, the symmetric functions of $X+Y$, $X-Y$ and XY are defined by setting

$$p_k(X+Y) = p_k(X)+p_k(Y)\,, \quad p_k(X-Y) = p_k(X)-p_k(Y) \quad p_k(XY) = p_k(X)p_k(Y)\,, \tag{2}$$

and then by expressing any symmetric function as a polynomial in the power sums. Therefore, the symmetric functions of $(q-1)X = qX - X$ are the images of those of X under the ring homomorphism $p_k \mapsto (q^k - 1)p_k$.

The Hecke algebra $H_n(q)$ was introduced by Iwahori in [48]. The original definition was as follows. Let $G = GL(n, \mathbb{F}_q)$ and B be the subgroup of G formed by upper triangular matrices. Then, G acts on the vector space $M = \mathbb{C}G/B$ spanned by the left cosets of B (which can be identified with complete flags in $\mathbb{F}_q^n$). To decompose this representation into irreducibles, one can use Schur's lemma and look at the centralizer of $\mathbb{C}G$ in $\mathrm{End}\,(M)$. It turns out that this centralizer is isomorphic to $H_n(q)$.

From the knowledge of the irreducible representations of $H_n(q)$, one can in principle obtain the characters of the irreducible representations of G occuring in the spectrum of M. The characters of these representations, now called *unipotent representations*, were in fact first obtained by Steinberg [102], by a different method, quite similar to the one used by Frobenius to determine the characters of symmetric groups.

The unipotent representations are parametrized by partitions λ of n. Their name comes from the fact that these are exactly as many as the conjugacy classes of unipotent elements in G, which are also parametrized by partitions μ specifying the sizes of their Jordan cells, so that a unipotent character is determined by its values on unipotent classes.

Steinberg's result was that the value χ^λ_μ of the unipotent character λ on the unipotent class μ is a polynomial in q, which has since been recognized as a Kostka-Foulkes polynomial [41, 76]

$$\chi^\lambda_\mu = \tilde{K}_{\lambda\mu}(q)\,. \tag{3}$$

The Kostka-Foulkes polynomials are the coefficients of the transition matrices between the bases of Schur functions and of Hall-Littlewood functions. The Hall-Littlewood Q-functions are defined, for N variables $x_1, \dots, x_N$, by the orbit sums

$$Q_\mu(x_1,\dots,x_N;t) = \frac{(1-t)^{\ell(\mu)}}{[m_0]_t!} \sum_{\sigma\in\mathfrak{S}_n} \sigma\left(x^\mu \frac{\Delta_N(t)}{\Delta_N(1)}\right) \tag{4}$$

where $m_0 = N-\ell(\mu)$, $[m]_t = (1-t^m)/(1-t)$ and $\Delta_N(t) = \prod_{i<j}(tx_i - x_j)$. The P-functions are just scaled versions of the Q's

$$P_\mu = \frac{1}{(1-t)^{\ell(\mu)}[m_1]_t!\cdots[m_n]_t!} Q_\mu\,, \tag{5}$$

where m_i is the multiplicity of i in μ. Then,

$$s_\lambda(X) = \sum_\mu K_{\lambda\mu}(t) P_\mu(X;t)\,, \tag{6}$$

and

$$Q'_\mu(X;t) = Q_\mu(X/(1-t);t) = \sum_\lambda K_{\lambda\mu}(t) s_\lambda(X) \tag{7}$$

where Q' is the adjoint basis of P for the standard scalar product of Sym, and finally

$$\tilde{Q}'_\mu(X;t) = t^{n(\mu)} Q'_\mu(X;t) = \sum_\lambda \tilde{K}_{\lambda\mu}(t) s_\lambda\,. \tag{8}$$

The Hall-Littlewood functions can be compactly expressed by means of an action of the Hecke algebra $H_N(q)$ on the algebra of polynomials in $x_1, \dots x_N$ (here, $q = t^{-1}$) [16]. This action is defined in terms of the isobaric divided difference operators π_i

$$\pi_i(f) = \frac{x_i f - \sigma_i(x_i f)}{x_i - x_{i+1}}\,, \tag{9}$$

where σ_i is the ring involution exchanging x_i and x_{i+1}, by

$$T_i = (q-1)\pi_i + \sigma_i\,. \tag{10}$$

The Hall-Littlewood functions are then, up to a scalar factor, the images of the monomials by the full symmetrizer

$$S^{(N)} = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \,, \tag{11}$$

that is,

$$S^{(N)}(x^\mu) = q^{\binom{N}{2}}(1-q^{-1})^{-\ell(\mu)}[m_0]_{1/q}!Q_\mu(x_1,\dots,x_N;q^{-1})\,. \tag{12}$$

The symmetric functions $\tilde{Q}'_\mu$ describe the characters of certain graded versions of the permutation representations of $\mathfrak{S}_n$, i.e. the coefficient of q^k is the characteristic of the homogeneous component of degree k (these are the Springer representations, in the cohomology rings of unipotent varieties, see [47, 78, 63]). The simplest case $\mu = (1^n)$ is already of interest. It describes the usual graded version of the regular representation, which is realized for example in the coinvariants, in the cohomology of the flag manifold, or in the space of $\mathfrak{S}_n$-harmonic polynomials. There is a closed formula for this graded character

$$\tilde{Q}'_{(1^n)}(X;q) = (q)_n h_n\left(\frac{X}{1-q}\right) \tag{13}$$

which implies that the Kostka-Foulkes polynomials for $\mu = (1^n)$ are essentially the principal specializations of Schur functions, i.e.

$$\tilde{K}_{\lambda\,(1^n)}(q) = (q)_n s_\lambda(1,q,q^2,\cdots)\,. \tag{14}$$

The combinatorial properties of symmetric functions, e.g., the Littlewood-Richardson rule for multiplying Schur functions, or the combinatorial interpretation of Kostka-Foulkes polynomials as generating functions of sets of Young tableaux according to a certain statistic (charge or cocharge [68], see also [63]), rely essentially on the Robinson-Schensted correspondence, which can be used to define a multiplicative structure on the set of Young tableaux (the plactic monoid of Lascoux and Schützenberger [69]). The plactic monoid over an ordered alphabet A is the quotient of the free monoid A^* by the relations $xzy = zxy$ $(x \le y < z)$, $yxz = yzx$ $(x < y \le z)$. These relations can now be understood in terms of crystals. If one considers the letters of A as the vertices of the crystal graph of the vector representation V of $U_q(gl_n)$, and words of length N as the vertices of the crystal graph of $V^{\otimes N}$, then, two words are equivalent under the plactic relations iff they label corresponding vertices of two isomorphic connected components of the graph. This allows to define a similar monoid for other Lie algebras, including the classical ones [66, 72]. The essential facts are that plactic classes correspond to Young tableaux, and that the plactic Schur functions S_λ, defined as the sum of all tableaux of the same shape λ, span a commutative subalgebra, isomorphic to $Sym(x_1,\dots,x_n)$. The Littlewood-Richardson rule, for example, follows straightforwardly from these facts.

To conclude, let us mention that the main objects of current interest in the theory of symmetric functions are the Macdonald polynomials [78]. We will not touch this subject here, since the proper noncommutative analogues of Macdonald polynomials have not yet been worked out in general. However, the other properties that have been alluded to in this section will all find some kind of noncommutative analogue.

2 The Hopf algebra of noncommutative symmetric functions

2.1 Algebraic generators

Imitating the description of Sym as a polynomial algebra on independent indeterminates e_k, graded by $\deg(e_k) = k$, one defines the algebra **Sym** of formal noncommutative symmetric functions as the free associative algebra on an infinite sequence Λ_k of noncommuting indeterminates (the noncommutative elementary functions), graded by $\deg(\Lambda_k) = k$. The coefficients are taken in some field $\mathbb{K}$ of characteristic 0, which is assumed to contain the rational functions in all extra variables $t, q, z, \ldots$ used for generating functions or as deformation parameters.

As in the commutative case, one introduces the generating series

$$\lambda(t) = \sum_{n\geq 0} \Lambda_n t^n \,, \tag{15}$$

where t is an indeterminate in the ground field $\mathbb{K}$. The complete homogeneous symmetric functions S_n are then naturally defined as the coefficients of the series

$$\sigma(t) = \lambda(-t)^{-1} = \sum_{n\geq 0} S_n t^n \,. \tag{16}$$

A concrete realization $\mathbf{Sym}(A)$ of this algebra can be given by taking an infinite sequence $A = \{a_n | n \geq 1\}$ of noncommuting indeterminates of degree 1, by setting

$$\lambda(A;t) = \sum_{n\geq 0} \Lambda_n(A) t^n = \overleftarrow{\prod_{i\geq 1}} (1 + t a_i) \,, \tag{17}$$

so that $\Lambda_n(A)$ gets identified with the sum of all strictly decreasing words of length n, and $S_n(A)$ with the sum of all nondecreasing words of the same length, which are respectively represented as column-shaped and row shaped Young tableaux.

The algebra homomorphism $\mathbf{Sym}(A) \to Sym(X)$ defined by $a_i \mapsto x_i$, called the *commutative image* $F \mapsto \underline{F}$, maps Λ_n to e_n, so that **Sym** is actually a noncommmtative lifting of the algebra of symmetric functions. One can object however, that the $\Lambda_n(A)$ are not invariant under the symmetric group. At least, not for the usual action. But we shall see later that they are indeed symmetric for a more subtle one.

The first really interesting question is “what are the noncommutative power sums ?”. There are indeed several possibilities. If one starts from the classical expression

$$\sigma(t) = \exp\left\{ \sum_{k\geq 1} p_k \frac{t^k}{k} \right\} \,, \tag{18}$$

one can choose to define noncommutative power sums Φ_k by the same formula

$$\sigma(t) = \exp\left\{ \sum_{k\geq 1} \Phi_k \frac{t^k}{k} \right\} \,, \tag{19}$$

but a noncommutative version of the Newton formulas

$$n h_n = h_{n-1} p_1 + h_{n-2} p_2 + \cdots + p_n \tag{20}$$

which are derived by taking the logarithmic derivative of (18) lead to different noncommutative power-sums Ψ_k inductively defined by

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + \Psi_n \,. \tag{21}$$

Introducing the generating function $\psi(t) = \sum_{k\geq 1} \Psi_k t^{k-1}$, one may regard $\sigma(t)$ as the unique solution of the differential equation

$$\frac{d}{dt}\sigma(t) = \sigma(t)\psi(t) \tag{22}$$

satisfying the initial condition $\sigma(0) = 1$. The generating function of the Φ_k, taken in the form

$$\Phi(t) = \sum_{k\geq 1} \Phi_k \frac{t^k}{k} \tag{23}$$

is then the logarithm of this solution. From this, one realizes that the relation between the two kinds of noncommutative power-sums is of a rather complicated nature. The expression of $\Phi(t)$ as a function of the Ψ_k is known as the *continuous Baker-Campblell-Hausdorff* series [80, 5, 84]. It can be written as a Chen series (iterated integrals) in a quite explicit form (to be discussed later), and it is usually interpreted as expressing the logarithm of the "evolution operator" $\sigma(t)$ in terms of the "Hamiltonian" $\psi(t)$ [5]. It is known that the continuous BCH series is a Lie series, so that the Φ_k are elements of the free Lie algebra $\mathcal{L}$ generated by the Ψ_k, of which they form another system of generators. In fact, any sequence (F_n) of generators of $\mathcal{L}$, with $\deg(F_n) = n$ can be shown to provide an admissible family of noncommutative power sums, in the sense that the commutative image of F_n is a nonzero multiple of p_n. Moreover, the Poincaré-Birkhoff-Witt theorem shows that **Sym** can be identified with the universal enveloping algebra $U(\mathcal{L})$ of $\mathcal{L}$. As such, **Sym** is endowed with a canonical comultiplication Δ, for which $\mathcal{L}$ is the space of primitive elements (Friedrich's theorem, see [94]). In particular,

$$\Delta\Psi_k = \Psi_k \otimes 1 + 1 \otimes \Psi_k \qquad \Delta\Phi_k = \Phi_k \otimes 1 + 1 \otimes \Phi_k \,, \tag{24}$$

and also

$$\Delta\Lambda_n = \sum_{k=0}^{n} \Lambda_k \otimes \Lambda_{n-k} \,, \quad \Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \,. \tag{25}$$

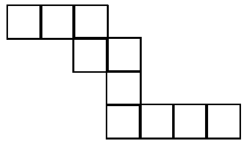
That is, the comultiplication of **Sym** is mapped onto the usual one on *Sym* under the commutative image homomorphism.

There is also an analogue of the canonical involution $\omega : e_n \leftrightarrow h_n$, defined in the same way by $\omega(\Lambda_n) = S_n$, and it is easy to check that the signed version $\tilde{\omega}(\Lambda_n) = (-1)^n S_n$ is an antipode for Δ.

2.2 Linear bases

As for ordinary symmetric functions, we first define linear bases by taking monomials in the various families of algebraic generators, such as $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$. Here, our generators do not commute, so that basis elements of the homogeneous component $\mathbf{Sym}_n$ of degree n will be labelled by *compositions* of n, i.e., ordered sequences $I = (i_1, \ldots, i_r)$ of positive integers. For a sequence (G_n) of homogenous generators with $\deg(G_n) = n$, we set $G^I = G_{i_1} G_{i_2} \cdots G_{i_r}$. Therefore, we already have the four bases Λ^I, S^I, Φ^I and Ψ^I.

A composition I of n is conveniently pictured as a *ribbon diagram*, which is a rim-hook shaped skew Young diagram whose succesive rows have lengths $i_1, i_2, \ldots, i_r$ (read from top to bottom in the French convention). For example, the ribbon diagram of shape $I = (3, 2, 1, 4)$ is



The number of compositions of n is equal to 2^{n-1}. A useful way to realize this is to encode the ribbon diagram of a composition I of n by the subset $\mathrm{Des}\,(I) = \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{r-1}\}$ of $\{1, 2, \ldots, n-1\}$. The elements of $\mathrm{Des}\,(I)$ are called the *descents* of the composition.

The next basis that should be defined, to pursue the parallel with the commutative theory, would be the analogue of monomial symmetric functions. However, we have no way to achieve this at this point. This is because in the classical case, the monomial basis (m_λ) is dual to the homogeneous one (h_λ), of which we already have the noncommutative analogue (S^I). But since the comultiplication Δ of **Sym** is obviously cocommutative, **Sym** cannot be self-dual, and the analogues of the monomial functions will have to live in the dual Hopf algebra $\mathbf{Sym}^*$, to be discussed in the next section.

On another hand, we can define the analogues of Schur functions. These are the so-called *ribbon Schur functions*. Their original definition was given in terms of quasi-determinants, but one can as well define them as follows.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4). The ribbon Schur function (R_I) is defined by the alternating sum

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)}\, S^J \qquad \left(\Longleftrightarrow S^I = \sum_{J \preceq I} R_J \right), \tag{26}$$

where $\ell(I)$ denotes the length of I. Clearly, (R_I) is also a homogenous basis of **Sym**.

The commutative image of a ribbon Schur function R_I is the corresponding ordinary ribbon Schur function, which will be denoted by by r_I. The r_I were first introduced by MacMahon, in his analysis of Simon Newcomb's problem [79]. They arise also, for example, in a generalization by Lascoux and Pragacz [67] of the Giambelli formula (expressing general Schur functions as determinants of ribbons instead of just hooks), or as sl_n-characters of the irreducible components of the Yangian representations in level 1 modules of $\widehat{sl}_n$ [54].

An important property of the ribbon Schur functions is their very simple multiplication formula (already known to MacMahon in the commutative case)

$$R_I\, R_J = R_{I \triangleright J} + R_{I \cdot J} \tag{27}$$

where $I \triangleright J$ denotes the composition $(i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$ and $I \cdot J$ the composition $(i_1, \ldots, i_r, j_1, \ldots, j_s)$.

The transition matrices between the above bases can be worked out quite explicitely (see [32]).

2.3 Duality and quasi-symmetric functions

The next step is to have a look at the dual Hopf algebra $\mathbf{Sym}^*$. As pointed out in the preceding section, we need it to define the proper generalization of monomial symmetric functions. In the commutative case, the duality between monomial symmetric functions m_λ and homogeneous products h_λ comes from the Cauchy-type identity

$$\sigma(XY;1) = \prod_{i,j}(1-x_iy_j)^{-1} = \sum_\lambda m_\lambda(X)h_\lambda(Y)\,, \tag{28}$$

where the sum runs over all partitions. By analogy, let us consider the ordered product

$$\sigma(XA;1) := \prod_{i\geq 1}^{\rightarrow} \sigma(A;x_i) = \prod_{i\geq 1}^{\rightarrow}\prod_{j\geq 1}^{\rightarrow}(1-x_ia_j)^{-1} \tag{29}$$

where A is a totally ordered set of noncommutative variables, and X a totally ordered set of commutative variables, also commuting with those of A. This is a natural choice, since we already know that the dual of $\mathbf{Sym}$ will be commutative. Expanding the product, we find

$$\sigma(XA;1) = \sum_I M_I(X)S^I(A)\,, \tag{30}$$

where the polynomials M_I are defined by

$$M_I(X) = \sum_{j_1<j_2<\ldots<j_r} x_{j_1}^{i_1}x_{j_2}^{i_2}\cdots x_{j_r}^{i_r}\,. \tag{31}$$

Thus, the M_I are pieces of monomial symmetric functions. That is, m_λ is equal to the sum of the M_I labelled by the distinct rearrangements I of the partition λ. One can show that the M_I form the basis of a subalgebra of $\mathbb{K}[X]$, which has been introduced by Gessel [39] under the name "quasi-symmetric functions". It will be denoted by $QSym$. It is naturally graded, and its homogeneous component $QSym_n$ has, like $\mathbf{Sym}_n$, dimension 2^{n-1}. One can then define a pairing $\langle\ ,\ \rangle$ between $QSym$ and $\mathbf{Sym}$ by requiring

$$\langle M_I\,,\,S^J\rangle = \delta_{IJ}\,. \tag{32}$$

With this at hand, one can show that $QSym$ is actually the (graded) dual Hopf algebra of $\mathbf{Sym}$ ([82], see also [32, 59]). That is,

$$\langle f\otimes g\,,\,\Delta(P)\rangle = \langle fg\,,\,P\rangle\,, \tag{33}$$

$$\langle \gamma(f)\,,\,P\otimes Q\rangle = \langle f\,,\,PQ\rangle\,, \tag{34}$$

where the comutiplication γ of $QSym$ maps a quasi-symmetric function $f = f(X)$ to the function $f(X\hat{+}Y)$, where $X\hat{+}Y$ denotes the ordered sum of two disjoint totally ordered sets, and $u(X)v(Y)$ is identified with $u\otimes v$. Also, the antipode $\tilde{\nu}$ of $QSym$, defined on the adjoint basis (F_I) of (R_I) by $\tilde{\nu}(F_I) = F_{I^\sim}$, where $I^\sim$ is the conjugate composition, is the adjoint of $\tilde{\omega}$.

The basis (F_I), which is in some sense a quasi-symmetric analogue of the Schur basis, has the following simple expression

$$F_I = \sum_{J\succeq I} M_J \tag{35}$$

in terms of the quasi-monomial functions M_J.

In fact, as observed recently by Hivert [44], quasi-symmetric functions are actually symmetric, but with respect to an unusual action of the symmetric group on polynomials. Suppose that $X = \{x_1, \ldots, x_n\}$. Let $\mathcal{P}(X)$ (resp. $\mathcal{P}_k(X)$) be the power set (resp. the set of k-elements subsets) of X. To a monomial $m = x_1^{m_1} \cdots x_n^{m_n}$, one associates its *support* $A \in \mathcal{P}(X)$, which is the set of variables x_i occuring in m with a nonzero exponent. Let I be the composition obtained by removing the zeros of the sequence $(m_1, \ldots, m_n)$. Then, we encode m by the symbol A^I. The image of m by a permutation $\sigma \in \mathfrak{S}_n$ is then

$$\sigma \cdot m = (\sigma A)^I \,, \tag{36}$$

where σA denotes the image of A by the usual action of permutations on subsets of X. For example, $s_1 \cdot x_1 x_2^2 = x_1 x_2^3$ and $s_2 \cdot x_1 x_2^3 = x_1 x_3^3$.

Then, it can be shown that the quasi-symmetric polynomials are exactly the invariants of this action of $\mathfrak{S}_n$ on $\mathbb{K}[X]$. Clearly, the quasi-monomial functions appear as the orbit sums

$$M_I(X) = \frac{1}{r!(n-r)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot (\{x_1, \ldots, x_r\})^I \,. \tag{37}$$

Moreover, the quasi-symmetric Schur functions F_I (quasi-ribbons) admit an expression analogous to Jacobi's original definition of Schur functions as ratios of alternants

$$s_\lambda(x_1, \ldots, x_n) = J(x^\lambda) := \frac{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma(x^{\lambda+\rho})}{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma(x^\rho)} \tag{38}$$

which is now recognized as the Weyl character formula for gl_n. As observed by Demazure [13] and Bernstein-Gelfand-Gelfand [4], the Jacobi-Weyl symmetrizer J can be factored into a product of elementary operators

$$\pi_i = (x_i - x_{i+1})^{-1}(x_i - x_{i+1}\sigma_i) \tag{39}$$

where σ_i is the automorphism exchanging x_i and x_{i+1}. These operators satisfy the braid relations, so that for any permutation $\sigma \in \mathfrak{S}_n$, there is a well-defined operator $\pi_\sigma = \pi_{i_1}\pi_{i_2} \cdots \pi_{i_r}$, where $s_{i_1}s_{i_2} \cdots s_{i_r}$ is any reduced decomposition of σ. Then, if ω denotes the longest permutation of $\mathfrak{S}_n$,

$$J = \pi_\omega \,. \tag{40}$$

It is this expression that can be extended to the quasi-symmetric case. Let

$$\pi_i' = (x_i - x_{i+1})^{-1}(x_i - x_{i+1}\sigma_i') \tag{41}$$

where $\sigma_i'(f) = s_i \cdot f$ is the image of f under the quasi-symmetrizing action of the elementary transposition s_i. It can be shown that the braid relations are also satisfied by these operators. Then [44, 45],

$$F_I(x_1, \ldots, x_n) = \pi_\omega'(x^I) \,. \tag{42}$$

Note that this result allows one to define F_I when I is not a composition, that is, when some of the i_k may be 0. The standardization rule is as follows. If $I = (i_1, \ldots, i_r, 0, \ldots, 0) \in \mathbb{N}^n$, with $i_r \neq 0$,

$$F_I(x_1, \ldots, x_n) = (-1)^{c_I} \sum_J F_J(X) \,, \tag{43}$$

where c_I is the number of $k < r$ such that $i_k = 0$, and the sum runs over all compositions obtained by replacing all maximal blocks $00\ldots0j$ (with $j \neq 0$) of I by a composition of j. If no such composition exists, the sum is understood to be zero. For example,

$$\begin{aligned} F_{005022} = \quad & -F_{113112} - F_{122112} - F_{131112} \\ & -F_{212112} - F_{221112} - F_{311112} \,. \end{aligned}$$

Finally, since any symmetric function f is in particular quasi-symmetric, one can expand it on the various bases of $QSym$. A useful property is then

$$\langle f \, , \, G \rangle = \langle f \, , \, g \rangle \,, \tag{44}$$

where on the right-hand side, $g = \underline{G}$ is the commutative image of G and the brackets denote the ordinary scalar product of symmetric functions [39].

2.4 Descent algebras and internal product

It remains to introduce one very important algebraic operation on **Sym**, namely, the noncommutative analogue of the internal product $*$ of symmetric functions. Although this operation is usually defined in terms of characters of symmetric groups, it also has a more fundamental characterization as the dual of the natural comultiplication $\delta(f) = f(XY)$, where, as usual, XY is the set of products $x_i y_j$ and $u(X)v(Y)$ is identified with $u \otimes v \in Sym \otimes Sym$. Therefore, the internal product occurs in algebraic identities such as

$$\sigma(XYZ; 1) := \prod_{i,j,k} (1 - x_i y_j z_k)^{-1} = \sum_{\lambda,\mu} s_\lambda(X) s_\mu(Y) (s_\lambda * s_\mu)(Z) \tag{45}$$

which generalizes the Cauchy identity to three sets of variables.

The comultiplication $f \mapsto f(XY)$ makes sense as well for quasi-symmetric functions, if ones takes care of the order on the product set XY. That is, X and Y need to be totally ordered, and one can set $x_i y_j < x_k y_l$ iff $(i, j) < (k, l)$ for the lexicographic order. Let us denote by $X \hat{\times} Y$ the product XY endowed with this order. Then, we can extend δ from Sym to $QSym$ by setting [39]

$$\delta(f) = f(X \hat{\times} Y) \,. \tag{46}$$

The internal product $*$ of **Sym** can now be defined as the dual of δ, that is,

$$\langle f \, , \, P * Q \rangle = \langle \delta(f) \, , \, P \otimes Q \rangle \,. \tag{47}$$

Clearly, each homogeneous component $\mathbf{Sym}_n$ is a $*$-subalgebra. In fact, it follows from results of Gessel [39] that this algebra is *anti*-isomorphic to the so-called *descent algebra* Σ_n of $\mathfrak{S}_n$. The descent algebras have been introduced by Solomon [99] for general finite Coxeter groups in the following way. Let (W, S) be a Coxeter system. One says that $w \in W$ has a descent at $s \in S$ if w has a reduced word ending by s. For $W = \mathfrak{S}_n$ and $s_i = (i, i+1)$, this means that $w(i) > w(i+1)$, whence the terminology. In this case, we rather say that i is a descent of w. Let $\mathrm{Des}\,(w)$ denote the descent set of w, and for a subset $E \subseteq S$, set

$$D_E = \sum_{\mathrm{Des}\,(w) = E} w \quad \in \mathbb{Z}W \,. \tag{48}$$

Then, Solomon's result is that the D_E span a $\mathbb{Z}$-subalgebra of $\mathbb{Z}W$. Moreover

$$D_{E'}D_{E''} = \sum_E c^E_{E'E''} D_E \tag{49}$$

where the coefficients $c^E_{E'E''}$ are nonnegative integers.

The canonical anti-isomorphism $\alpha : \Sigma_n \to \mathbf{Sym}_n$ maps the descent class D_E to the ribbon Schur function R_I, with I such that $E = \mathrm{Des}\,(I)$. From one of Solomon's formulas, one obtains the following multiplication rule.

Let $i = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ be two compositions of n. Then,

$$S^I * S^J = \sum_{M \in \mathrm{Mat}\,(I,J)} S^M \tag{50}$$

where $\mathrm{Mat}\,(I, J)$ denotes the set of matrices of nonnegative integers $M = (m_{ij})$ of order $p \times q$ such that $\sum_s m_{rs} = i_r$ and $\sum_r m_{rs} = j_s$ for $r \in [1,p]$ and $s \in [1,q]$, and where

$$S^M = S_{m_{11}}\, S_{m_{12}} \,\cdots\, S_{m_{1p}} \,\cdots\, S_{m_{q1}} \cdots S_{m_{qp}} \ .$$

Note that by definition, if F and G are homogeneous of different degrees, $F * G = 0$, and that S_n is the unit element of the $*$-subalgebra $\mathbf{Sym}_n$.

Let $h_\lambda = h_I = \underline{S^I}$ and $h_\mu = h_J = \underline{S^J}$ be the commutative images of S^I and S^J. From the known expression of $h_\lambda * h_\mu$ in the commutative case, one can see that $\underline{(S^I * S^J)} = \underline{S^I} * \underline{S^J}$, so that in general

$$\underline{F * G} = \underline{F} * \underline{G}\,, \tag{51}$$

that is, the commutative image is a homomorphism for the internal products.

From equation (50), one derives a fundamental formula, whose commutative version is just a special case of the Mackey formula for the restriction of an induced character. Let $F_1, F_2, \ldots, F_r, G \in \mathbf{Sym}$. Then,

$$(F_1 F_2 \cdots F_r) * G = \mu_r\left[(F_1 \otimes \cdots \otimes F_r) * \Delta^r G\right] \tag{52}$$

where in the right-hand side, μ_r denotes the r-fold ordinary multiplication and $*$ stands for the operation induced on $\mathbf{Sym}^{\otimes n}$ by $*$ [32].

In the commutative case, the power-sums p_n and more generally the power-sum products are quasi-idempotents (i.e., idempotents up to a scalar factor) for the internal product. Precisely,

$$p_\mu * p_\mu = z_\mu\, p_\mu \tag{53}$$

where for $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$, $z_\mu = \prod_i i^{m_i} \cdot m_i!$. Therefore, the commutative images of noncommutative power sums and their products are quasi-idempotents, and one may wonder whether there are true quasi-idempotents among them. Thanks to the anti-isomorphism α with the descent algebra, we could then use them to construct idempotents in the group algebras of symmetric groups.

As an illustration of the formalism, let us try this program with the power sums Ψ_n. We want to know whether $\Psi_n * \Psi_n = n\Psi_n$. To this end, we can write a generating function for the $*$-squares in the form

$$\sum_{n \geq 1} (xy)^{n-1} (\Psi_n * \Psi_n) = \psi(x) * \psi(y)\,, \tag{54}$$

since $\Psi_i * \Psi_j = 0$ for $i \neq j$. Now, writing (22) in the form

$$\psi(t) = \sum_{n\geq 1} t^{n-1}\Psi_n = \lambda(-t)\sigma'(t)$$

and applying the splitting formula (52), we get

$$\begin{aligned}\sum_{n\geq 1}(xy)^{n-1}(\Psi_n * \Psi_n) &= \lambda(-x)\sigma'(x) * \lambda(-y)\sigma'(-y) \\ &= \mu\left[(\lambda(-x)\otimes\sigma'(x)) * (\psi(y)\otimes 1 + 1\otimes\psi(y))\right] \\ &= \mu\left[(\lambda(-x) * 1)\otimes(\sigma'(x) * \psi(y))\right] ,\end{aligned}$$

(since $\sigma'(x)$ has no constant term)

$$= \left(\sum_{n\geq 1} nx^{n-1}\, S_n\right) * \left(\sum_{n\geq 1} y^{n-1}\,\Psi_n\right) = \sum_{n\geq 1} (xy)^{n-1}\, n\,\Psi_n \ ,$$

the last equality following from the fact that $S_n * F = F$ for $F \in \mathbf{Sym}_n$.

Hence, $\Psi_n * \Psi_n = n\Psi_n$, so that $\theta_n = \alpha^{-1}(\Psi_n)$ is a quasi-idempotent of Σ_n. To see what it looks like, we have to expand Ψ_n on the ribbon basis. The linear recurrence (21) together with the multiplication formula for ribbons (27) (recall that $S_k = R_k$) easily yields

$$\Psi_n = R_n - R_{1,n-1} + R_{1,1,n-2} - \cdots = \sum_{k=0}^{n-1}(-1)^k R_{1^k,n-k}\,, \tag{55}$$

which is analogous to the classical expression of p_n as the alternating sum of hook Schur functions. Therefore, in the descent algebra,

$$\theta_n = \sum_{k=0}^{n-1}\,(-1)^k\, D_{\{1,2,\ldots,k\}}\,. \tag{56}$$

On this expression, we can recognize a famous element of the group algebra of the symmetric group, namely, Dynkin's left-bracketing operator [20] (see also [100, 109, 29, 2, 94]). The *standard left bracketing* of a word $w = x_1x_2\cdots x_n$ is the Lie polynomial

$$L_n(w) = [\cdots[[[x_1,x_2],x_3],x_4],\ldots,x_n]\ . \tag{57}$$

This formula defines a linear operator L_n on the homogeneous component $\mathbb{K}_n\langle A\rangle$ of the free associative algebra $\mathbb{K}\langle A\rangle$. In terms of the right action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{K}_n\langle A\rangle$, defined on words by

$$x_1\,x_2\,\cdots\,x_n\cdot\sigma = x_{\sigma(1)}\,x_{\sigma(2)}\,\cdots\,x_{\sigma(n)}\ , \tag{58}$$

one can write

$$L_n(w) = \sum_{\sigma\in\mathfrak{S}_n} a_\sigma\,(w\cdot\sigma) = w\cdot\theta_n$$

the coefficient a_σ being ± 1 or 0, according to whether σ is a "hook permutation" or not. To see this, one just has to write the permutations appearing in the first θ_i as ribbon tableaux and then to argue by induction. For example,

$$\theta_3 = [[1,2],3] = 123\ -\ \begin{array}{ll}2 & \\ 1 & 3\end{array}\ -\ \begin{array}{ll}3 & \\ 1 & 2\end{array}\ +\ \begin{array}{l}3\\2\\1\end{array}$$

and it is clear that when expanding $\theta_4 = [\theta_3, 4]$ one will only get those (signed) tableaux obtained from the previous ones by adding 4 at the end of the last row, minus those obtained by adding 4 on the top of the first column. Thus, we have proved that $\frac{1}{n}L_n$ is a projector, whose image is obviously a subspace of the free Lie algebra. By iteration of Jacobi's identity, it is easy to see that any Lie element can be written as a linear comnbination of standard left bracketings, so that what we have actually obtained is a proof of Dynkin's characterization of Lie elements: a noncommutative polynomial $P \in \mathbb{K}\langle A\rangle$ is a Lie polynomial iff $L_n(P) = nP$ [20, 94, 7].

Idempotents such as $\frac{1}{n}\theta_n$, acting as projectors onto the free Lie algebra are usually called *Lie idempotents* [29, 2, 94]. They play an important role in the analysis of the Hausdorff series, or as the building blocks of other idempotents, such as the Eulerian idempotents, used in Hodge-type decompositions of certain cohomology theories (see [38, 73, 75, 90, 88, 89]).

So far, our formalism has just led us to an exotic proof of a classical result. Let us now see whether the method contains the germ of some generalization. One ingredient in our proof was the analogue (55) of the expansion of a power sum as an alternating sum of hook Schur functions. This expression has a well known q-analogue, namely, the one involved in the character formula for Hecke algebras (1). It can be written in the form

$$\frac{h_n((1-q)X)}{1-q} = \sum_{k=0}^{n-1}(-q)^k s_{n-k,1^k}\,. \tag{59}$$

Let us look for a noncommutative analogue of this expression. To this aim, it will be convenient to extend as much as possible the λ-ring notation of the classical theory. Given two totally ordered sets A and B of noncommuting variables, we can define the virtual alphabet $A-B$ by specifying its complete symmetric functions $S_n(A-B)$. Their generating series is defined by

$$\sigma(A-B;t) := \lambda(-B;t)\sigma(A;t)\,. \tag{60}$$

One also defines the symmetric functions of $A+B$ by

$$\sigma(A+B;t) := \sigma(A;t)\sigma(B;t) \tag{61}$$

where now, A and B can be either real or virtual, and for a real ordered commutative alphabet X, the virtual alphabet XA is defined by

$$\sigma(XA;t) = \prod_{i\geq 1}^{\rightarrow} \sigma(A, tx_i)\,. \tag{62}$$

These definitions imply a definition of quasi-symmetric functions of a difference $f(X-Y)$, which is the same as the one obtained by composing the comultiplication and the antipode, and we can now give a meaning to the noncommutative symmetric functions of all virtual alphabets of the type $(X \pm Y)(A \pm B)$.

The case we have in mind corresponds to $X = \{1\}$ and $Y = \{q\}$. According to our definitions,

$$\begin{aligned}\sigma((1-q)A;t) &= \sigma(A-qA;t) = \lambda(qA;-t)\sigma(A;t)\\ &= \textstyle\sum_{k\geq 0} t^k(-q)^k\Lambda_k(A)\sum_{l\geq 0} t^l S_l(A)\,.\end{aligned}$$

Taking into account the fact that $\Lambda_k = R_{1^k}$ and applying the multiplication rule for ribbons, we obtain

$$S_n((1-q)A) = (1-q)\sum_{k=0}^{n-1}(-q)^k R_{1^k,n-k}\,, \tag{63}$$

the q-analogue we were looking for. The symmetric function

$$\Theta_n(q) := \frac{S_n((1-q)A)}{1-q} \tag{64}$$

corresponds to a natural q-analogue of Dynkin's element in the descent algebra. Indeed, writing permutations as words in the letters $1, 2, \ldots, n$, it is easy to see that the image of $\Theta_n(q)$ under the isomorphism α^{-1} is the left q-bracketing of the standard word $1\,2\,\ldots\,n$, that is,

$$\alpha^{-1}(\Theta_n(q)) = \theta_n(q) = [[\ldots[[\,1,\,2\,]_q,\,3\,]_q,\,\ldots]_q,\,n\,]_q$$

where $[R, S]_q = RS - q\,SR$. Now, we can prove the following q-analog of Dynkin's theorem, which, according to what we have already seen, can be understood as describing the left q-bracketing of a homogeneous Lie polynomial:

$$\Theta_n(q) * \Psi_n = [n]_q\,\Psi_n\ . \tag{65}$$

The proof works exactly as the previous one. Let

$$\vartheta(t) = \sum_{n\geq 1}\Theta_n(q)t^{n-1} = \frac{\sigma((1-q)A;t)-1}{(1-q)t}\,. \tag{66}$$

It is easy to see that

$$\vartheta(t) = \lambda(A;-qt)\sigma'_q(A;t) \tag{67}$$

where σ'_q denotes the the q-derivative

$$\sigma'_q(t) := \frac{\sigma(t)-\sigma(qt)}{(1-q)t} \tag{68}$$

with respect to t. Then, one can write

$$\vartheta(t)*\Psi_n = \mu((\lambda(A;-qt)\otimes\sigma'_q(A;t))*\Delta(\Psi_n)) = \mu((\lambda(A;-qt)\otimes\sigma'_q(A;t))*(1\otimes\Psi_n+\Psi_n\otimes 1))\,,$$

from which it follows that

$$\vartheta(t)*\Psi_n = (\lambda(A;-qt)*1)\ (\sigma'_q(A;t)*\Psi_n) = [n]_q\,\Psi_n\,t^{n-1}\ .$$

Equation (65) means that homogeneous Lie polynomials of degree n are again eigenvectors of the left q-bracketing operator, now with the q-integer $[n]_q$ as eigenvalue. They actually constitute the $[n]_q$ eigenspace. However, the q-Dynkin element $\theta_n(q)$ is invertible in the group algebra for generic values of q, and its other eigenvalues are nonzero. It would therefore be of interest to investigate its spectral decomposition. This will be done in Section 4

In the last two examples, we have been interpreting noncommutative symmetric functions as linear operators on the free algebra $\mathbb{K}\langle A\rangle$, by means of the identification of $\mathbf{Sym}_n$

with Σ_n and of the right action of permutations. If we extend the action of $\mathfrak{S}_n$ to all words by deciding that $w\sigma = 0$ if w is not of length n, we obtain in this way a representation

$$\rho : \mathbf{Sym} \longrightarrow \mathrm{End}^{\mathrm{gr}}(\mathbb{K}\langle A\rangle) \tag{69}$$

where $\mathrm{End}^{\mathrm{gr}}$ is the algebra of degree-preserving endomorphism. Under the representation ρ, the internal product $*$ is mapped to the composition $\circ$, and, as shown by Reutenauer [94], the ordinary product becomes the *convolution product* $\star$ of $\mathrm{End}^{\mathrm{gr}}(\mathbb{K}\langle A\rangle)$, which is defined by

$$f \star g = m \circ (f \otimes g) \circ C \tag{70}$$

where $m : u \otimes v \mapsto uv$ is the multiplication map of $\mathbb{K}\langle A\rangle$ and C is its standard comultiplication, defined by $C(x) = x \otimes 1 + 1 \otimes x$ for $x \in A$ and $C(uv) = C(u)C(v)$.

3 Representation theoretical interpretations

The noncommutative ribbon Schur functions R_I and the quasi-symmetric functions F_I share many properties with ordinary Schur functions. In particular, the structure constants of **Sym** and $QSym$ in these bases are nonnegative integers. This suggests the existence of representation theoretical interpretations of these generalized symmetric functions. Such interpretations have actually been found [19, 17, 60, 61], and turned out to be related to the specialization $q = 0$ in certain q-deformations of the classical structures related to Schur functions.

3.1 The 0-Hecke algebra

The first intepretation involves the type A Hecke algebras $H_n(q)$ at $q = 0$. Recall that when q is a generic complex number, i.e. neither zero nor a root of unity, the $H_n(q)$ are semi-simple (in fact isomorphic to $\mathbb{C}\mathfrak{S}_n$) and the direct sum

$$\mathcal{R}(q) = \bigoplus_{n\geq 0} R(H_n(q)) \tag{71}$$

where $R(H_n(q))$ is the vector space spanned by isomorphism classes $[M]$ of finite dimensional $H_n(q)$-modules, with addition induced by direct sum, can be identified with Sym, the simple modules $S_\lambda(q)$ (q-Specht modules) being represented by Schur functions. This defines a characteristic map $\mathrm{ch} : R(H_n(q)) \to Sym_n$. The usual product of symmetric functions corresponds then to induction from $H_m \otimes H_n$ to H_{m+n}, i.e., for $\lambda \vdash m$ and $\mu \vdash n$,

$$\mathrm{ch}\,\left(S_\lambda(q) \otimes S_\mu(q) \uparrow_{H_m(q)\otimes H_n(q)}^{H_{m+n}(q)}\right) = s_\lambda s_\mu\,. \tag{72}$$

This statement summarizes a good deal about the representation theory of Hecke algebras in the generic case.

For non generic values of the parameter, when $H_n(q)$ is not semi-simple, it is a difficult task to describe $R(H_n(q))$ as defined above, since this would amount to understand *all* the indecomposable representations up to isomorphism. The usual strategy to investigate a non semi-simple algebra $\mathcal{A}$ is to introduce two kinds of *Grothendieck groups* (cf. [11]). The first one, usually denoted by $G_0(\mathcal{A})$ is the quotient of $R(\mathcal{A})$ by the relations $[M] = [M'] + [M'']$ whenever there is a short exact sequence $0 \to M' \to M \to M'' \to 0$. In $G_0(\mathcal{A})$, $[M] = [N]$ whenever M and N have the same simple composition factors, occuring with the same multiplicities. The second one, denoted by $K_0(\mathcal{A})$, is the free abelian group generated by isomorphism classes of finite dimensional projective $\mathcal{A}$-modules. Here, addition corresponds to direct sum. In the sequel, we will rather mean by G_0 and K_0 the complexified Grothendieck groups.

A particularly interesting example of this situation occurs for $\mathcal{A} = H_n(\zeta)$, where ζ is a primitive k-th root of unity. Then, the direct sums

$$\mathcal{G}(\zeta) = \bigoplus_{n\geq 0} G_0(H_n(\zeta)) \quad \text{and} \quad \mathcal{K}(\zeta) = \bigoplus_{n\geq 0} K_0(H_n(\zeta)) \tag{73}$$

can be respectively identified with a quotient and with a subalgebra of Sym. Precisely, if we denote by $\mathcal{J}_k$ the ideal of Sym generated by the power sums p_{nk}, $n \geq 1$, and by $\mathcal{S}_k$ the subalgebra generated by the p_m such that $m \not\equiv 0 \bmod k$, then

$$\mathcal{G}(\zeta) \simeq Sym/\mathcal{J}_k \quad \text{and} \quad \mathcal{K}(\zeta) \simeq \mathcal{S}_k\,. \tag{74}$$

The characteristic map ch : $\mathcal{G}(\zeta) \to Sym/\mathcal{J}_k$ realizing the first isomorphism is compatible with the previous one, in the sense that the specialized Specht module $S_\lambda(\zeta)$ is mapped to the class $\bar{s}_\lambda = s_\lambda \bmod \mathcal{J}_k$. The induction formula remains valid, too, but this does not tell us that much about the representation theory of $H_n(\zeta)$. This is because $S_\lambda(\zeta)$ is usually not irreducible, and essential information is contained in the multiplicities $d_{\lambda\mu}$ of its simple composition factors D_μ. These numbers are called the decomposition numbers, and they form the *decomposition matrix* of $H_n(\zeta)$. These matrices have been determined only quite recently [65, 1], the starting point being the identification of $\mathcal{G}(\zeta)$ and $\mathcal{K}(\zeta)$ with the basic representation of the affine Lie algebra $\widehat{sl}_k$. In principle, these numbers could also be obtained by Schur-Weyl duality from the Lusztig conjecture [77] proved by Kazhdan and Lusztig [52, 53].

On the other hand, the non semi-simple Hecke algebras $H_n(0)$ are quite well-understood. The representation theory of 0-Hecke algebras for general type has been worked out by Norton [86], and special combinatorial features of type A have been described by Carter [9].

There are 2^{n-1} simple $H_n(0)$-modules, which are all one-dimensional [9, 86]. To see this, it is sufficient to observe that $(T_iT_{i+1} - T_{i+1}T_i)^2 = 0$. Thus, all the commutators $[T_i, T_j]$ are in the radical of $H_n(0)$. But the quotient of $H_n(0)$ by the ideal generated by these elements is the commutative algebra generated by $n-1$ elements $t_1, \ldots, t_{n-1}$ such that $t_i^2 = -t_i$. It is easy to check that this algebra has no nilpotent elements, so that it is $H_n(0)/\mathrm{rad}\,(H_n(0))$. The irreducible representations are obtained by sending a set of generators to -1 and its complement to 0. We shall label these representations by compositions of n rather than by subsets of generators. Let I be a composition of n and let $\mathrm{Des}\,(I)$ the associated subset of $\{1, \ldots, n-1\}$. The irreducible representation φ_I of $H_n(0)$ is defined by

$$\varphi_I(T_i) = \begin{cases} -1 & \text{if } i \in \mathrm{Des}\,(I)\ , \\ 0 & \text{if } i \notin \mathrm{Des}\,(I)\ . \end{cases} \tag{75}$$

The corresponding $H_n(0)$-module is denoted by $\mathbf{C}_I$. These modules (when I runs over all compositions of n) form a complete system of simple $H_n(0)$-modules.

The direct sum of the Grothendieck groups

$$\mathcal{G}(0) = \bigoplus_{n \geq 0} G_0(H_n(0)) \tag{76}$$

has therefore a canonical basis, the classes $[\mathbf{C}_I]$ of the simple modules, which is naturally labelled by compositions. One may then look for a characteristic map with values in **Sym** or in $QSym$. The correct choice is to identify $\mathcal{G}(0)$ with the algebra of quasi-symmetric functions, the characteristic map being given by

$$\mathrm{ch}\,([\mathbf{C}_I]) = F_I\ . \tag{77}$$

This map is again compatible with the characterstic map of the generic case. One has a decomposition map d : $\mathcal{G}(q) \to \mathcal{G}(0)$ sending the class $[S_\lambda(q)]$ of a generic Specht module to the class $[S_\lambda(0)]$ of its 0-specialization, which is usually not irreducible, nor even semi-simple. The choice (77) has the property that

$$\mathrm{ch} \circ d = d \circ \mathrm{ch} \tag{78}$$

that is, $\mathrm{ch}\,(S_\lambda(0)) = s_\lambda$, regarded as a quasi-symmetric function.

From this, we can easily recover Carter's combinatorial description of the decomposition matrix of $H_n(0)$. The multiplicity $d_{\lambda I}$ of the simple module $\mathbf{C}_I$ as a composition factor of $S_\lambda(0)$ is equal to the coefficient of F_I in the quasi-symmetric expansion

$$s_\lambda = \sum_I d_{\lambda I} F_I \tag{79}$$

so that by duality, $d_{\lambda I} = \langle s_\lambda\,,\, R_I \rangle$, and by (44), this is equal to the ordinary scalar product of symmetric functions $\langle s_\lambda\,,\, r_I\rangle$. By the Littlewood-Richardson rule, this is the number of Yamanouchi words y of weight λ and ribbon shape I, i.e., such that $\mathrm{Des}\,(y) = \mathrm{Des}\,(I)$. These Yamanouchi words can be encoded by standard tableaux of shape λ, for example by their Q-symbols in the Robinson-Schensted correspondence. These are exactly the tableaux obtained by Carter in [9].

The characteristic map at $q = 0$ is also compatible with the induction product, that is, once again,

$$\mathrm{ch}\,\left([M \otimes N] \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}\right) = \mathrm{ch}\,([M])\mathrm{ch}\,([N]) \tag{80}$$

In particular, the composition factors of the induction product of two simple modules $\mathbf{C}_I$ and $\mathbf{C}_J$ are described by the product $F_I F_J$, wich is given by an interesting combinatorial formula.

To state it, let us first define the shape composition $I = C(\sigma)$ of a permutation σ as the unique composition of n such that $\mathrm{Des}\,(\sigma) = \mathrm{Des}\,(I)$. Now, suppose that $|I| = m$ and $|J| = n$. Take any permutation $\alpha \in \mathfrak{S}_m$ such that $C(\alpha) = I$ and any permutation $\beta \in \mathfrak{S}_n$ such that $C(\beta) = J$. We consider permutations as words on the letters $1, 2, \ldots$, and we denote by $\beta[m]$ the shifted word

$$\beta[m] = (\beta_1 + m)\cdot(\beta_2 + m)\cdots(\beta_n + m)\,. \tag{81}$$

Recall that the *shuffle* product on words is defined inductively by

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)\,, \quad a, b \in A \;\; u, v \in A^* \tag{82}$$

the initial condition being that the empty word is the unit element. The product formula is then [81]

$$F_I F_J = \sum_\sigma (\alpha \sqcup\!\sqcup \beta[m]\,,\, \sigma) F_{C(\sigma)} \tag{83}$$

where $(\alpha \sqcup\!\sqcup \beta[m]\,,\, \sigma)$ denotes the coefficient of σ in $\alpha \sqcup\!\sqcup \beta[m]$.

The simple modules can be realized as minimal left ideals of $H_n(0)$. To describe the generators, we associate with a composition I of n two permutations $\alpha(I)$ and $\omega(I)$ of $\mathfrak{S}_n$ defined as follows. The permutation $\alpha(I)$ is obtained by filling the columns of the ribbon diagram of I from bottom to top and from left to right with the numbers $1, 2, \ldots, n$, and $\omega(I)$ is the permutation obtained by filling the rows of the same diagram from left to right and from bottom to top with $1, 2, \ldots, n$.

For example, with $I = (22113)$, the fillings of the diagram corresponding to $\alpha(I)$ and $\omega(I)$ are

$$\begin{array}{ccccc} 1 & 3 & & & \\ & 2 & 7 & & \\ & & 6 & & \\ & & 5 & & \\ & & 4 & 8 & 9 \end{array} \qquad\qquad \begin{array}{ccccc} 8 & 9 & & & \\ & 6 & 7 & & \\ & & 5 & & \\ & & 4 & & \\ & & 1 & 2 & 3 \end{array}$$

$\alpha(22113)$ $\qquad$ $\omega(22113)$

Thus $\alpha(22113) = 132765489$ and $\omega(22113) = 896754123$.

Let us now set $\Box_i = 1 + T_i$. These elements verify the braid relations, together with $\Box_i^2 = \Box_i$. As the $\Box_i$ satisfy the braid relations, one can as usual associate to each permutation $\sigma \in \mathfrak{S}_n$ the element $\Box_\sigma$ of $H_n(0)$ defined by $\Box_\sigma = \Box_{i_1} \ldots \Box_{i_r}$ where $\sigma_{i_1} \ldots \sigma_{i_r}$ is an arbitrary reduced decomposition of σ.

For a composition $I = (i_1, \ldots, i_r)$ we denote by $\overline{I} = (i_r, \ldots, i_1)$ its mirror image and by $I^\sim$ its conjugate composition, i.e. the composition obtained by writing from right to left the heights of the columns of the ribbon diagram of I, e.g., $\overline{(3,2,1)} = (1,2,3)$ and $(3,2,1)^\sim = (2,2,1,1)$.

Then, the simple $H_n(0)$ module $\mathbf{C}_I$ is isomorphic to the minimal left ideal $H_n(0)\,\eta_I$, where

$$\eta_I = T_{\omega(\overline{I})}\,\Box_{\alpha(I^\sim)}\,. \tag{84}$$

Let us now look at the K_0-groups of the 0-Hecke algebras. The indecomposable projective $H_n(0)$-modules have also been classified by Norton (cf. [9, 86]). From the general theory of finite dimensional algebras, one knows that for each simple module S, there is a unique indecomposable projective module M such that $S = M/\mathrm{rad}\,(M)$. Thus, one associates to each composition I of n the unique indecomposable projective $H_n(0)$-module $\mathbf{M}_I$ such that $\mathbf{M}_I/\mathrm{rad}\,(\mathbf{M}_I) \simeq \mathbf{C}_I$. The canonical duality between $G_0(\mathcal{A})$ and $K_0(\mathcal{A})$, defined by the pairing

$$\langle N\,,\,M\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(M,N) \tag{85}$$

tells us that the direct sum of Grothendieck groups

$$\mathcal{K}(0) = \bigoplus_{n\ge 0} K_0(H_n(0)) \tag{86}$$

has to be identified with the algebra of noncommutative symmetric functions. The functorial duality between induction and restriction implies that the characteristic map

$$\mathrm{Ch}\;:\quad \mathcal{K}(0) \longrightarrow \mathbf{Sym} \tag{87}$$

$$[\mathbf{M}_I] \longmapsto R_I \tag{88}$$

is a ring homomorphism, i.e.,

$$Ch(\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}) = R_I\,R_J = R_{I\cdot J} + R_{I\triangleright J}\,. \tag{89}$$

The $\mathbf{M}_I$ are the indecomposable summands of the left regular representation. As a left ideal, $\mathbf{M}_I = H_n(0)\,\nu_I$ where

$$\nu_I = T_{\alpha(I)}\,\Box_{\alpha(\overline{I}^\sim)}\,. \tag{90}$$

Since $\alpha(I^\sim)^{-1} = \omega_n\,\omega(\overline{I})$, one gets $\mathrm{Des}\,(\alpha(\overline{I}^\sim)^{-1}) = [1, n{-}1] - \mathrm{Des}\,(I)$. It follows that the generator ν_I of $\mathbf{M}_I$ is different from 0 and that a basis of $\mathbf{M}_I$ is

$$\{\;T_\sigma\,\Box_{\alpha(\overline{I}^\sim)},\ \mathrm{Des}\,(\sigma) = \mathrm{Des}\,(I)\;\} = \{\;T_\sigma\,\Box_{\alpha(\overline{I}^\sim)},\ \sigma \in [\alpha(I), \omega(I)]\;\}\;. \tag{91}$$

Hence, the dimension of $\mathbf{M}_I$ is equal to the cardinality of the descent class

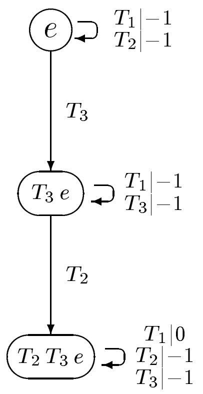
$$[\alpha(I), \omega(I)] = \{\sigma | C(\sigma) = I\}\,.$$

The family $(\mathbf{M}_I)_{|I|=n}$ forms a complete system of indecomposable projective $H_n(0)$-modules, and the decomposition of the left regular representation

$$H_n(0) = \bigoplus_{|I|=n} \mathbf{M}_I \tag{92}$$

is multiplicity free.

As an illustration, let us take $I = (1,1,2)$. Then $I^\sim = (1,3)$, $\overline{I} = (2,1,1)$, $\overline{I}^\sim = (3,1)$, $\alpha(I) = 3214$ and $\alpha(\overline{I}^\sim) = 1243$. Hence $\nu_{112} = T_2\,T_1\,T_2\,\square_3$. The module $\mathbf{M}_{112}$ is described by the following graph. An arrow labelled T_i going from f to g means that $T_i \cdot f = g$, and a loop on the vertex f labelled $T_i|\epsilon$ (with $\epsilon = 0$ or $\epsilon = -1$) means that $T_i \cdot f = \epsilon\, f$.



This is also the graph of the interval $[3214, 4312] = D_{112}$ in the permutohedron of $\mathfrak{S}_4$. The (-1)-loops at a vertex correspond to the descents of the inverse of the permutation labelling this vertex.

Let M be a finite-dimensional $H_n(0)$-module, and consider a composition series, i.e. a decreasing sequence $M = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_k \supseteq M_{k+1} = 0$, such that each M_i/M_{i+1} is simple. One can refine the characteristic $\mathrm{ch}\,(M)$ into a graded version (a q-analogue), at least when M is a cyclic module i.e. when $M = H_n(0)\,e$. In this case, the length filtration

$$H_n(0)^{(k)} = \bigoplus_{l(\sigma)\geq k} \mathbb{C}\,T_\sigma$$

of the 0-Hecke algebra induces a filtration $(M^{(k)})_{k\in\mathbb{N}}$ of M, defined by $M^{(k)} = H_n(0)^{(k)}\,e$. This suggests to define the *graded characteristic* $\mathrm{ch}_q(M)$ of M by

$$\mathrm{ch}_q(M) = \sum_{k\geq 0} q^k\,\mathrm{ch}\,(M^{(k)}/M^{(k+1)})\ .$$

The ordinary characteristic $\mathrm{ch}\,(M)$ is then the specialization of $\mathrm{ch}_q(M)$ at $q = 1$.

The graded characteristic is in particular defined from the modules induced by tensor products of simple 1-dimensional modules

$$M_{I_1,\ldots,I_r} = \mathbf{C}_{I_1} \otimes \cdots \otimes \mathbf{C}_{I_r} \uparrow_{H_{n_1}(0)\otimes\cdots\otimes H_{n_r}(0)}^{H_{n_1+\cdots+n_r}(0)}\ ,$$

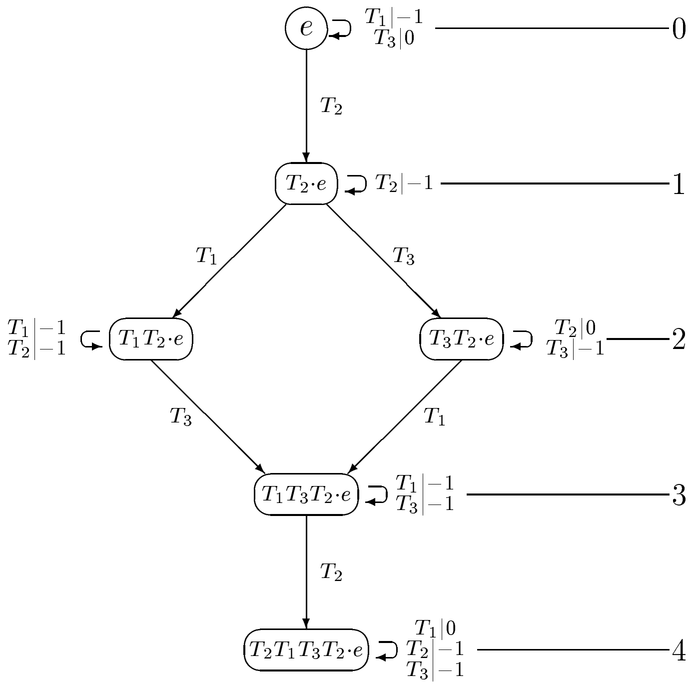
whose characteristics are the products $F_{I_1}\,\cdots\,F_{I_r}$. The induction formula can be refined to take into account the filtrations, and this leads to a q-analogue of the algebra of quasi-symmetric functions. This q-analogue is defined in terms of the q-shuffle product [17].

Let A be an alphabet and let q be an indeterminate commuting with A. The q-shuffle is the bilinear operation of $\mathbb{N}[q]\langle A\rangle$ denoted by $⧢_q$ and recursively defined on words by the relations

$$\begin{cases} 1 ⧢_q u = u ⧢_q 1 = u \ , \\ (a\,u) ⧢_q (b\,v) = a\,(u ⧢_q b\,v) + q^{|au|}\, b\,(a\,u ⧢_q v) \ , \end{cases}$$

where 1 is the empty word, $u, v \in A^*$, $a, b \in A$ and $|w|$ denotes the length of a word w. One can show that $⧢_q$ is associative (cf. [17]).

For example, let $M_{(11),(2)}$ denote the $H_4(0)$-module obtained by inducing to $H_4(0)$ the $H_2(0) \otimes H_2(0)$-module $\mathbf{C}_{11} \otimes \mathbf{C}_2$, identifying $H_2(0) \otimes H_2(0)$ with the subalgebra of $H_4(0)$ generated by T_1 and T_3. This $H_4(0)$-module is generated by a single element e on which T_1 and T_3 act by $T_1 \cdot e = -e$ and by $T_3 \cdot e = 0$. The following graph gives a complete description of this module. The vertices correspond to a basis of $M_{(11),(2)}$ formed by the images of e under the action of some elements of $H_4(0)$.



The graph is graded by the distance $d(f)$ of a vertex f to the origin e as indicated on the picture. This grading is precisely the one described by ch_q. That is, if we associate with each vertex f the composition $I(f)$ of 4 whose associated subset of $[1,3]$ is $D(f) = \{\, i \in [1,3] \mid T_i \cdot f = -f \,\}$, we find

$$\mathrm{ch}_q(M_{(11,2)}) = \sum_f q^{d(f)}\, F_{I(f)} = F_{13} + q\,F_{22} + q^2\,(F_{112} + F_{31}) + q^3\,F_{121} + q^4\,F_{211} \ .$$

This equality can also be read on the q-shuffle of 21 and 34:

$$21 ⧢_q 34 = 2134 + q\,2314 + q^2\,2341 + q^2\,3214 + q^3\,3241 + q^4\,3421 \ .$$

One obtains the graded characteristic by replacing each permutation σ in this expansion by the quasi-symmetric function $F_{C(\sigma)}$.

This example illustrates the general fact that the graded characteristic of an induced module as above is always given by a q-shuffle. As this is an associative operation, one obtains in this way a q-deformation of the ring of quasi-symmetric functions [19, 17, 60, 104]. More precisely, let I and J be compositions of m and n. Let also $\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$ be such that $C(\alpha) = I$ and $C(\beta) = J$. The $H_{m+n}(0)$-module obtained by inducing to $H_{m+n}(0)$ the $H_m(0) \otimes H_n(0)$-module $\mathbf{C}_I \otimes \mathbf{C}_J$ is cyclic, and its canonical filtration is described by its graded characteristic, given by

$$\mathrm{ch}_q(\mathbf{C}_I \otimes \mathbf{C}_J \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}) = \sum_{\sigma \in \mathfrak{S}_{m+n}} (\alpha \sqcup\!\sqcup_q \beta[m] \, , \, \sigma) F_{C(\sigma)} \, . \tag{93}$$

The algebra $QSym_q$ of *quantum quasi-symmetric functions* [104] can now be defined as the associative algebra with generators F_I labelled by all compositions, and multiplication rule

$$F_I F_J = \sum_\sigma (\alpha \sqcup\!\sqcup_q \beta[m] \, , \, \sigma) F_{C(\sigma)} \tag{94}$$

given by the natural q-analogue of (83).

This algebra is not commutative, and it is free for generic q. In this case, it is isomorphic to the algebra of noncommutative symmetric functions. The natural isomorphism $\mathbf{Sym} \to QSym_q$, denoted by $P \mapsto \hat{P}$, is given by $\hat{S}_n = F_n$. Therefore, $\mathbf{Sym}$ gets identified with a q-analogue of its dual. This suggest to define a scalar product on $QSym_q$ by

$$(F_I | \hat{R}_I) = \delta_{IJ} \, . \tag{95}$$

The scalar products $(\hat{R}_I | \hat{R}_J)$ are then q-analogues of the Cartan invariants of $H_n(0)$ (i.e., the multiplicities of the simple composition factors $\mathbf{C}_I$ in the indecomposable projective modules $\mathbf{M}_J$), and one can show that they describe, up to a shift, their canonical filtrations.

When q is a root of unity, the isomorphism breaks down, and the scalar product degenerates. There are, however, other singular values of q which are still not understood.

The quantum quasi-symmetric functions can also be realized as a subalgebra of the quantum polynomial ring $\mathbb{C}_q[X]$, generated by variables x_i such that $x_j x_i = q x_i x_j$ for $j > i$ [104].

3.2 The 0-quantum GL_n

We shall now look for an analogue of the interpretation of Schur functions as characters of polynomial representations of general linear groups. Polynomial representations are dual to comodules over the Hopf algebra $\mathbb{C}[GL_n]$ of polynomial functions on the algebraic group GL_n, and one may think of putting $q = 0$ in the quantized function algebra $GL_q(n) = \mathbb{C}_q[GL_n]$. However, the standard version of this algebra [23] involve factors such as $q - q^{-1}$ in its defining relations, and cannot be defined at $q = 0$. One has to use a different version, introduced by Dipper and Donkin [15], which is also a specialization of the two-parameter quantization defined by Takeuchi [103].

Let

$$V = \bigoplus_{i=1}^{n} \mathbb{C}(q) \, e_i \tag{96}$$

be the $\mathbb{C}(q)$-vector space with basis (e_i). The right action of $\mathfrak{S}_N$ on $V^{\otimes N}$ is defined as usual by $v_1 \otimes \cdots \otimes v_N \, \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}$. Let $\mathbf{v} = e_{k_1} \otimes \cdots \otimes e_{k_N}$. Following [50, 15], one defines a right action of $H_N(q)$ on $V^{\otimes N}$ by

$$\left\{\begin{array}{lcl} \mathbf{v}\cdot T_i &=& \mathbf{v}\sigma_i \qquad\qquad \text{if } k_i < k_{i+1}\ , \\ \mathbf{v}\cdot T_i &=& q\,\mathbf{v} \qquad\qquad \text{if } k_i = k_{i+1}\ , \\ \mathbf{v}\cdot T_i &=& q\,\mathbf{v}\sigma_i + (q-1)\,\mathbf{v} \qquad \text{if } k_i > k_{i+1}\ . \end{array}\right. \tag{97}$$

The quantum matrix algebra $A_q(n)$ is defined as the $\mathbb{C}(q)$-algebra generated by the n^2 elements $(x_{ij})_{1\le i,j\le n}$ subject to the relations

$$\left\{\begin{array}{rcll} x_{jk}\,x_{il} &=& q\,x_{il}\,x_{jk} & \text{for } i<j,\ k\le l\ , \\ x_{ik}\,x_{il} &=& x_{il}\,x_{ik} & \text{for every } i,k,l\ , \\ x_{jl}\,x_{ik} - x_{ik}\,x_{jl} &=& (q-1)\,x_{il}\,x_{jk} & \text{for } i<j,\ k<l\ . \end{array}\right.$$

This is the quantization of the Hopf algebra of polynomial functions on the variety of $n\times n$ matrices introduced by Dipper and Donkin in [15]. It is not isomorphic to the classical quantization of Faddeev-Reshetikin-Takhtadzhyan [23], and although for generic values of q both versions play essentially the same rôle, an essential difference is that the Dipper-Donkin algebra is defined for $q=0$.

$A_q(n)$ is a Hopf algebra with comultiplication Δ defined by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik}\otimes x_{kj}\ .$$

Moreover one can define a left coaction δ of $A_q(n)$ on $V^{\otimes N}$ by

$$\delta(e_i) = \sum_{j=1}^{n} x_{ij}\otimes e_j$$

and the left coaction δ of $A_q(n)$ on $V^{\otimes N}$ commutes with the right action of $H_N(q)$ on $V^{\otimes N}$. This property still holds for $q=0$. Thus, for any $h\in H_N(0)$, $V^{\otimes N}h$ will be a sub-$A_0(n)$-comodule of $V^{\otimes N}$. For later reference, note that the defining relations of $A_0(n)$ are

$$\left\{\begin{array}{rcll} x_{jk}\,x_{il} &=& 0 & \text{for } i<j,\ k\le l\ , \\ x_{ik}\,x_{il} &=& x_{il}\,x_{ik} & \text{for every } i,k,l\ , \\ x_{jl}\,x_{ik} &=& x_{ik}\,x_{jl} - x_{il}\,x_{jk} & \text{for } i<j,\ k<l\ . \end{array}\right. \tag{98}$$

The quantum matrix algebra $A_q(n)$ has an interesting subalgebra, the quantum diagonal algebra (or *quantum torus*) $\Delta_q(n)$, which is the subalgebra generated by the diagonal coordinates $x_{11},\ldots,x_{nn}$. It is the specialization of this subalgebra which will provide us with the analogue of the plactic algebra for quasi-symmetric functions.

Recall that the *plactic algebra* on a totally ordered alphabet A is the $\mathbb{C}$-algebra $Pl(A)$, quotient of $\mathbb{C}\langle A\rangle$ by the relations

$$\left\{\begin{array}{lll} aba = baa\ , & bba = bab & \text{for } a<b\ , \\ acb = cab\ , & bca = bac & \text{for } a<b<c\ . \end{array}\right.$$

These relations, which were obtained by Knuth [56], generate the equivalence relation identifying two words which have the same P-symbol under the Robinson-Schensted correspondence. Though Schensted had shown that the construction of the P-symbol is an associative operation on words, the monoid structure on the set of tableaux has been mostly studied by Lascoux and Schützenberger [69] under the name 'plactic monoid'.

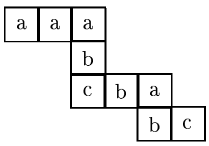
It turns out that the specialization of the quantum diagonal algebra at $q = 0$ is a remarkable quotient of the plactic algebra that we shall now introduce.

The *hypoplactic algebra* $HPl(A)$ is the quotient of the plactic algebra $Pl(A)$ by the quartic relations

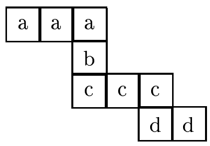
$$\begin{cases} baba = abab \quad , \quad baca = abac & \text{for } a < b < c \,, \\ cacb = acbc \quad , \quad cbab = bacb & \text{for } a < b < c \,, \\ badc = dbca \quad , \quad acbd = cdab & \text{for } a < b < c < d \,. \end{cases} \tag{99}$$

The combinatorial objects playing the rôle of Young tableaux are called ribbons and quasi-ribbons. Let I be a composition. A *quasi-ribbon tableau* of shape I is a ribbon diagram r of shape I filled by letters of A in such a way that each row of r is nondecreasing from left to right, and each column of r is strictly increasing from *top to bottom*. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-ribbon diagram of shape I.

For example, the word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-ribbon tableau, as one can see on the picture. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3, 1, 3, 2)$. The quasi-ribbon tableau corresponding to v is also given below.



aacbabbac



aacbacdcd

In the classical case, the crucial point of the theory of the plactic monoid is that each plactic class contains exactly one tableau. Similarly, one can show that each hypoplactic class contains exactly one quasi-ribbon word, so that the classes of the quasi-ribbon words form a linear basis of the hypoplactic algebra $HPl(A)$ [60].

The ring homomorphism defined by $\varphi : a_i \longrightarrow x_{ii}$ is an isomorphism between the hypoplactic algebra $HPl(A)$ and the specialization $\Delta_0(n)$ of the quantum diagonal algebra. One can show that the hypoplactic quasi-ribbons

$$F_I = \sum_{w \in QR(I)} w \quad \in HPl(A) \tag{100}$$

where the sum runs over the classes of all quasi-ribbon words of shape I, span a commutative subalgebra of $HPl(A)$, isomorphic to the algebra of quasi-symmetric functions.

This will be a consequence of the following representation theoretical considerations. A direct combinatorial proof has also been given by Novelli [87].

In the classical case, the character $\chi(M)$ of a polynomial representation $\rho : GL_n \to GL(M)$ of GL_n is usually defined as the symmetric polynomial $\mathrm{tr}_M(\rho(X))$, where $X = \mathrm{diag}\,(x_1, \ldots, x_n)$ is a generic diagonal matrix. One might also consider the function $g \mapsto \mathrm{tr}_M(\rho(g))$ as a polynomial in the matrix entries x_{ij}. This is this second notion which is best suited to deal with comodules, especially in the quantum case, where an interesting phenomenon occurs.

Let M be a finite dimensional $A_q(n)$-comodule with coaction δ. Let $(m_i)_{i=1,m}$ be a basis of M. There exists elements $(a(i,j))_{1\le i,j\le m}$ of $A_q(n)$ such that

$$\delta(m_i) = \sum_{j=1}^{m} a(i,j)\otimes m_j$$

for $i \in [1,m]$. The element

$$\sum_{i=1}^{m} a(i,i)$$

of $A_q(n)$ is independent of the choice of the basis (m_i). It will be denoted by $\chi(M)$ and called the *character* of the $A_q(n)$-comodule M. It has the standard properties of a character:

1. If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, $\chi(M) = \chi(M') + \chi(M'')$;
2. $\chi(M\otimes N) = \chi(M)\,\chi(N)$;
3. if $M \simeq N$, then $\chi(M) = \chi(N)$.

The interesting quantum phenomenon, which has no classical analogue, is that for generic values of q, the character of an $A_q(n)$-comodule is always an element of the quantum diagonal algebra. This follows from an expression of the quantum determinant of $A_q(n)$ as an iterated q-commutator in the diagonal generators [60]

$$\begin{aligned}\left|\begin{matrix} x_{11} & x_{12} & \dots & x_{1n}\\ x_{21} & x_{22} & \dots & x_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ x_{n1} & x_{n2} & \dots & x_{nn}\end{matrix}\right|_q &\overset{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\, x_{1\sigma(1)}\,\dots\, x_{n\sigma(n)}\\ &= \frac{1}{(1-q)^{n-1}}\,[x_{nn},[\dots,[x_{22},x_{11}]_q\dots]_q]_q\end{aligned} \tag{101}$$

where $[P,Q]_q = P\,Q - q\,Q\,P$.

From now on, we set $q=0$, so that the quantum diagonal algebra becomes the hypoplactic algebra. Recall that to a composition I of N, we associated the element $\eta_I = T_{\omega(\overline{I})}\,\square_{\alpha(I^\sim)}$ of $H_N(0)$ which generates the one-dimensional left $H_N(0)$-module $\mathbf{C}_I$. One can use it to construct the $A_0(n)$-comodule

$$\mathbf{D}_I = V^{\otimes N}\cdot \eta_I \ .$$

Denote by $F_I(A)$ the sum of all quasi-ribbon words of shape I in the free algebra over A. According to a result of Gessel [39], the commutative image of $F_I(A)$ is the quasi-symmetric function F_I. For example, the quasi-ribbon tableaux of shape $I=(2,1)$ over $\{a<b<c\}$ are

$$\begin{array}{|c|c|}\hline a & a\\\hline \multicolumn{1}{c|}{} & b\\\cline{2-2}\end{array}\qquad \begin{array}{|c|c|}\hline a & a\\\hline \multicolumn{1}{c|}{} & c\\\cline{2-2}\end{array}\qquad \begin{array}{|c|c|}\hline a & b\\\hline \multicolumn{1}{c|}{} & c\\\cline{2-2}\end{array}\qquad \begin{array}{|c|c|}\hline b & b\\\hline \multicolumn{1}{c|}{} & c\\\cline{2-2}\end{array}$$

Thus $F_{21}(a,b,c) = aba + aca + acb + bcb$, and the commutative image of $F_{21}(a,b,c)$ is equal to $M_{21} + M_{111} = F_{21}$.

One can prove that the hypoplactic quasi-symmetric functions are the characters of the comodules $\mathbf{D}_I$ [60]

$$\chi(\mathbf{D}_I) = F_I(x_{11}, \dots, x_{nn}) \, . \tag{102}$$

For example, with $n = 3$, $N = 4$ and $I = (3,1)$ we have $\eta_{31} = T_3\,T_2\,T_1\,(1+T_2)\,(1+T_3)\,(1+T_2)$ and $\mathbf{D}_{31} = V^{\otimes 4}\cdot\eta_{31}$. By computing the images under η_{31} of the basis vectors of $V^{\otimes 4}$, one gets

$$\mathbf{D}_{31} = \mathbb{C}\,a_1a_2a_3a_2\cdot\eta_{31} \oplus \mathbb{C}\,a_2a_2a_3a_2\cdot\eta_{31} \oplus \mathbb{C}\,a_1a_1a_3a_2\cdot\eta_{31} \oplus \mathbb{C}\,a_1a_1a_3a_1\cdot\eta_{31} \oplus \mathbb{C}\,a_1a_1a_2a_1\cdot\eta_{31} \, .$$

Thus,

$$\begin{aligned} \chi(\mathbf{D}_{31}) &= x_{22}x_{11}x_{11}x_{11} + +x_{33}x_{11}x_{11}x_{11} + x_{33}x_{22}x_{22}x_{22} + x_{33}x_{11}x_{11}x_{22} + x_{33}x_{11}x_{22}x_{22} \\ &= x_{11}x_{11}x_{22}x_{11} + +x_{11}x_{11}x_{33}x_{11} + x_{22}x_{22}x_{33}x_{22} + x_{11}x_{11}x_{33}x_{22} + x_{11}x_{22}x_{33}x_{22} \, . \end{aligned}$$

This last expression is exactly the sum of the quasi-ribbons words associated with the five quasi-ribbon tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & 2 \\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \multicolumn{2}{c|}{} & 3 \\ \cline{3-3}\end{array}$$

Hence, $\chi(\mathbf{D}_{31}) = F_{31}(x_{11}, x_{22}, x_{33})$.

The $\mathbf{D}_I$ are irreducible, pairwise non-isomorphic $A_0(n)$-comodules. and $(\mathbf{D}_I)_I$ (where I runs through all compositions) is a complete system of irreducible $A_0(n)$-comodules [60, 61].

Performing now the same construction, starting from the generators $\nu_I = T_{\alpha(I)}\,\square_{\alpha(\bar{I}^\sim)}$ of the indecomposable projective left $H_n(0)$-modules $\mathbf{M}_I$, one obtains the $A_0(n)$-comodules

$$\mathbf{N}_I = V^{\otimes N}\cdot\nu_I \ .$$

The $\mathbf{N}_I$ are the indecomposable direct summands of $V^{\otimes N}$

$$V^{\otimes N} = \sum_{|I|=N} \mathbf{N}_I \tag{103}$$

(the sum is multiplicity free), and their characters are the hypoplactic ribbon Schur functions

$$\chi(\mathbf{N}_I) = R_I(x_{11}, \dots, x_{nn}) \, , \tag{104}$$

which are defined as follows. A word is said to be of *ribbon shape* I if it can be obtained by reading from left to right and from top to bottom the columns of a skew Young tableau of ribbon shape I. The ribbon Schur function $R_I(A)$ is equal to the sum of all words of A^* of ribbon shape I.

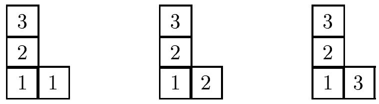
For example, let $n = 3$, $N = 4$ and $I = (1,1,2)$. Then $\nu_{112} = T_1\,T_2\,T_1\,(1+T_3)$ and $\mathbf{N}_{112} = V^{\otimes 4}\cdot\nu_{112}$. By computing the action of ν_{211} on the standard basis of $V^{\otimes 4}$, one gets

$$\mathbf{N}_{112} = \mathbb{C}\,a_3\,a_2\,a_1\,a_1\cdot\nu_{112} \ \oplus \ \mathbb{C}\,a_3\,a_2\,a_1\,a_2\cdot\nu_{112} \ \oplus \ \mathbb{C}\,a_3\,a_2\,a_1\,a_3\cdot\nu_{112} \ .$$

Then,

$$\chi(\mathbf{N}_{112}) = x_{33}\, x_{22}\, x_{11}\, x_{11} + x_{33}\, x_{22}\, x_{11}\, x_{22} + x_{33}\, x_{22}\, x_{11}\, x_{33}\ .$$

This expression is the sum of the ribbon words associated with the 3 ribbon tableaux



and $\chi(\mathbf{N}_{112}) = R_{112}(x_{11}, x_{22}, x_{33})$.

In the classical case ($q = 1$), the Robinson-Schensted correspondence is the combinatorial counterpart of the decomposition of $V^{\otimes N}$ into $GL_n(\mathbb{C}) \times \mathfrak{S}_N$-bimodules. On the other hand, for $q = 0$, there are two natural ways of decomposing $V^{\otimes N}$ into $A_0(n) \times H_N(0)$-bicomodules. This leads to two different Robinson-Schensted type correspondences, involving now ribbon and quasi-ribbon diagrams.

The first one corresponds to the decomposition

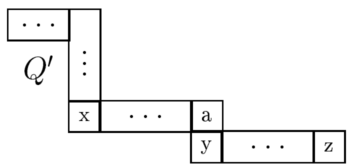
$$V^{\otimes N} = \bigoplus_{I \vdash N} \mathbf{N}_I\,. \tag{105}$$

It is essentially the identity map, since it just associates to a word w its ribbon diagram.

The second one, which is more interesting, is related to the simple composition factors of $V^{\otimes N}$. These are exactly the comodules $\mathbf{D}_I$ for $|I| = n$, each of them occuring $|QR(I)|$ times. But $\mathbf{D}_I$ considered as a left $H_N(0)$-module is isomorphic to $\mathbf{M}_I$. It follows that there exists a basis of $V^{\otimes N}$ indexed by pairs (Q, R) where Q is a quasi-ribbon word of shape I and where R is a standard ribbon word of the same shape. The corresponding Robinson-Schensted type algorithm which associates to each word $w \in A^*$ the pair (Q, R) is described below.

Let Q be a quasi-ribbon diagram and let $a \in A$. Let Q' be the diagram obtained from Q by deleting its last row and let x (resp. z) be the first (resp. last) letter of the last row of Q. The result $\mathcal{Q}$ of the insertion of a in Q is defined by the following rules:

- if $z \leq a$, $\mathcal{Q}$ is obtained by adding a box containing a at the end of the last row of Q
- if $x \leq a < z$, let y be the first letter of the last row of Q which is strictly greater than a. The quasi-ribbon diagram $\mathcal{Q}$ is then



;

- if $a < x$, $\mathcal{Q}$ is obtained by inserting a in Q' and glueing under the quasi-ribbon obtained in this way the last row of Q.

Let $w = a_1 \ldots a_n$ be a word of length n. The pair (Q, R) associated with w can be defined as follows. The quasi-ribbon diagram Q is obtained by inserting the letters of w (from left to right), starting from an empty diagram. The standard ribbon diagram R is iteratively constructed by putting at each step $i \in [1, n]$ of the algorithm the number i in the box that contains at this moment in Q the letter a_i inserted at this step. Let us illustrate again this correspondence on $w = baccb$.

$$(\emptyset,\emptyset) \to \boxed{b}\,,\boxed{1} \to \begin{array}{l}\boxed{a}\\ \boxed{b}\end{array},\begin{array}{l}\boxed{2}\\ \boxed{1}\end{array} \to \begin{array}{ll}\boxed{a} & \\ \boxed{b} & \boxed{c}\end{array},\begin{array}{ll}\boxed{2} & \\ \boxed{1} & \boxed{3}\end{array} \to \begin{array}{lll}\boxed{a} & & \\ \boxed{b} & \boxed{c} & \boxed{c}\end{array},\begin{array}{lll}\boxed{2} & & \\ \boxed{1} & \boxed{3} & \boxed{4}\end{array} \to \begin{array}{lll}\boxed{a} & & \\ \boxed{b} & \boxed{b} & \\ & \boxed{c} & \boxed{c}\end{array},\begin{array}{lll}\boxed{2} & & \\ \boxed{1} & \boxed{5} & \\ & \boxed{3} & \boxed{4}\end{array}$$

The correspondence $w \to (Q, R)$ is clearly a bijection. In fact, the quasi-ribbon diagram Q associated with w is of shape $C(\sigma^{-1})$ where $\sigma = \mathrm{std}\,(w)$ [60].

One can show that two words $u, v \in A^*$ correspond to the same quasi-ribbon Q under the second algorithm iff $u \equiv v$ with respect to the hypoplactic congruence.

This is a hypoplactic analogue of Knuth's theorem [56]. In other words, the hypoplactic relations play, for quasi-ribbons, the same rôle as the plactic relations for Young tableaux. The hypoplactic analogue of the Robinson-Schensted-Knuth correspondence has been worked out by Ung [107]. It provides, as in the classical case, a combinatorial proof of the noncommutative Cauchy identity

$$\sigma(XA; 1) = \sum_I F_I(X) R_I(A) \tag{106}$$

as well as of other analogues of various Schur function identitites, such as

$$\sum_I F_I(X) = \frac{1}{2}\left[\prod_{i \geq i} \frac{1+x_i}{1-x_i} + 1\right] , \tag{107}$$

an analogue of Schur's identity for the sum of all Schur functions, or

$$\left(\sum_I F_I(X)\right)^{-1} = 1 + \sum_{n \geq 0} (-1)^{n+1} \sum_{|J|=2n+1} c_J F_J(X) \tag{108}$$

where c_I is the number of permutations σ such that $C(\sigma) = I$ and $C(\sigma^{-1}) = (1\,2^n)$ [107].

3.3 The 0-quantized enveloping algebra $\mathcal{U}_0(gl_n)$

It remains to find an analogue of the interpretation of Schur functions as characters of the Lie algebra gl_n. To do this, we again need a variant of the quantized enveloping algebra $U_q(gl_n)$ which displays the same behaviour for generic q, but can also be specialized at $q = 0$. This algebra is essentially the specialization $(r, s) = (q, 1)$ of Takeuchi's two-parameter deformation $U_{r,s}(gl_N)$ [103], with some slight modifications. To allow the specialization $q = 0$, we have to use non-invertible generators for the Cartan part, to introduce some extra relations, and to define it only as a bialgebra (no antipode). This algebra, as well as other multiparameter deformations, can be realized by specialization of the difference operators of [37].

Let q be an indeterminate or a complex parameter. The crystalizable quantum analogue $\mathcal{U}_q(gl_n)$ of the universal enveloping algebra of the Lie algebra gl_n is the algebra over $\mathbb{C}(q)$ generated by the elements $(e_i)_{1 \leq i \leq n-1}$, $(f_i)_{1 \leq i \leq n-1}$ and $(k_i)_{1 \leq i \leq n}$ with relations:

$$k_i\, k_j = k_j\, k_i \qquad \text{for } 1 \leq i, j \leq n , \tag{109}$$

$$\left\{ \begin{array}{rcll} q\, k_i\, e_{i-1} & = & e_{i-1}\, k_i & \text{for } 2 \leq i \leq n-1 , \\ k_i\, e_i & = & q\, e_i\, k_i & \text{for } 1 \leq i \leq n-1 , \\ k_i\, e_j & = & e_j\, k_i & \text{for } j \neq i-1, i , \end{array} \right. \tag{110}$$

$$\left\{\begin{array}{rcll} k_i\, f_{i-1} &=& q\, f_{i-1}\, k_i & \text{for } 2 \le i \le n-1\ , \\ q\, k_i\, f_i &=& f_i\, k_i & \text{for } 1 \le i \le n-1\ , \\ k_i\, f_j &=& f_j\, k_i & \text{for } j \ne i-1, i\ , \end{array}\right. \tag{111}$$

$$[e_i, f_j] \;=\; \delta_{ij}\, \frac{k_i - k_{i+1}}{q-1} \qquad \text{for } 1 \le i,j \le n\ , \tag{112}$$

$$\left\{\begin{array}{rcll} q\, e_{i+1}\, e_i^2 - (1+q)\, e_i\, e_{i+1}\, e_i + e_i^2\, e_{i+1} &=& 0 & \text{for } 1 \le i \le n-2\ , \\ q\, e_{i+1}^2\, e_i - (1+q)\, e_{i+1}\, e_i\, e_{i+1} + e_i\, e_{i+1}^2 &=& 0 & \text{for } 1 \le i \le n-2\ , \end{array}\right. \tag{113}$$

$$\left\{\begin{array}{rcll} f_{i+1}\, f_i^2 - (1+q)\, f_i\, f_{i+1}\, f_i + q\, f_i^2\, f_{i+1} &=& 0 & \text{for } 1 \le i \le n-2\ , \\ f_{i+1}^2\, f_i - (1+q)\, f_{i+1}\, f_i\, f_{i+1} + q\, f_i\, f_{i+1}^2 &=& 0 & \text{for } 1 \le i \le n-2\ , \end{array}\right. \tag{114}$$

$$\left\{\begin{array}{rcll} [e_i, e_j] &=& 0 & \text{for } |i-j| > 1\ , \\ [f_i, f_j] &=& 0 & \text{for } |i-j| > 1\ , \end{array}\right. \tag{115}$$

$$\left\{\begin{array}{rcll} e_i\, e_{i+1}\, k_{i+1} &=& k_{i+1}\, e_i\, e_{i+1} & \text{for } 1 \le i \le n-2\ , \\ f_{i+1}\, f_i\, k_{i+1} &=& k_{i+1}\, f_{i+1}\, f_i & \text{for } 1 \le i \le n-2\ . \end{array}\right. \tag{116}$$

Relations (116) are consequences of the other relations when $q \ne 0$ and do not play any rôle in the generic case. One has to add them in order to endow $\mathcal{U}_q(gl_n)$ with a bialgebra structure at $q = 0$. Note also that the q-Serre relations (113) are those obtained by Ringel [95] in his construction of $U_q(\mathfrak{n}_+)$ via the Hall algebras associated to quivers. That is, $\mathcal{U}_q(\mathfrak{n}_+)$ is the (untwisted) Hall algebra of the quiver of type A_{n-1}.

$\mathcal{U}_q(gl_n)$ is a $\mathbb{C}$-bialgebra for the comultiplication Δ and the co-unit ε defined by

$$\left\{\begin{array}{rcllrcll} \Delta(e_i) &=& 1 \otimes e_i + e_i \otimes k_i, & \varepsilon(e_i) &=& 0 & \text{for } 1 \le i \le n-1\ , \\ \Delta(f_i) &=& k_{i+1} \otimes f_i + f_i \otimes 1, & \varepsilon(f_i) &=& 0 & \text{for } 1 \le i \le n-1\ , \\ \Delta(k_i) &=& k_i \otimes k_i, & \varepsilon(k_i) &=& 1 & \text{for } 1 \le i \le n\ . \end{array}\right. \tag{117}$$

Let $(\xi_i)_{1 \le i \le n}$ be the canonical basis of $V = \mathbb{C}^n$, and let E_{ij} be the endomorphism defined by $E_{ij}\xi_k = \delta_{jk}\xi_i$. One can define an algebra morphism ρ_V of $\mathcal{U}_q(gl_n)$ in $End_{\mathbb{C}}(V)$ by setting

$$\left\{\begin{array}{rcll} \rho_V(e_i) &=& E_{i,i+1} & \text{for } 1 \le i \le n-1\ , \\ \rho_V(f_i) &=& E_{i+1,i} & \text{for } 1 \le i \le n-1\ , \\ \rho_V(k_i) &=& q\, E_{i,i} + \sum_{j \ne i} E_{j,j} & \text{for } 1 \le i \le n\ . \end{array}\right.$$

The pair (ρ_V, V) is called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_q(gl_n)$. Since $\mathcal{U}_q(gl_n)$ is a bialgebra, one can define its N-th tensor power $(\rho_{N,n}, V^{\otimes N})$ by $\rho_{N,n} = \rho_V^{\otimes N} \circ \Delta^{(N)} : \mathcal{U}_q(gl_n) \longrightarrow End_{\mathbb{C}}(V^{\otimes N})$, where $\Delta^{(N)}$ denotes the N-fold comultiplication $\mathcal{U}_q(gl_n) \to \mathcal{U}_q(gl_n)^{\otimes N}$.

We denote by $A'_q(n)$ the $\mathbb{C}(q)$-algebra generated by the n^2 elements $(x_{ij})_{1 \le i,j \le n}$ subject to the defining relations:

$$\left\{\begin{array}{rcll} x_{il}\, x_{jk} &=& q\, x_{jk}\, x_{il} & \text{for } 1 \le i \le j \le n,\ 1 \le k < l \le n\ , \\ x_{ik}\, x_{jk} &=& x_{jk}\, x_{ik} & \text{for } 1 \le i,j,k \le n\ , \\ x_{jl}\, x_{ik} - x_{ik}\, x_{jl} &=& (q-1)\, x_{jk}\, x_{il} & \text{for } 1 \le i < j \le n,\ 1 \le k < l \le n\ . \end{array}\right. \tag{118}$$

It is the specialization $GL_{q,1}(n)$ of Takeuchi's algebra [103]. Up to the symmetry $x_{ij} \leftrightarrow x_{ji}$, $A'_q(n)$ is the Dipper-Donkin quantization $A_q(n)$ introduced in the preceding section.

The comultiplication comultiplication $\overline{\Delta}$ and counity η of $A'_q(n)$ are defined by

$$\overline{\Delta}(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \qquad \text{and} \qquad \eta(x_{ij}) = \delta_{ij} \ .$$

The quantum diagonal algebra (or quantum torus) $\Delta'_q(n)$ of $A'_q(n)$ is defined as above as the subalgebra generated by the x_{ii}. For $q = 1$, $\Delta'_1(N)$ is just an algebra of commutative polynomials and for $q = 0$ it is again isomorphic to the hypoplactic algebra (in fact there is no need to consider both $A_q(n)$ and $A'_q(n)$, we just do it for the sake of compatibility with the existing literature) .

For any $q \in \mathbb{C}$, the bialgebras $A'_q(n)$ and $\mathcal{U}_q(gl_n)$ are in duality, in a sense to be made precise below. According to the general theory, the graded dual $\mathcal{U}_q(gl_n)^*$ of $\mathcal{U}_q(gl_n)$ has a canonical algebra structure, defined by the convolution product

$$\varphi \cdot \psi = \mu \circ (\varphi \otimes \psi) \circ \Delta \tag{119}$$

for $\varphi, \psi \in \mathcal{U}_q(gl_n)^*$, where μ is the multiplication map $\mu(g \otimes h) = g\,h$, $g, h \in \mathcal{U}_q(gl_n)$.

Define linear functionals $(a_{ij})_{1 \le i,j \le n} \in \mathcal{U}_q(gl_n)^*$ as the matrix coefficients of the vector representation, i.e. $\rho_V(g) = (a_{ij}(g))_{1 \le i,j \le n}$ for $g \in \mathcal{U}_q(gl_n)$.

Then, the n^2 linear functionals $(a_{ij})_{1 \le i,j \le n}$ of $\mathcal{U}_q(gl_n)^*$ satisfy the quantum relations (118).

A fundamental property of the standard version of $U_q(gl_n)$ is the existence of a q-analogue of the Schur-Weyl duality involving the Hecke algebra instead of the symmetric group [50]. Such a duality can also be worked out for $\mathcal{U}_q(gl_n)$. The double commutant theorem will break down at $q = 0$, but one can still give a weaker statement, which will allow us to interpret quasi-symmetric functions and noncommutative symmetric functions as characters of $\mathcal{U}_0(gl_n)$.

We denote by $\pi_{N,n}$ the representation of $H_N(q)$ in $End_{\mathbb{C}}(V^{\otimes N})$ defined by (97). One can check that $\rho_{N,n}$ commutes with $\pi_{N,n}$. For generic q, one has a stronger property.

Let q be a nonzero complex number which is not a non trivial k-th root of unity for some $k \le N$. The two subalgebras $\pi_{N,n}(H_N(q))$ and $\rho_{N,n}(\mathcal{U}_q(gl_n))$ of $End_{\mathbb{C}}(V^{\otimes N})$ are then commutant of each other.

This result is well-known for the standard version of $U_q(gl_N)$ [50] and can be obtained by straightforward modifications of its standard proofs (see also [61]).

Let us now define a polynomial $\mathcal{U}_q(gl_n)$-module of degree N as a submodule of $V^{\otimes N}$. For generic q, the polynomial $\mathcal{U}_q(gl_n)$ modules are just the duals of the $A'_q(n)$-comodules. We now define the character of a polynomial module in the same way as in the preceding section.

Let M be a polynomial $\mathcal{U}_q(gl_n)$-module of degree N and let $(m_i)_{1 \le i \le m}$ be a $\mathbb{C}(q)$-linear basis of M. Since M is a $\mathcal{U}_q(gl_n)$-module, there exists a family $(\mu_{ij})_{1 \le i,j \le m}$ of linear functionals of $\mathcal{U}_q(gl_n)^*$ such that for $g \in \mathcal{U}_q(gl_n)$

$$g \cdot m_i = \sum_{j=1}^{m} \mu_{ij}(g)\, m_j \ .$$

One can check that the trace of the action of $\mathcal{U}_q(gl_n)$ on M, i.e. the element

$$c(M) = \sum_{i=1}^{m} \mu_{ii} \in \mathcal{U}_q(gl_n)^*$$

is independent on the choice of the basis. Moreover $c(M)$ belongs to the subalgebra generated by the $(a_{ij})_{1\le i,j\le n}$. It follows that there exists elements $p_{IJ}(q)$ of $\mathbb{C}(q)$ such that

$$c(M) = \sum_{I,J\in[1,n]^N} p_{IJ}(q)\, a_{IJ} \ .$$

Since the map Ψ is injective on the linear subspace of $A'_q(n)$ spanned by homogeneous elements of fixed length, we can lift this formula to $A'_q(N)$. Let now x_{IJ} stand for the monomial $x_{i_1j_1}\,\ldots\,x_{i_Nj_N}$ in $A_q(n)$. The element

$$\chi(M) = \sum_{I,J\in[1,n]^N} p_{IJ}(q)\, x_{IJ}$$

of $A'_q(N)$ will be called the *character* of the polynomial $\mathcal{U}_q(gl_n)$-module M. By duality, M is also an $A'_q(N)$ comodule, and $\chi(M)$ is equal to its character in the previous sense. Therefore, for $q \ne 1$, $\chi(M)$ belongs to the diagonal algebra $\Delta'_q(n)$. From now on, we set $q = 0$. The character of a polynomial $\mathcal{U}_0(gl_n)$-module is therefore an element of the hypoplactic algebra. However, $\rho_{N,n}(\mathcal{U}_0(gl_n))$ is not semisimple, and we have to look for the simple composition factors of $V^{\otimes N}$ (the irreducible polynomial modules) and its indecomposable direct summands. These modules can be constructed in the same way as the simple modules and the indecomposable projective modules of the 0-Hecke algebra. Since the action of $\mathcal{U}_0(gl_n)$ on $V^{\otimes N}$ commutes with the right action of $H_N(0)$, one can use η_I (cf. (84) to construct the $\mathcal{U}_0(gl_n)$-module

$$\mathbf{D}_I = V^{\otimes N}\cdot\,\eta_I \ .$$

Then, $\mathbf{D}_I$ is an irreducible $\mathcal{U}_0(gl_n)$-module. Its character is equal to the sum $F_I(x_{11},\ldots,x_{nn})$ of all quasi-ribbon words over $\{\,x_{11} < \ldots < x_{nn}\,\}$.

Here is a sketch of the proof. For $w = w_1\cdots w_N$ a word on the alphabet $\{1,\ldots,n\}$, denote by $\mathbf{w}$ the tensor $\xi_{w_1}\otimes\cdots\otimes\xi_{w_n}$. Let $QR(I)$ be the set of quasi-ribbon words of shape I over the same alphabet. It is shown in [60] that the $\mathbf{d}_w = \mathbf{w}\eta_I$ for $w\in QR(I)$ form a basis of $\mathbf{D}_I$.

Now, let $\tilde e_i$, $\tilde f_i$ be Kashiwara's crystal graph operators [51], acting on words considered as vertices of the crystal graph of the N-th tensor power of the vector representation of the standard $U_q(gl_n)$. We have

$$f_i\mathbf{d}_w = \begin{cases} \mathbf{d}_{\tilde f_i(w)} & \text{if } \tilde f_i(w)\in QR(I),\\ 0 & \text{otherwise.}\end{cases} \tag{120}$$

and similarly

$$e_i\mathbf{d}_w = \begin{cases} \mathbf{d}_{\tilde e_i(w)} & \text{if } \tilde e_i(w)\in QR(I),\\ 0 & \text{otherwise.}\end{cases} \tag{121}$$

Let $\Gamma_n(I)$ be the directed graph having as vertices the quasi-ribbon words of shape I over $\{1,\ldots,n\}$, and with arrows $w \xrightarrow{f_i} w'$ and $w \xleftarrow{e_i} w'$ when $\mathbf{d}_{w'} = f_i\mathbf{d}_w$ (which is equivalent to $\mathbf{d}_w = e_i\mathbf{d}_{w'}$). This graph is called the *quasi-crystal graph* of $\mathbf{D}_I$ (it is actually a subgraph of a crystal graph, corresponding to a quasi-symmetric piece of a character). It should be noted that although crystal graphs describe in general only the combinatorial skeleton of a generic module, the quasi-crystal graph $\Gamma_n(I)$ encodes the full structure of the $\mathcal{U}_0(gl_n)$-module $\mathbf{D}_I$. The above considerations imply that $\Gamma_n(I)$ is strongly connected, which proves the irreducibility of $\mathbf{D}_I$.

Finally, from the definition of $\chi(\mathbf{D}_I)$ and (120), (121), we obtain

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} x_{ww} = F_I(x_{11}, \ldots, x_{nn}) \ ,$$

as required. The character of $\mathbf{D}_I$ is therefore the hypoplactic quasi-symmetric function $F_I(A)$.

The indecomposable summand of $V^{\otimes N}$ labelled by the composition I of N is (cf. (90))

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I \ .$$

Its character is the sum $R_I(x_{11}, \ldots, x_{NN})$ of all words of ribbon shape I over $\{ x_{11} < \ldots < x_{NN} \}$. Moreover, $\mathbf{N}_I$ has a canonical filtration, whose levels are described by the quantum quasi-symmetric functions $\hat{R}_I$ of [104].

Hivert's formula (42) can now be interpreted as a Weyl character formula for the irreducible representation $\mathbf{D}_I$. Moreover, the partial symmetrizers π'_w for all $w \in \mathfrak{S}_n$ give, as in the classical case, the characters of the Demazure modules of $\mathcal{U}_0(gl_n)$.

Let us first remark that the basis vectors $\mathbf{d}_w$ are also weight vectors for the usual action of gl_n on $V^{\otimes N}$. The $\mathcal{U}_0(gl_n)$-weight of an element of $V^{\otimes N}$ is defined as its gl_n-weight, and the *formal character* $\mathrm{ch}\,(M)$ of a polynomial modules M will be the generating function $\sum_\lambda \dim M_\lambda \, x^\lambda$. It is the commutative image $x_{ii} \mapsto x_i$ of its character $\chi(M)$ is the previous sense.

The $\mathcal{U}_0(gl_n)$-weights of polynomial modules are integer vectors $H \in \mathbb{N}^n$. In this context, a weight H is said to be *dominant* if there exists $r \le n$ such that $h_k > 0$ for $k \le r$ and $h_k = 0$ for $k \ge r$. That is, H is of the form $I \cdot 0 \cdots 0$, where I is some composition. To simplify the notation, we shall discard the final zeros and identify H with the composition I.

The Weyl group action on $\mathcal{U}_0(gl_n)$-weights corresponds to the quasi-symmetrizing action on polynomials. That is, the image $\sigma(H)$ of a weight H by $\sigma \in \mathfrak{S}_n$ is defined by

$$x^{\sigma(H)} = \sigma'(x^H) \,, \tag{122}$$

where σ' denotes the quasi-symmetrizing action (36).

The irreducible module $\mathbf{D}_I$ has a unique highest weight vector v_I, i.e., $\mathbf{D}_I = \mathcal{U}^- v_I$, $\mathcal{U}^+ v_I = 0$, where $\mathcal{U}^-$ (resp $\mathcal{U}^+$) is the subalgebra generated by the f_i (resp. by the e_i), and the weight of v_I is the dominant weight I.

On the other hand, the indecomposable summands of $V^{\otimes N}$ are not highest-weight modules.

As in the classical case, let us define an extremal weight vector of a module M with highest weight H as a weight vector whose weight K is in the Weyl group orbit of H. For a composition I and a permutation $\sigma \in \mathfrak{S}_n$, the Demazure module $D(I, \sigma)$ is the $\mathcal{U}^+$ module generated by an extremal weight vector of weight $\sigma(I)$ (which is unique up to a scalar factor). Then, we have the following analogue of the classical Demazure formula [44, 45]

$$\mathrm{ch}\, D(I, \sigma) = \pi'_\sigma(X^I) \,, \tag{123}$$

where $X^I = x_1^{i_1} \ldots x_r^{i_r}$ and π'_σ is the operator defined in (41).

Recall now that the classical divided difference operators π_i can be used to define an action of the Hecke algebra $H_n(q)$ on polynomials in n variables, by formula (10), and that the Hall-Littlewood functions are essentially the images of monomials under the action of the full Hecke symmetrizer $S^{(n)}$ (12). This approach to Hall-Littlewood functions can be extended to the quasi-symmetric case, and by duality, to the noncommutative case.

The relevant operators here are

$$\overline{\pi}_i = \pi'_i - 1\,. \tag{124}$$

These operators satisfy the braid relations, and $\overline{\pi}_i^2 = -\overline{\pi}_i$ The Hecke algebra action is defined by

$$T_i = (1-q)\overline{\pi}_i + q\sigma'_i\,. \tag{125}$$

Then, the quasi-symmetric analogues of the Hall-Littlewood functions $P_\lambda(X;q)$ are

$$G_I(X;q) = \frac{1}{[r]_q![n-r]_q!} S^{(n)}(X^I) \tag{126}$$

where $S^{(n)}$ acts by the representation (125), and the noncommutative analogues $H_I(A;q)$ of the $Q'_\lambda(X;q)$ are defined as the dual basis of G_I. The expansions of GI on the basis (F_J) and of H_I on the basis (R_J) are explicitly known [44, 45]. The coefficients are just powers of q, with signs in the first case. In the second case, we see that the analogues of the Kostka-Foulkes polynomials in the noncommutative theory reduce to well-understood monomials.

The multiplication rule and the specialization at roots of unity of the H_I have also been worked out [45]. It remains to understand the quasi-symmetric and noncommutative analogues of the Macdonald polynomials (work in progress by F. Hivert).

4 Selected applications

4.1 Lie idempotents and the Hausdorff series

Let $\mathbb{K}\langle A\rangle$ and $L(A)$ be the free associative algebra and the free Lie algebra over an alphabet $A = \{a_1, a_2, \ldots, a_N\}$, and let $\mathbb{K}_n\langle A\rangle$ and $L_n(A)$ denote their homogeneous components of degree n. Recall that elements of $\mathbb{K}\mathfrak{S}_n$ can be interpreted as endomorphisms of $\mathbb{K}_n\langle A\rangle$ via the right action of permutations on words, and that we defined a *Lie idempotent* as an element of $\mathbb{K}\mathfrak{S}_n$ acting as a projector onto $L_n(A)$.

Such elements arise naturally in the investigation of the Hausdorff series

$$H(a_1, a_2, \ldots, a_N) = \log\left(e^{a_1}e^{a_2}\cdots e^{a_N}\right) = \sum_{n\geq 0} H_n(A) \tag{127}$$

which is known to be a *Lie series*, i.e., each homogeneous component $H_n(A) \in L_n(A)$. This is known as the Baker-Campbell-Hausdorff (BCH) theorem. It follows immediately from the characterization of $L(A)$ as the space of primitive elements of the standard comultiplication of $\mathbb{K}\langle A\rangle$ (Friedrich's criterion).

However, there is no obvious way to expand H_n as a linear combination of commutators. For example, it is not immediate at first sight that

$$\begin{aligned}
H_3(a_1,a_2,a_3) \;=\;& \frac{1}{12}a_1a_1a_2 + \frac{1}{12}a_1a_1a_3 - \frac{1}{6}a_1a_2a_1 + \frac{1}{12}a_1a_2a_2 + \frac{1}{3}a_1a_2a_3 - \frac{1}{6}a_1a_3a_1\\
&+\frac{1}{12}a_1a_3a_3 + \frac{1}{12}a_2a_1a_1 - \frac{1}{6}a_2a_1a_2 - \frac{1}{6}a_2a_1a_3 + \frac{1}{12}a_2a_2a_1 + \frac{1}{12}a_2a_2a_3\\
&-\frac{1}{6}a_2a_3a_1 - \frac{1}{6}a_2a_3a_2 + \frac{1}{12}a_2a_3a_3 + \frac{1}{12}a_3a_1a_1 - \frac{1}{6}a_3a_1a_2 - \frac{1}{6}a_3a_1a_3\\
&+\frac{1}{3}a_3a_2a_1 + \frac{1}{12}a_3a_2a_2 - \frac{1}{6}a_3a_2a_3 + \frac{1}{12}a_3a_3a_1 + \frac{1}{12}a_3a_3a_2
\end{aligned}$$

can be rewritten as

$$\begin{aligned}
H_3(a_1,a_2,a_3) \;=\;& = \frac{1}{12}[a_1,[a_1,a_2]] + \frac{1}{12}[[a_1,a_2],a_2] + \frac{1}{12}[a_1,[a_1,a_3]] + \frac{1}{12}[[a_1,a_3],a_3]\\
&+\frac{1}{12}[a_2,[a_2,a_3]] + \frac{1}{12}[[a_2,a_3],a_3] + \frac{1}{6}[a_1,[a_2,a_3]] + \frac{1}{6}[[a_1,a_2],a_3]\,.
\end{aligned}$$

The first systematic procedure for expanding the Hausdorff series in terms of commutators was found by Dynkin [20] and independently by Specht [100] and Wever [109]. It amounts to the fact, that we already proved in Section 2, that

$$\frac{1}{n}\theta_n = \frac{1}{n}[\ldots[[[1,2],3],\ldots,],n] \tag{128}$$

is a Lie idempotent. Therefore, writing $H_n = \frac{1}{n}H_n\theta_n$ gives the required expression.

To actually compute it, we need to know H_n as a linear combination of words

$$H_n(A) = \sum_{w\in A^n} c_w w\,. \tag{129}$$

There is a closed formula, due to Goldberg (see [94]), for the coefficients c_w. One way to obtain it is to express the whole polynomial $H_n(A)$ as the image of the homogeneous component $E_n(A)$ of the product of exponentials

$$E(A) = e^{a_1} e^{a_2} \cdots e^{a_N} = \sum_{n\geq 0} E_n(A)\,. \tag{130}$$

under an element of $\mathbb{K}\mathfrak{S}_n$

$$H_n(A) = E_n(A)\cdot\phi_n\,, \tag{131}$$

where

$$\phi_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{\binom{n-1}{d(\sigma)}}\,\sigma \qquad (d(\sigma) = |\mathrm{Des}\,(\sigma)|)\,. \tag{132}$$

This formula is due to Solomon [98] and independently to Bialynicki-Birula, Mielnik and Plebański [5]. It can be shown, although no one of these properties is clearly apparent on the expression (132), that ϕ_n is a Lie idempotent. It is clearly in the descent algebra Σ_n, and one can show that the corresponding noncommutative symmetric function is

$$\alpha(\phi_n) = \frac{1}{n}\Phi_n\,. \tag{133}$$

We can now write the Hausdorff polynomials

$$H_n(A) = \frac{1}{n}E_n(A)\phi_n\theta_n = \frac{1}{n}\sum_{w\in A^n} c_w\cdot w\theta_n \tag{134}$$

as linear combinations of commutators $w\theta_n$. However, these elements are far from being linearly independent, and one would be interested in an expansion of H_n on a *basis* of the free Lie algebra. One way to achieve this is to use *Klyachko's basis*. It is defined by taking the images of Lyndon words (words which are lexicographically minimal among their circular shifts) by Klyachko's idempotent

$$\kappa_n = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} \omega^{\mathrm{maj}\,(\sigma)}\sigma \tag{135}$$

where $\omega = e^{2i\pi/n}$ and $\mathrm{maj}\,(\sigma) = \sum_{j\in\mathrm{Des}\,(\sigma)} j$ is the *major index* of σ. This element is clearly in the descent algebra. It can be shown that it is a Lie idempotent [55]. Its relevance to the expansion of the Hausdorff series comes from its compatibility with circular permutations: $\gamma_n\kappa_n = \omega^{-1}\kappa_n$, where $\gamma_n = (12\ldots n)$. This allows to construct a basis of $L_n(A)$ by applying κ_n to a set of representatives of circular classes of words.

The corresponding noncommutative symmetric function can be shown to be $\frac{1}{n}K_n(\omega)$, where

$$K_n(q) = \sum_{|I|=n} q^{\mathrm{maj}\,(I)} R_I = (q)_n S_n\left(\frac{A}{1-q}\right) \tag{136}$$

is the natural noncommutative analogue of the Hall-Littlewood function

$$\tilde{Q}'_{1^n}(X;q) = \sum_{\lambda\vdash n} \tilde{K}_{\lambda,1^n}(q) s_\lambda = (q)_n h_n\left(\frac{X}{1-q}\right)\,. \tag{137}$$

which describes the graded character of the representation of $\mathfrak{S}_n$ in the space of coinvariants. The specialization of $\tilde{Q}'_{1^n}(X;q)$ at $q=\omega$ is known to be equal to p_n. This is the simplest example of a series of specialization properties of Hall-Littlewood functions at roots of unity [64]. The other specializations of $\tilde{Q}'_{1^n}(X;q)$ are related to the decomposition into irreducibles of the $\mathfrak{S}_n$-representations induced by a transitive cyclic subgroup (see the next subsection).

The sequence of noncommutative symmetric functions $K_n(\omega_n)$, where $\omega_n = \exp(2\pi i/n)$ can be regarded as a family of noncommutative power sums. Indeed, the last equality in (136) shows that the coproduct of $K_n(q)$ is given by

$$\Delta K_n(q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q K_r(q) \otimes K_{n-r}(q) \tag{138}$$

and under the specialization $q = \omega_n$, the q-binomial coefficients vanish, except for $r = 0$ or n.

To summarize, the investigation of the Hausdorff series brought to the fore three Lie idempotents, all of them in the descent algebra, and being mapped to a noncommutative symmetric function in the primitive Lie algebra, with commutative image $\frac{1}{n}p_n$.

It can be proved that these two conditions actually characterize the Lie idempotents of the descent algebras [32]. This is a powerful result, since it provides us at once with a large supply of Lie idempotents, by just playing with the various algebraic operations available in **Sym**. For example, consider the element

$$\varphi_n(q) \longleftrightarrow \frac{1}{n}(1-q^n)\Psi_n\left(\frac{A}{1-q}\right) . \tag{139}$$

Clearly, its commutative image is $\frac{1}{n}p_n$, and it is easy to check that it is a primitive element. Therefore, it corresponds to a Lie idempotent $\varphi_n(q)$ of Σ_n. A somewhat lengthy calculation gives its expansion on the ribbon basis, and one obtains

$$\varphi_n(q) = \frac{1}{n} \sum_{|I|=n} \frac{(-1)^{d(\sigma)}}{\begin{bmatrix} n-1 \\ d(\sigma) \end{bmatrix}_q} q^{\mathrm{maj}\,(\sigma) - \binom{d(\sigma)+1}{2}} \sigma \tag{140}$$

so that $\varphi_n(q)$ appears as a natural q-analogue of Solomon's idempotent (132), that is, $\phi_n = \varphi_n(1)$. More suprising are the specializations

$$\varphi_n(0) = \frac{1}{n}\theta_n \quad \text{and} \quad \varphi_n(\omega_n) = \kappa_n . \tag{141}$$

Therefore, we have found a one-parameter family interpolating between the three previous examples [18, 59]. Moreover, the limit $q \to \infty$ exists and corresponds to the standard right bracketing. Other interesting families, with one or more parameters, can be constructed in the same way [59].

Finally, let us explain the relevance of the power-sums Φ_n to the calculation of the Hausdorff series in terms of the interpretation of **Sym** as an algebra of graded endomorphisms of $\mathbb{K}\langle A\rangle$ discussed at the end of Section 2. The identity map of $\mathbb{K}\langle A\rangle$ gets identified to $\rho(\sigma(1))$, since $\rho(S_n)$ is the orthogonal projector onto the component $\mathbb{K}_n\langle A\rangle$. Then, one can write

$$\begin{aligned} \log(e^{a_1}a^{a_2}\cdots e^{a_N}) &= \log\left[\rho(\sigma(1))(e^{a_1}a^{a_2}\cdots e^{a_N})\right] \\ &= \rho(\log\sigma(1))\left[(e^{a_1}a^{a_2}\cdots e^{a_N})\right] \\ &= \rho\left(\sum_{n\geq 1}\frac{\Phi_n}{n}\right)\left[e^{a_1}a^{a_2}\cdots e^{a_N}\right] . \end{aligned}$$

To obtain (132), it remains to express Φ_n on the basis of ribbon Schur functions. Note that the product of exponentials plays no role in this calculation, and that we have just calculated the logarithm of the identity in the convolution algebra. Therefore, this calculation applies as well to the continuous Baker-Campbell-Hausdorff series

$$\Omega(t) = \log U(t)\,, \tag{142}$$

where $U(t)$ is the unique solution of the operator evolution equation

$$\frac{dU}{dt} = \mathcal{H}(t)U(t) \tag{143}$$

satisfying $U(0) = 1$. To obtain $\Omega(t) = \sum_n \Omega_n(t)$ as a series of iterated integrals, one first writes $U(t)$ as a Volterra series

$$U(t) = 1 + \int_0^t dt_1 \mathcal{H}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{H}(t_1)\mathcal{H}(t_2) + \cdots \tag{144}$$

and one obtains Ω_n by acting on the nth term of (144) by ϕ_n, permutations acting on the subscripts i of the variables t_i. This yields the continuous BCH formula of [5]

$$\Omega_n(t) = \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{\sigma\in\mathfrak{S}_n} \frac{(-1)^{d(\sigma)}}{n} \binom{n-1}{d(\sigma)}^{-1} \mathcal{H}(t_{\sigma(1)}) \cdots \mathcal{H}(t_{\sigma(n)}) \tag{145}$$

Also, in the above discussion, the logarithm plays no particular role. It could be replaced by any function f analytic in a neighbourhood of 1

$$f(1+z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{146}$$

Indeed, $f(U(t))$ or $f(e^{a_1}a^{a_2}\cdots e^{a_N})$ can again be obtained by applying to the Volterra series of $U(t)$ or to the power series expansion of $e^{a_1}a^{a_2}\cdots e^{a_N}$ the operator $\rho(f(\sigma(1)))$. Now,

$$\begin{aligned} f(\sigma(1)) &= \sum_{n\geq 1} a_n(\sigma(1)-1)^n \\ &= \sum_{n\geq 1} \frac{1}{2\pi i} \oint_{z=0} \frac{f(1+z)}{z^{n+1}} (\sigma(1)-1)^n \\ &= \frac{1}{2\pi i} \oint_{z=0} \frac{dz}{z} \frac{f(1+z)}{1 - z^{-1}(\sigma(1)-1)}\,. \end{aligned}$$

Now, using the generating function of the noncommutative Eulerian polynomials given in [32], one can see that

$$\frac{1}{z-(\sigma(1)-1)} = \sum_I \frac{(1+z)^{|I|-\ell(I)+1}}{z^{|I|+1}} R_I \tag{147}$$

and obtain from this the operator $\rho(f(\sigma(1)))$ as a linear combination of permutations. For example, the Eulerian idempotents are obtained from the case $f(z) = z^x$, by writing $\sigma(1)^x = \sum_{k\geq 0} x^k E^{[k]}$. The Eulerian idempotent $E_n^{[k]}$ is the term of degree n in $E^{[k]}$.

4.2 Noncommutative cyclic characters of symmetric groups

We have already encountered the specialization of the Hall-Littlewood function $\tilde{Q}'_{1^n}(X;q)$ at $q=\omega_n$, a primitive n-th root of unity. The specializations $q=\omega_n^k$ at other roots of unity are also known, and the result can be shown to be equivalent to the following formula for the reduction modulo $1-q^n$

$$\tilde{Q}'_{1^n}(X;q) \bmod 1-q^n = \sum_{k=0}^{n-1} q^k \ell_n^{(k)} \tag{148}$$

where the symmetric function $\ell_n^{(k)}$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ induced by the character $\gamma \mapsto \omega_n^k$ of the cyclic subgroup generated by a n-cycle γ. Their expression on the basis of power-sums is [25]

$$\ell_n^{(k)} = \frac{1}{n}\sum_{d|n} c(k,d) p_d^{n/d}, \tag{149}$$

where $c(k,d)$ is the sum of the k-th powers of the primitive d-th roots of unity (called Ramanujan sums or von Steneck functions). Since $\tilde{Q}'_{1^n}(X;q)$ is the commutative image of $K_n(q)$, we have

$$\ell_n^{(k)} = \sum_{\mathrm{maj}\,(I)\equiv k \mod n} r_I\ , \tag{150}$$

as the sum of all ribbons parametrized by compositions I whose major index is congruent to k mod n. This is equivalent to the combinatorial description of these characters given by Kraskiewicz and Weyman [57]. We will take this equation as the starting point for defining noncommutative analogues of the $\ell_n^{(k)}$ in the algebra of noncommutative symmetric functions. It will turn out that there are natural noncommutative analogues of the power sum products $p_d^{n/d}$ which give an expansion corresponding to (149).

Imitating (150) we put

$$L_n^{(k)} := \sum_{\mathrm{maj}\,(I)\equiv k \mod n} R_I\ . \tag{151}$$

The elements $L_n^{(k)}$ will be called *noncommutative cyclic characters.*

Another basis of the subspace spanned by the $L_n^{(k)}$ is given by

$$K_n^{(k)} := K_n(\omega^k) = \sum_i \omega^{ik} L_n^{(i)}\ , \tag{152}$$

and the transition matrix $\Xi(L,K)$ from the elements $L_n^{(k)}$ to the $K_n^{(j)}$ is the character table of the cyclic group of order n

$$\Xi(L,K) = (\omega^{ik})_{i,k}\ .$$

The inverse transformation is described by

$$\Xi(K,L) = \Xi(L,K)^{-1} = \frac{1}{n}(\omega^{-ik})_{k,i}\ ,$$

i.e.

$$L_n^{(k)} = \frac{1}{n}\sum_i \omega^{-ik} K_n^{(i)}\ , \tag{153}$$

When ω^k is a primitive n-th root of unity, *i.e.* $n\wedge k=1$, we have seen that $\frac{1}{n}K_n^{(k)}$ is an idempotent corresponding to Klyachko's idempotent in the descent algebra. In particular, $K_n^{(k)}$ is primitive for Δ. When ω^k is not primitive, $K_n^{(k)}$ is a product of primitive elements. More precisely, let ζ be any r-th root of unity. Let $n=ar+b$ with $a,b\in\mathbb{N}, b<r$. Then

$$K_n(\zeta)=K_r(\zeta)^a K_b(\zeta)\,. \tag{154}$$

From this, one can show that the $K_n^{(m)}$ span a subalgebra, with the multiplication rule

$$K_n^{(k)} * K_n^{(l)} = \begin{cases} (n/d)!\,d^{n/d}K_n^{(l)} & \text{if } n\wedge k = n\wedge l = d \\ 0 & \text{otherwise }\ . \end{cases} \tag{155}$$

In particular,

$$\mathcal{C}_n := \langle\langle K_n^{(k)} \mid k=1,\dots n\rangle\rangle = \langle\langle L_n^{(k)} \mid k=1,\dots n\rangle\rangle \tag{156}$$

is a subalgebra of **Sym** with respect to the internal product $*$. It is noncommutative for $n\geq 3$.

From this result the computation of $L_n^{(k)} * L_n^{(l)}$ is straightforward, and one finds

$$L_n^{(k)} * L_n^{(l)} = \sum_{m=1}^{n} \langle \ell_n^{(k)}, \ell_n^{(m-l)}\rangle L_n^{(m)}\,. \tag{157}$$

In particular, the commutative image

$$\ell_n^{(k)} * \ell_n^{(l)} = \sum_{m=1}^{n} \langle \ell_n^{(k)}, \ell_n^{(m-l)}\rangle \ell_n^{(m)} \tag{158}$$

amounts to an identity on Ramanujan sums

$$\frac{1}{d}\sum_{m=1}^{d} c(\ell-m,d)c(m,d) = c(\ell,d)\,. \tag{159}$$

It gives the decomposition of the product of two cyclic characters into a sum of characters of the same type.

4.3 Diagonalization of the left q-bracketing

We have seen that the Lie polynomials are eigenvectors of the standard left q-bracketing $x_1x_2\cdots x_n \longmapsto [\dots[x_1\,,\,x_2]_q\,,\dots x_n]_q$. It turns out that this operator is semisimple, with $p(n)$ distinct eigenvalues, and that its spectral projectors provide an example of a complete family of orthogonal idempotents of the descent algebra constructed from a sequence of Lie idempotents.

Instead of $\Theta_n(q)$, it will be more convenient to deal with $S_n((1-q)A)$ considered as left or right operator for the internal product, i.e. with the operators

$$F \longrightarrow S_n((1-q)A) * F \quad \text{or} \quad F \longrightarrow F * S_n((1-q)A) \tag{160}$$

where $F\in \mathbf{Sym}_n$.

The eigenvalues of both operators (left or right) are the polynomials

$$p_\lambda(1-q) = (1-q^{\lambda_1})\,(1-q^{\lambda_2})\,\dots\,(1-q^{\lambda_r}) \tag{161}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ runs over all partitions of n. The algebraic tools presented in Section 2 allows one to prove the following facts [59].

First, there exists a unique family $\pi(q) = (\pi_n(q))_{n\geq 1})$ of Lie idempotents (with $\pi_n(q) \in \mathbf{Sym}_n$ for all n) characterized by the property

$$\pi_n(q)((1-q)A) = (1-q^n)\,\pi_n(q)(A)\ . \tag{162}$$

Here are the first idempotents $\pi_n(q)$ for $n \leq 4$:

$$\pi_1(q) = \Psi_1\ ,\quad \pi_2(q) = \frac{\Psi_2}{2}\ ,\quad \pi_3(q) = \frac{\Psi_3}{3} + \frac{1}{6}\,\frac{1-q}{(1+2q)}\,[\Psi_2, \Psi_1]\ ,$$

$$\pi_4(q) = \frac{\Psi_4}{4} + \frac{1}{12}\,\frac{(1-q)(2q+1)}{(1+q+2q^2)}\,[\Psi_3, \Psi_1] + \frac{1}{24}\,\frac{(1-q)^2}{(1+q+2q^2)}\,[[\Psi_2, \Psi_1], \Psi_1]\,.$$

Next, from any sequence (F_n) of Lie idempotents such that $F_n \in \mathbf{Sym}_n$, one can construct a complete set of orthogonal idempotents $E_\lambda(F)$ for each descent algebra. We define as usual

$$F^I = F_{i_1}\,F_{i_2}\,\ldots\,F_{i_n} \tag{163}$$

for any composition $I = (i_1, i_2, \ldots, i_n)$. The family (F^I) is a basis of $\mathbf{Sym}$ and we can write

$$S_n = \sum_{|I|=n} p_{n,I}\,F^I \tag{164}$$

for some scalar coefficients $(p_{n,I})$. Then we associate to a partition λ of n the element

$$E_\lambda(F) = \sum_{I\in\mathfrak{S}(\lambda)} p_{n,I}\,F^I \tag{165}$$

where $\mathfrak{S}(\lambda)$ is the set of all permutations of λ. Hence we have the following decomposition of the identity S_n

$$S_n = \sum_{\lambda\vdash n} E_\lambda(F)\,, \tag{166}$$

and one can show that the family $(E_\lambda(F))_{\lambda\vdash n}$ is a complete family of orthogonal idempotents of $\mathbf{Sym}_n$. When applied to the sequence $F_n = \frac{1}{n}\Phi_n$, this construction gives the Garsia-Reutenauer idempotents [30].

Let now $E_\lambda(\pi(q))$ be the orthogonal idempotents associated with the family $\pi(q)$ by the above process. Then, the element $E_\lambda(\pi(q))$ is the spectral projector of $S_n((1-q)A)$ associated with the eigenvalue $p_\lambda(1-q)$. Therefore, $S_n((1-q)A)$ is semisimple.

The first spectral projectors of the operators $S_n((1-q)A)$ are

$$E_1 = \Psi_1\ ,\ E_2 = \frac{\Psi_2}{2}\ ,\ E_{11} = \frac{\Psi^{11}}{2}\ ,$$

$$E_3 = \frac{\pi_3(q)}{3}\ ,\ E_{21} = \frac{1}{2\,(1+2q)}\,(q\,\Psi^{21} + (1+q)\,\Psi^{12})\ ,\ E_{111} = \frac{\Psi^{111}}{6}\ ,$$

$$E_4 = \frac{\pi_4(q)}{4}\ ,\ E_{31} = \frac{1}{1+q+2q^2}\,(q^2\,\pi^{31}(q) + (q^2+q+1)\,\pi^{13}(q))\ ,\ E_{22} = \frac{\Psi^{22}}{8}\ ,$$

$$E_{211} = \frac{1}{4\,(1+2q)}\,(q\,\Psi^{211} + 2q\,\Psi^{121} + (q+2)\,\Psi^{112})\ ,\ E_{1111} = \frac{\Psi^{1111}}{24}\ .$$

One can recover from the above results the character of the left ideal $\mathbb{C}\mathfrak{S}_n\theta_n(q)$, as computed in [8], or, which amounts to the same, the character $f_n(q)$ of the linear group $GL(V)$ of a vector space V in $T^n(V)\theta_n(q)$.

For q not a root of unity of order n or less, the element $S_n(A/(1-q))$ is well defined, and invertible for the internal product, with inverse $S_n((1-q)A)$. So in this case, $f_n(q) = p_1^n$.

When $q = \omega$ is a root of unity, $W := T^n(V)\theta_n(\omega)$ is the direct sum of the eigenspaces $W_\lambda := T^n(V)e_\lambda(\omega)$ corresponding to nonzero eigenvalues (here, the e_λ are the elements of the group algebra of $\mathfrak{S}_n$ corresponding to the spectral projectors constructed in this section). Since these eigenvalues are $t_\lambda = p_\lambda(1-\omega)$, one finds that the character of $GL(V)$ in the image $W = T^n(V)\theta_n(\omega)$ of the iterated q-bracketing at a root of unity is

$$f_n(\omega) = \sum_{w_\lambda(\omega)\neq 0} L_\lambda \tag{167}$$

where $w_\lambda(q) = p_\lambda(1-q)/(1-q)$ and $L_\lambda = h_{m_1}[\ell_1^{(1)}]h_{m_2}[\ell_2^{(1)}]\cdots$ for $\lambda = (1^{m_1}2^{m_2}\cdots)$. Here $f[g]$ denotes the plethysm of g by f, and the $\ell_k^{(1)}$ are the cyclic characters of the preceding subsection. If ω is a primitive k-th root of unity, the generating series of the characters is

$$\sum_{n\geq 0} f_n(\omega) = \sigma_1\left[\sum_{m\not\equiv 0 \mod k} \ell_m\right].$$

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