# RSOS models and Jantzen-Seitz representations of Hecke algebras at roots of unity.

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**Abstract**

A special family of partitions occurs in two apparently unrelated contexts: the evaluation of 1-dimensional configuration sums of certain RSOS models, and the modular representation theory of symmetric groups or their Hecke algebras $H_m$. We provide an explanation of this coincidence by showing how the irreducible $H_m$-modules which remain irreducible under restriction to $H_{m-1}$ (Jantzen-Seitz modules) can be determined from the decomposition of a tensor product of representations of $\widehat{\mathfrak{sl}}_n$.

## 1 Introduction

The solution of a class of "restricted-solid-on-solid" (RSOS) models by the corner transfer matrix method leads to the evaluation of weighted sums of combinatorial objects called paths [1]. The Kyoto group realized that these combinatorial sums are branching functions of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [6], and was able to define similar models associated with other affine Lie algebras, in particular with $\widehat{\mathfrak{sl}}_n$ [16].

For the models associated with the cosets $(\widehat{\mathfrak{sl}}_n)_1 \times (\widehat{\mathfrak{sl}}_n)_1/(\widehat{\mathfrak{sl}}_n)_2$, a different description of the branching functions as generating series of certain sets of partitions has been obtained in [9], and was used to derive fermionic expressions for the configuration sums.

It turns out that exactly the same partitions arise in the modular representation theory of the symmetric groups: as conjectured by Jantzen and Seitz [14] and established recently by Kleshchev [20], such partitions label the irreducible representations of a symmetric group $\mathfrak{S}_m$ over a field of characteristic $n$ which remain irreducible under restriction to $\mathfrak{S}_{m-1}$.

The aim of this Letter is to provide an explanation of this seemingly mysterious coincidence.

The first point is to replace symmetric groups in characteristic $n$ by Hecke algebras of type $A$ over $\mathbb{C}$ at an $n$th root of unity. Indeed, the representation theories of $\mathbb{F}_n[\mathfrak{S}_m]$ and $H_m(\sqrt[n]{1})$ have many formal similarities, but the consideration of Hecke algebras removes the restriction of $n$ being a prime, which does not appear on the statistical mechanics side.

Moreover, a connection between the representation theory of $H_m(\sqrt[n]{1})$ and the level 1 representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ has been pointed out in [22]. Building on a conjecture of [22], recently proved by Ariki and Grojnowski, we show that the Jantzen-Seitz type problem for $H_m(\sqrt[n]{1})$ is equivalent to the decomposition via crystal bases of tensor products of level 1 $\widehat{\mathfrak{sl}}_n$-modules. Using the results of [9], we can then characterize the Hecke algebra modules of Jantzen-Seitz type and explain the occurence of their partition labels in the configuration sums of RSOS-models.

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As an application, we express the generating function of the Jantzen-Seitz partitions having a given $n$-core in terms of branching functions of $\widehat{\mathfrak{sl}}_n$. This is to be compared with a well-known result on blocks of Hecke algebras. Indeed, the blocks of $H_m(\sqrt[n]{1})$ are labelled by $n$-cores, and the dimension of a block is the number of $n$-regular partitions of $m$ with the corresponding $n$-core. Using a formula first proved in the fifties by Robinson (in the symmetric group case) one can compute the generating series of the dimensions of all blocks labelled by a given $n$-core, and recognize the string function of the level 1 $\widehat{\mathfrak{sl}}_n$-modules. Our result shows that some branching functions of $\widehat{\mathfrak{sl}}_n$ other than the level 1 string function arise in a natural way in the representation theory of $H_m(\sqrt[n]{1})$.

## 2 Characters and branching functions of $\widehat{\mathfrak{sl}}_n$

Using the notion of paths, it was shown in [5] that the characters of the integrable highest weight modules of $\widehat{\mathfrak{sl}}_n$ may be obtained by enumerating certain *coloured multipartitions*. In this Letter, we are interested only in the case when the highest weight is of level one, and therefore the multipartitions are simply partitions.

As is usual, we define a partition $\lambda$ of $m$ to be a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = m$. If $i > k$, we understand that $\lambda_i = 0$. The set of all partitions of $m$ is denoted $\Pi(m)$ and we write

$$\Pi = \bigcup_{m \geq 0} \Pi(m).$$

Occasionally, it will be convenient to use the exponent notation

$$\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})$$

where here $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$, and $a_i > 0$ specifies the multiplicity of $\lambda_i$ in $\lambda$. If $a_i < n$ for $1 \leq i \leq r$ then we say that the partition $\lambda$ is $n$-regular. The set of all $n$-regular partitions of $m$ is denoted $\Pi_n(m)$, and we define

$$\Pi_n = \bigcup_{m \geq 0} \Pi_n(m).$$

The Young diagram associated with the partition $\lambda$ is an array of $k$ left-adjusted rows of nodes (or boxes) in which the $i$th row contains $\lambda_i$ nodes. For example if $\lambda = (4, 3, 1)$, the corresponding Young diagram is:

$$F^{(4,3,1)} = \begin{array}{|c|c|c|c|}\hline \phantom{0} & \phantom{0} & \phantom{0} & \phantom{0} \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

We will not distinguish between a partition and its Young diagram. The partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ conjugate to $\lambda$ is defined such that $\lambda'_j$ is the length of the $j$th column of $\lambda$, reading from left to right.

A coloured partition is a Young diagram in which each node $\gamma$ is filled by its *colour charge* $c(\gamma)$ given by $c(\gamma) = (j - i) \bmod n$, when $\gamma$ is the node at the intersection of the $i$th row and the $j$th column. For example, in the case $n = 3$, the coloured partition $\lambda = (5, 5, 4, 1, 1)$ appears as follows:

$$\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & 2 & 0 \\ \hline 1 & 2 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 0 & \multicolumn{4}{c}{} \\ \cline{1-1} 2 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}.$$

Let $m_i$ be the number of nodes of $\lambda$ with colour charge $i$. The energy of $\lambda$ is defined by $E(\lambda) = m_0$, and its weight by $\mathrm{wt}\,(\lambda) = \Lambda_0 - \sum_{i=0}^{n-1} m_i \alpha_i$. (We use freely the standard notations for roots, weights, etc. of $\widehat{\mathfrak{sl}}_n$, see *e.g.* [5].) Then, the formal character of the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$ is

$$\mathrm{ch}\, V(\Lambda_0) = \sum_{\lambda \in \Pi_n} e^{\mathrm{wt}\,(\lambda)}.$$

A realisation of $V(\Lambda_0)$ that has a basis naturally indexed by the set of $n$-regular partitions $\Pi_n$ was given in [5].

In [15], it was also shown how to describe combinatorially the branching functions of tensor products of irreducible $\widehat{\mathfrak{sl}}_n$-modules. The branching function $b_{\Lambda,\Lambda'}^{\Lambda''}(q)$ is defined such that, for each $i \geq 0$, the multiplicity of the module $V(\Lambda'' - i\delta)$ in the tensor product $V(\Lambda) \otimes V(\Lambda')$ is given by the coefficient of $q^i$ in $b_{\Lambda,\Lambda'}^{\Lambda''}(q)$. Therefore, in terms of characters, we have

$$\mathrm{ch}\,(V(\Lambda))\mathrm{ch}\,(V(\Lambda')) = \sum_{\Lambda'' \in P+} b_{\Lambda,\Lambda'}^{\Lambda''}(e^{-\delta})\mathrm{ch}\,(V(\Lambda'')).$$

The result of [15] relies on the notion of a path which we describe in the case where $\Lambda' = \Lambda_0$. A path $p$ is a sequence $p = (p_0, p_1, \ldots)$, where $p_k \in P$, the weight lattice of $\widehat{\mathfrak{sl}}_n$. Given a dominant integral weight $\Lambda$ of level $l$ for some $l \geq 0$, and $\lambda \in \Pi_n$, a path $p = p(\lambda)$ is determined as follows. For $k \geq \lambda_1$, define $p_k = \Lambda + \Lambda_k$. Then for $0 < k \leq \lambda_1$, recursively obtain $p_{k-1}$ from $p_k$ by

$$p_{k-1} = p_k - \epsilon_{k-1-\lambda'_k},$$

where $\epsilon_i = \Lambda_{i+1} - \Lambda_i$ and as usual the indices are understood modulo $n$. We write $\lambda \in \mathcal{Y}(\Lambda, \Lambda_0)$ if and only if all the coordinates $p_k$ of the path $p(\lambda)$ are dominant weights.

**Theorem 2.1** [15]

$$b_{\Lambda,\Lambda_0}^{\Lambda'}(q) = \sum_{\substack{\lambda \in \mathcal{Y}(\Lambda,\Lambda_0) \\ \mathrm{wt}\,(\lambda) = \Lambda' - \Lambda\ (\mathrm{mod}\,\delta)}} q^{E(\lambda)}.$$

In the case when $\Lambda$ is of level one, the partitions appearing in this sum were characterised in [9]. We define the set $FOW(n,j) \subset \Pi_n$ as follows. Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r}) \in \Pi_n$. Then $\lambda \in FOW(n,j)$ if and only if either $r = 1$ or

$$a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod{n},$$

for $i = 1, 2, \ldots, r-1$ and $j = (\lambda_1 - a_1) \bmod n$.

**Theorem 2.2** [9]

$$\mathcal{Y}(\Lambda_j, \Lambda_0) = FOW(n,j).$$

If we now define $FOW(n,j,k)$ to be the subset of $FOW(n,j)$ comprising those partitions $\lambda$ for which $\mathrm{wt}\,(\lambda) = \Lambda_k + \Lambda_{j-k} - \Lambda_j\,(\mathrm{mod}\,\delta)$, then Theorem 2.1 immediately yields the following:

**Corollary 2.3** [9] *Let $0 \leq j, k < n$. Then the branching function $b_{\Lambda_j,\Lambda_0}^{\Lambda_k+\Lambda_{j-k}}(q)$ is given by:*

$$b_{\Lambda_j,\Lambda_0}^{\Lambda_k+\Lambda_{j-k}}(q) = \sum_{\lambda \in FOW(n,j,k)} q^{E(\lambda)}.$$

By means of this expression, the following was obtained:

**Theorem 2.4** [9] *Let $0 \leq j < n$ and $\Lambda = \Lambda_s + \Lambda_t$ with $0 \leq s \leq t < n$. In addition, let $C$ be the Cartan matrix of $\mathfrak{sl}_n$, $e_i$ be the $(n-1)$-dimensional unit vector $(0, \cdots, 0, \overset{i}{1}, 0 \cdots, 0)^t$ and we set $e_n = 0$. Then*

$$b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q) = \sum_{m} \frac{q^{m^t C^{-1} m - m^t C^{-1} e_{s-t+n} + st/n}}{\prod_{i=1}^{n-1} (q)_{m_i}},$$

*where the sum is taken over all $m \in (\mathbb{Z}_{\geq 0})^{\times(n-1)}$ satisfying $t + \sum_{i=1}^{n-1} i m_i = 0 \,(\bmod n)$, and $(q)_k = (1-q)(1-q^2)\cdots(1-q^k)$.*

# 3 Modular representations

The set of partitions $FOW(n, j, k)$ has a definite meaning in modular representation theory. Indeed, $\bigcup_{j,k} FOW(n, j, k)$ labels the modular representations of $\mathfrak{S}_m$ in characteristic $n$ that remain irreducible under reduction to $\mathfrak{S}_{m-1}$ [14, 20].

However, in the case of $FOW(n, j, k)$, $n$ can be any positive integer, whereas in the case of $\mathfrak{S}_m$, it has to be a prime number. This difference can be removed by working in the context of Hecke algebras at an $n$th root of unity, where the Jantzen-Seitz problem still makes sense.

The Hecke algebra $H_m(v)$ of type $A_{m-1}$, is the $\mathbb{C}(v)$-algebra generated by the elements $T_1, \ldots, T_{m-1}$ subject to the relations

$$\begin{aligned} &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \\ &T_i T_j = T_j T_i \qquad |i-j| > 1; \\ &T_i^2 = (v-1) T_i + v. \end{aligned}$$

In the case $v = 1$, $H_m(v)$ may be identified with the group algebra of the symmetric group $\mathfrak{S}_m$, through identifying $T_i$ with the simple transposition $(i, i+1) \in \mathfrak{S}_m$. In fact, for generic values of $v$, $H_m(v)$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_m$. Hence, it is semisimple, and its irreducible representations are parametrised by $\Pi(m)$. A convenient realisation of the representation labelled by $\lambda$ is the Specht module $S^\lambda$ in which the entries of the representation matrices of the generators are elements of $\mathbb{Z}[v]$.

In the non-generic case when $v$ is a primitive $n$th root of unity, $H_m(v)$ is no longer semisimple in general, and its representations are not necessarily completely reducible. The full set of irreducible representations is indexed by $\Pi_n(m)$ (see [4]). The irreducible module labelled by $\mu \in \Pi_n(m)$ is denoted by $D^\mu$.

Define $JS(n)$ to be the set of all partitions that label the irreducible representations of $H_m(\sqrt[n]{1})$ which remain irreducible under reduction to $H_{m-1}(\sqrt[n]{1})$. One of the aims of this work is to show that the partitions in $JS(n)$ are defined by the same conditions as in the case of symmetric groups, and to explain why

$$JS(n) = \bigcup_{j,k} FOW(n, j, k).$$

To tackle this problem, it is convenient to consider, as was done in [22, 2], the Grothendieck group $G_0(H_m(\sqrt[n]{1}))$ of the category of finitely generated $H_m(\sqrt[n]{1})$-modules. The elements of $G_0(H_m(\sqrt[n]{1}))$ are classes $[M]$ of modules, where $[M_1] = [M_2]$ if and only if the simple composition factors of $M_1$ and $M_2$ occur with identical multiplicities (the order of the composition factors in the series is disregarded). The sum is defined by $[M] + [N] = [M \oplus N]$. It is known that this is a free abelian group with as basis the set $\{[D^\mu]\}$ of classes of irreducible $H_m(\sqrt[n]{1})$-modules. Then define $\mathcal{G}(n)$ as the direct sum $\mathcal{G}(n) = \bigoplus_m G_0(H_m(\sqrt[n]{1}))$.

As observed in [22], $\mathcal{G}(n)$ can be identified with the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$, the action of the Chevalley generators $e_i$, $f_i$ being given by the $i$-restriction and $i$-induction operators, as defined in the fifties by G. de B. Robinson in the case of symmetric groups.

As an integrable highest weight module of an affine algebra, $\mathcal{G}(n)$ has a canonical basis in the sense of Lusztig and Kashiwara. It turns out that this canonical basis coincides with the natural basis given by the classes $[D(\mu)]$ of the irreducible $H_m(\sqrt[n]{1})$-modules.

To prove this, and to compute explicitly the canonical basis, one considers the Fock space representation $\mathcal{F}$ of $\widehat{\mathfrak{sl}}_n$, which contains $V(\Lambda_0)$ as its highest irreducible component:

$$\mathcal{F} = V(\Lambda_0) \oplus V_{\rm low} \ .$$

The standard basis $(v_\lambda)$ of $\mathcal{F}$ is labelled by all $\lambda \in \Pi$ and one has an $\widehat{\mathfrak{sl}}_n$-homomorphism

$$\begin{array}{rccc} d: & \mathcal{F} & \longrightarrow & V(\Lambda_0) \simeq \mathcal{F}/V_{\rm low} \\ & v_\lambda & \longmapsto & v_\lambda \ {\rm mod}\ V_{\rm low} \end{array}$$

If $\mathcal{F}$ is identified with the Grothendieck ring of all $H_m(v)$ for a generic $v$ by writing $[S^\lambda] = v_\lambda$, then $d$ coincides with the decomposition map of modular representation theory. In order to introduce the canonical basis, one needs to $q$-deform the picture and to consider the $q$-Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$.

# 4 The Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$

The $q$-Fock space $\mathcal{F}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ has been described in [11] using a $q$-analogue of the Clifford algebra. Another realization was given in [24, 19] in terms of semi-infinite $q$-wedge products. Let $(v_\lambda)$ denote the standard basis of weight vectors of $\mathcal{F}$. The Fock space $\mathcal{F}$ affords an integrable representation of $U_q(\widehat{\mathfrak{sl}}_n)$ whose decomposition into irreducible highest weight modules is given by

$$\mathcal{F} = \bigoplus_{k\geq 0} V(\Lambda_0 - k\delta)^{\oplus p(k)} \ . \qquad (1)$$

In [23], the lower crystal basis of $\mathcal{F}$ and its crystal graph structure were described. Let $A \subset \mathbb{Q}(q)$ denote the ring of rational functions without a pole at $q = 0$. The lower crystal basis at $q = 0$ of $\mathcal{F}$ is the pair $(L, B)$ where $L$ is the lower crystal lattice given by

$$L = \bigoplus_{\lambda\in\Pi} A\, v_\lambda \ ,$$

and $B$ is the basis of the $\mathbb{Q}$-vector space $L/qL$ given by

$$B = \{ v_\lambda \, {\rm mod}\, qL \mid \lambda \in \Pi \} \ .$$

The action of Kashiwara's operators $\tilde{f}_i$ on the element $v_\lambda \in B$ corresponds to adding to $\lambda$ a certain node of colour $i$ which is called a good addable $i$-node. Likewise the action of $\tilde{e}_i$ corresponds to the removal from $\lambda$ of a certain node of colour $i$ which is called a good removable $i$-node. As observed in [22], these nodes are precisely those used by Kleshchev in his modular branching rule for symmetric groups [21], whence the terminology.

For each $\lambda \in \Pi$, the largest integer $k$ such that $\tilde{e}_i{}^k(\lambda) \neq 0$ (*resp* $\tilde{f}_i{}^k(\lambda) \neq 0$) is denoted by $\varepsilon_i(\lambda)$ (*resp.* $\varphi_i(\lambda)$). The crystal graph $\Gamma_n$ of $\mathcal{F}$ is disconnected and reflects the decomposition (1). The connected component of the empty partition is the crystal graph of $V(\Lambda_0)$ and its vertices are labelled by $\Pi_n$.

We denote by $\{G(\mu), \mu \in \Pi_n\}$ the lower global basis of $V(\Lambda_0)$. It is the $\mathbb{Q}[q, q^{-1}]$-basis of the integral form $V_{\mathbb{Q}}(\Lambda_0)$ of $V(\Lambda_0)$ characterized by

$$G(\mu) \equiv v_\mu ( {\rm mod}\, qL), \qquad \overline{G(\mu)} = G(\mu).$$

Here, for $v = x v_\emptyset \in V(\Lambda_0)$, $\overline{v}$ is defined by $\overline{v} = \overline{x} v_\emptyset$, where $x \mapsto \overline{x}$ is the $\mathbb{Q}$-linear ring automorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$\overline{q} = q^{-1}, \qquad \overline{q^h} = q^{-h} \, , \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i,$$

for all $h \in P^\vee$ and $0 \le i < n$.

The upper global crystal basis $\{G^{up}(\mu), \mu \in \Pi_n\}$ of $V(\Lambda_0)$ [17] is the basis adjoint to $\{G(\mu)\}$ with respect to the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle v_\emptyset, v_\emptyset\rangle = 1 \qquad \langle q^h v, v'\rangle = \langle v, q^h v'\rangle\ , \qquad \langle e_i v, v'\rangle = \langle v, f_i v'\rangle$$

for all $v, v' \in V(\Lambda_0)$, $h \in P^\vee$ and $0 \le i < n$.

From now on we fix $n \ge 2$ and write $H_m$ instead of $H_m(\sqrt[n]{1})$. It has been conjectured in [22] and proved in [2] that at $q = 1$, the upper global crystal basis of the $U_q(\widehat{\mathfrak{sl}}_n)$-module $V(\Lambda_0)$ coincides in the identification $V(\Lambda_0) \simeq \mathcal{G}(n)$ with the basis $([D^\lambda])$ of $\mathcal{G}(n)$. This was also proved by Grojnowski, using the results of [10]. This implies that

**Theorem 4.1** *If $\lambda \in \Pi_n(m)$ and the coefficients $c_{\lambda\mu}(q)$ are defined by:*

$$\left(\sum_{i=0}^{n-1} e_i\right) G^{up}(\lambda) = \sum_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(q) G^{up}(\mu),$$

*then,*

$$[D^\lambda \downarrow^{H_m}_{H_{m-1}}] = \bigoplus_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(1)[D^\mu].$$

# 5 Jantzen-Seitz modules of Hecke algebras

We are now in position to prove the following result.

**Theorem 5.1** *Let $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r}) \in \Pi_n(m)$. Then $D^\lambda \downarrow^{H_m}_{H_{m-1}}$ is irreducible, i.e. $\lambda \in JS(n)$, if and only if either $r = 1$ or $a_i + \lambda_i - \lambda_{i+1} + a_{i+1} \equiv 0(\bmod n)$ for $i = 1, 2, \ldots, r-1$.*

*Proof:* By Theorem 4.1, $D^\lambda\!\downarrow$ is irreducible if and only if $e_i\, G^{up}(\lambda) = G^{up}(\nu)$ for some $\nu$. It is known (see [18], eqs. 5.3.8. – 5.3.10.) that

$$e_i\, G^{up}(\lambda) = [\varepsilon_i(\lambda)]_q G^{up}(\tilde{e}_i\lambda) + \sum_{\mu \ne \lambda} E^i_{\lambda\mu}(q)\, G^{up}(\mu) \tag{2}$$

where $E^i_{\lambda\mu}(q)$ is a Laurent polynomial invariant under $q \mapsto q^{-1}$ such that

$$E^i_{\lambda\mu}(q) \in q^{2-\varepsilon_i(\lambda)}\mathbb{Z}[q]\,.$$

If $\varepsilon_i(\lambda) = 1$, then $[\varepsilon_i(\lambda)]_q = 1$, and $E^i_{\lambda\nu}(q) = 0$ since for all $k$ the coefficients of $q^k$ and $q^{-k}$ in $E^i_{\lambda\mu}$ are equal. Therefore $e_i\, G^{up}(\lambda) = G^{up}(\nu)$ where $\nu = \tilde{e}_i\lambda$. On the other hand, if $e_i\, G^{up}(\lambda) = G^{up}(\nu)$ for some $\nu$, then (2) implies that $\varepsilon_i(\lambda) = 1$ so that $\nu = \tilde{e}_i\lambda$.

Hence we have obtained that $D^\lambda\!\downarrow$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$ for some $i \in \{0, 1, \ldots, n-1\}$ and $\varepsilon_j(\lambda) = 0$ for $j \ne i$.

Using crystal graph theory (see *e.g.* Lemma 5.1 and Theorem 5.3 of [15]), we see that $\epsilon_j(\lambda) \le \langle\Lambda', h_j\rangle$ for $j = 0, 1, \ldots, n-1$ if and only if $\lambda \in \mathcal{Y}(\Lambda', \Lambda_0)$. Therefore, $D^\lambda \downarrow$ is irreducible if and only if $\lambda \in \mathcal{Y}(\Lambda_i, \Lambda_0)$ for some $i$. Then, by Theorem 2.1, the Theorem is proved.

**Example 5.2** The argument considered in the above proof may be illustrated by reference to the $\emptyset$-connected component of the crystal graph $\Gamma_3$ of $\widehat{\mathfrak{sl}}_3$ given (up to weight 8) in Fig. 1. Here, those partitions $\lambda$ for which $\lambda \in JS(n)$ have been highlighted with an asterisk. As in the above proof, these partitions correspond to the nodes $b$ in this crystal graph for which, for some $j$, $\varepsilon_j(b) \le 1$ and $\varepsilon_i(b) = 0$ for all $i \ne j$.

Now define the set $JS(n,\mu,d)$ to be the subset of $JS(n)$ comprising those partitions with $n$-core $\mu$ and $n$-weight $d$ (see [13, 7] for definitions of $n$-core and $n$-weight). Then define the generating series

$$\chi_{n,\mu}(z) = \sum_{d\geq 0} \#JS(n,\mu,d)z^d.$$

**Theorem 5.3** (i) *If $\lambda \in JS(n)$ then the $n$-core $\mu$ of $\lambda$ is a rectangular partition $\mu = (k^l)$ such that $k + l \leq n$ (it is assumed here that if either $k = 0$ or $l = 0$ then $(k^l)$ means the empty partition).*

(ii) *If $k \neq 0 \neq l$ then*

$$\chi_{n,(k^l)}(z) = z^{-s}\, b^{\Lambda_k+\Lambda_{-l}}_{\Lambda_{k-l}\,\Lambda_0}(z)\,,$$

*where $s = \min(k,l)$.*

(iii)

$$\chi_{n,\emptyset}(z) = \left(\sum_{k=0}^{n-1} b^{\Lambda_k+\Lambda_0}_{\Lambda_k\,\Lambda_0}(z)\right) - (n-1)\,.$$

*Proof:* In the tensor product $V(\Lambda_i)\otimes V(\Lambda_0)$, all highest weights are of the form $\Lambda_k+\Lambda_{-l}-e\delta$ with $k-l = i \bmod n$ (for later convenience we use $-l$ and not $l$ here). We take $0\leq k,l<n$ here and can also assume that $k\leq -l \bmod n$ whereupon $k+l\leq n$, and $l=0$ only if $k=0$. By definition, the multiplicity of $V(\Lambda_k+\Lambda_{-l}-e\delta)$ is given by the coefficient of $z^e$ in $b^{\Lambda_k+\Lambda_{-l}}_{\Lambda_i,\Lambda_0}(z)$ and hence, by Corollary 2.3, by the number of partitions $\lambda\in JS(n)$ for which $\mathrm{wt}\,(\lambda) = \Lambda_k+\Lambda_{-l}-\Lambda_i-e\delta$. We claim that the $n$-core of such a $\lambda$ is the rectangular partition $\mu=(k^l)$, for which we calculate $\mathrm{wt}\,(\mu) = \Lambda_k+\Lambda_{-l}-\Lambda_{k-l}-s\delta = \Lambda_k+\Lambda_{-l}-\Lambda_i-s\delta$, where $s=\min(k,l)$ is the multiplicity of the colour charge 0 in $\mu$. This follows from the fact that for every string of weights $\Lambda,\Lambda-\delta,\Lambda-2\delta,\ldots$, of the $\widehat{\mathfrak{sl}}_n$-module $V(\Lambda_0)$ (where $\Lambda+\delta$ is not a weight of $V(\Lambda_0)$), those partitions having these weights have the same $n$-core, and this $n$-core has weight $\Lambda$. Then since $\mu=(k^l)$ with $k+l\leq n$ is manifestly an $n$-core, it follows that it is the $n$-core of $\lambda$, thence proving part (i).

Part (ii) follows immediately since in the case $k\neq 0$ (so that $l\neq 0$), the partitions enumerated by the branching function $b^{\Lambda_k+\Lambda_{-l}}_{\Lambda_{k-l}\Lambda_0}$ are precisely those elements $\lambda\in JS(n)$ having weight $\mathrm{wt}\,(\lambda) = \Lambda_k+\Lambda_{-l}-\Lambda_{k-l}-e\delta$, for some $e$, and hence $n$-core $(k^l)$.

Finally, for $k=0$ and arbitrary $l$, each partition counted by $b^{\Lambda_0+\Lambda_{-l}}_{\Lambda_{-l}\Lambda_0}$ has empty $n$-core, and hence contributes to $\chi_{n,\emptyset}$. However, the empty partition occurs for each $l$, hence an adjustment of $n-1$ is needed after summing over all $l$. No other partition $\lambda$ is repeated since, as indicated by Theorem 2.2, the $b^{\Lambda_0+\Lambda_{-l}}_{\Lambda_{-l}\Lambda_0}$ to which it contributes is uniquely determined by $-l \bmod n = (\lambda_1-a_1) \bmod n$. (The summation over $-l$ is replaced by one over $k$ to give the final result).

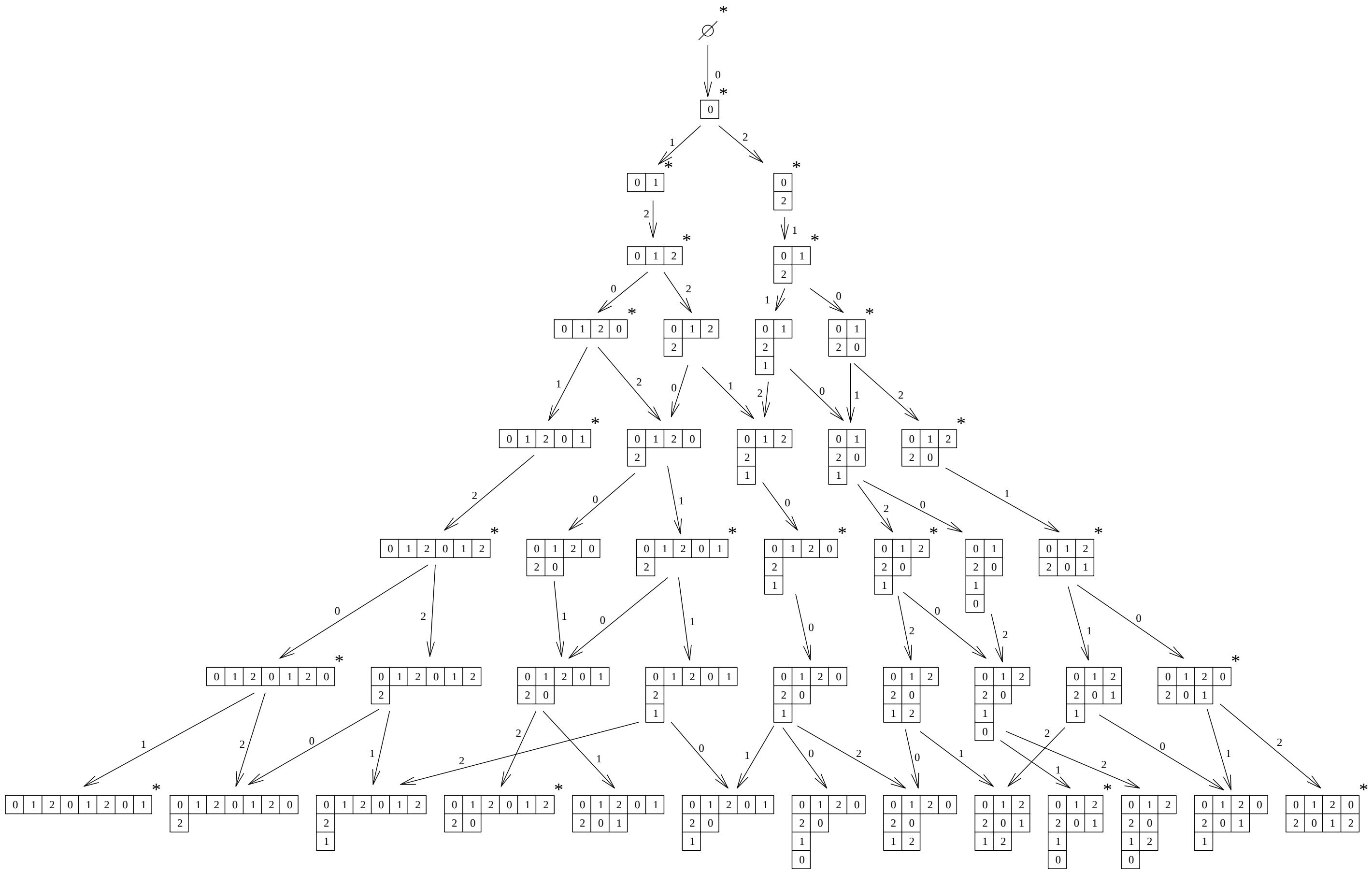

Figure 1.

**Example 5.4** To illustrate this result, consider again the case $n = 3$, where we have the following branching functions (to three terms):

$$\begin{aligned}
b^{2\Lambda_0}_{\Lambda_0,\Lambda_0} &= 1 + q^2 + \cdots;\\
b^{\Lambda_1+\Lambda_2}_{\Lambda_0,\Lambda_0} &= q + 2q^2 + 2q^3 + \cdots;\\
b^{\Lambda_1+\Lambda_0}_{\Lambda_1,\Lambda_0} &= 1 + q + 2q^2 + \cdots;\\
b^{2\Lambda_2}_{\Lambda_1,\Lambda_0} &= q + q^2 + 2q^3 + \cdots;\\
b^{\Lambda_2+\Lambda_0}_{\Lambda_2,\Lambda_0} &= 1 + q + 2q^2 + \cdots;\\
b^{2\Lambda_1}_{\Lambda_2,\Lambda_0} &= q + q^2 + 2q^3 + \cdots.
\end{aligned} \tag{3}$$

These are calculated using Corollary 2.3 which leads to the enumeration of the nodes of Fig. 1 labelled by asterisks. The only rectangular 3-cores are $\emptyset$, $(1)$, $(2)$ and $(1^2)$. Using Theorem 5.3, we thus obtain:

$$\begin{aligned}
\chi_{n,\emptyset} &= 1 + 2q + 5q^2 + \cdots;\\
\chi_{n,(1)} &= 1 + 2q + 2q^2 + \cdots;\\
\chi_{n,(2)} &= 1 + q + 2q^2 + \cdots;\\
\chi_{n,(1^2)} &= 1 + q + 2q^2 + \cdots.
\end{aligned} \tag{4}$$

These correspond to the following sets:

$$\begin{aligned}
JS(3,\emptyset,0) &= \{\emptyset\};\\
JS(3,\emptyset,1) &= \{(3),(21)\};\\
JS(3,\emptyset,2) &= \{(6),(51),(3^2),(41^2),(321)\},\\
JS(3,(1),0) &= \{(1)\};\\
JS(3,(1),1) &= \{(4),(2^2)\};\\
JS(3,(1),2) &= \{(7),(43)\},\\
JS(3,(2),0) &= \{(2)\};\\
JS(3,(2),1) &= \{(5)\};\\
JS(3,(2),2) &= \{(8),(3^21^2)\},\\
JS(3,(1^2),0) &= \{(1^2)\};\\
JS(3,(1^2),1) &= \{(32)\};\\
JS(3,(1^2),2) &= \{(62),(44)\}.
\end{aligned} \tag{5}$$

# 6 Discussion

We posed and solved the Jantzen-Seitz problem for Hecke algebras of type $A$. The solution is obtained by mapping the problem to a problem in exactly solvable lattice models, namely that of characterising the space of states of a certain class of restricted solid-on-solid model in terms of the states of the corresponding (unrestricted) solid-on-solid models. The latter was solved in [9]. The relationship between the two problems is based on the fact that both can be formulated in the same language of representation theory of $q$-affine algebra. The background to these two problems is discussed in greater detail in [7].

In a forthcoming paper [8], we plan to discuss the Jantzen-Seitz problem in the context of type-B Hecke algebras, and more generally, of Ariki-Koike (cyclotomic) Hecke algebras.

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*Note added* — A forthcoming preprint, [25], contains (among other things) an elementary purely combinatorial proof of part $(i)$ of 5.3.

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