RIBBON TABLEAUX, HALL-LITTLEWOOD FUNCTIONS, QUANTUM AFFINE ALGEBRAS AND UNIPOTENT VARIETIES *

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Abstract

We introduce a new family of symmetric functions, which are q-analogues of products of Schur functions, defined in terms of ribbon tableaux. These functions can be interpreted in terms of the Fock space representation $\mathcal{F}_q$ of $U_q(\widehat{\mathfrak{sl}}_n)$, and are related to Hall-Littlewood functions via the geometry of flag varieties. We present a series of conjectures, and prove them in special cases. The essential step in proving that these functions are actually symmetric consists in the calculation of a basis of highest weight vectors of $\mathcal{F}_q$ using ribbon tableaux.

Running head: Ribbon tableaux

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I Introduction

This article is devoted to the study of a new family of symmetric functions $H_\lambda^{(k)}(X;q)$, defined in terms of certain generalized Young tableaux, called ribbon tableaux, or rim-hook tableaux [1].

These objects, although unfamiliar, arise naturally in several contexts, and their use is implicit in many classical algorithms related to the symmetric groups (see *e.g.* Robinson's book [2]). In particular, they can be applied to the description of the power-sum plethysm operators $\psi^k : f(\{x_i\}) \mapsto f(\{x_i^k\})$ on symmetric functions [3, 4], and this point of view suggests the definition of a natural q-analogue ψ_q^k of ψ^k. This q-analogue turns out to make sense when the algebra of symmetric functions is interpreted as the bosonic Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_k)$. Indeed, one can prove, building on recent work by Kashiwara, Miwa and Stern [5, 6], that the image $\psi_q^k(f)$ of any symmetric function by this operator, is a highest weight vector for $U_q(\widehat{\mathfrak{sl}}_k)$. In particular, the images $\psi_q^k(h_\lambda)$ of products of complete homogeneous functions have a simple combinatorial description, and can be used as a convenient basis of highest weight vectors.

The space of symmetric functions is endowed with a natural scalar product (the same as in the Fock space interpretation), and one can consider the adjoint φ_q^k of the q-plethysm operator ψ_q^k. This operator divides degrees by k, and sends the Schur functions $s_{k\lambda}$ indexed by partitions of the form $k\lambda = (k\lambda_1, k\lambda_2, \ldots, k\lambda_r)$ onto a new basis, which is essentially the one considered in this paper. More precisely, $H_\lambda^{(k)}(X;q^{-2}) = \varphi_q^k(s_{k\lambda})$. It should be said, however, that our original definition was purely combinatorial, and that the connection with $U_q(\widehat{\mathfrak{sl}}_k)$ was understood only recently.

The H-functions are generalizations of Hall-Littlewood functions. We prove that for k sufficiently large, $H_\lambda^{(k)} = Q'_\lambda$, where (Q'_λ) is the adjoint basis of (P_λ) for the standard scalar product. Moreover, we conjecture that the differences $H_\lambda^{(k+1)} - H_\lambda^{(k)}$ are nonnegative on the Schur basis, *i.e.* that the H-functions form a filtration of the Q'-functions. In particular, the coefficients of H-functions on the Schur basis are conjectured to be polynomials with nonnegative integer coefficients.

The Q'-functions are known to be related to a variety of topics in representation theory [7, 8, 9, 10, 11], algebraic geometry [12, 13, 14, 15, 16, 17, 18, 19], combinatorics [20, 21, 22] and mathematical physics [23, 24, 25, 26]. As a general rule, q-analogues related to quantum groups admit interesting interpretations when the parameter q is specialized to the cardinality of a finite field, or to a complex root of unity. The Q'-functions are no exception. In the first case, the coefficients $\tilde{K}_{\lambda\mu}(q)$ of $\tilde{Q}'_\mu$ on the Schur basis are character values of the group $GL(n, \mathbb{F}_q)$ [15], while in the second one, a factorization property reminescent of Steinberg's tensor product theorem leads to combinatorial formulas for the Schur expansion of certain plethysms, in particular for $\psi^k(h_\mu)$ [8, 9].

On the basis of extensive numerical computations, we conjecture that the H-functions display the same behaviour with respect to specializations at roots of unity, giving this time plethysms $\psi^k(s_\lambda)$ of Schur functions by power sums. In fact, the H-functions were originally defined as q-analogues of products of Schur functions, being the natural generalization of those introduced in [27]. A combinatorial description of general H-functions on the Schur basis, similar to the one given in [27] in terms of Yamanouchi domino tableaux, would lead to a refined Littlewood-Richardson rule, compatible with cyclic symmetrization in the same way as the rule of [27] is compatible with symmetrized and antisymmetrized squares. This means that if one splits a tensor power $V_\lambda^{\otimes k}$ of an irreducible representation V_λ of $\mathrm{U}(n)$ into eigenspaces $E^{(i)}$ of the cyclic shift operator $v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto v_2 \otimes v_3 \otimes \cdots \otimes v_k \otimes v_1$, each $E^{(i)}$ is a representation of $\mathrm{U}(n)$ whose spectrum is given, according to the conjectures, by the coefficient of q^i in the reduction modulo $1 - q^k$ of $H^{(k)}_{\lambda^k}(q)$.

All the conjectures are proved for $k = 2$ (domino tableaux) and for k sufficiently large (the stable case). The case of domino tableaux follows from the combinatorial constructions of [27] and [28], while the stable case relies on the interpretation of Kostka-Foulkes polynomials in terms of characters of finite linear groups, and as Poincaré polynomials of certain algebraic varieties, in particular on the cell decompositions of these varieties found by N. Shimomura [18].

This article is structured as follows. In Section II, we recall some properties of Hall-Littlewood functions, in particular their interpretation in terms of affine Hecke algebras and their connection with finite linear groups. In Section III, we explain the connexion between plethysm and Hall-Littlewood functions at roots of unity, and in Section IV, we show how to translate these results in terms of ribbon tableaux. The application of ribbon tableaux to the construction of highest weight vectors in the Fock representation of $U_q(\widehat{\mathfrak{sl}}_n)$ is presented in Section V, and connected to a recent construction of Kashiwara, Miwa and Stern [6]. In Section VI, we define the H-functions, and summarize their known or conjectural properties. Section VII establishes these conjectures for H-functions of level 2, corresponding to domino tableaux. In Section VIII, we recall Shimomura's cell decompositions of unipotent varieties, and show the equivalence of his description of the Poincaré polynomials with a variant needed in the sequel. From this, we deduce that the H-functions of sufficiently large level are equal to Hall-Littlewood functions, which is also sufficient to prove all the conjectured properties in this case.

II Hall-Littlewood functions

Our notations for symmetric functions will be essentially those of the book [4], to which the reader is referred for more details.

The original definition of Hall-Littlewood functions can be reformulated in terms of an action of the affine Hecke algebra $\widehat{H}_N(q)$ of type A_{N-1} on the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ [29].

The affine Hecke algebra $\widehat{H}_N(q)$ is generated by T_i, $i = 1, \ldots, N-1$ and $y_i^{\pm 1}$, $i = 1, \ldots, N$, with relations

$$\begin{cases} T_i^2 = (q-1)T_i + q \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ T_i T_j = T_j T_i \quad (|j-i| > 1) \end{cases} \qquad \begin{cases} y_i y_j = y_j y_i \\ y_j T_i = T_i y_j \quad j \neq i, i+1 \\ y_j T_j = T_j y_{j+1} - (q-1) y_{j+1} \\ y_{j+1} T_j = T_j y_j + (q-1) y_{j+1} \end{cases} \tag{1}$$

If $\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_r}$ is a reduced decomposition of a permutation $\sigma \in \mathfrak{S}_N$, where $\sigma_i = (i, i+1)$, one sets as usual $T_\sigma = T_{i_1} T_{i_2} \ldots T_{i_r}$, the result being independent of the reduced decomposition.

Let $\Delta_N(q) = \prod_{1 \leq i < j \leq N} (qx_i - x_j)$. Then, on the one hand, the Hall-Littlewood polynomial $Q_\lambda(x_1, \ldots, x_N; q)$ indexed by a partition λ of length $\ell(\lambda) \leq N$ is defined by [30]

$$Q_\lambda(x_1, \ldots, x_N; q) = \frac{(1-q)^{\ell(\lambda)}}{[m_0]_q!} \sum_{\sigma \in \mathfrak{S}_N} \sigma \left(x^\lambda \frac{\Delta_N(q)}{\Delta_N(1)} \right) \tag{2}$$

where $m_0 = N - \ell(\lambda)$ and the q-integers are here defined by $[n]_q = (1-q^n)/(1-q)$.

On the other hand, $\widehat{H}_N(q)$ acts on $\mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by $y_i(f) = x_i f$ and $T_i = (q-1)\pi_i + \sigma_i$, where π_i is the isobaric divided difference operator

$$\pi_i(f) = \frac{x_i f - x_{i+1}\sigma_i(f)}{x_i - x_{i+1}} ,$$

and it is shown in [29] that if one defines the q-symmetrizing operator $S^{(N)} \in \widehat{H}_N(q)$ by

$$S^{(N)} = \sum_{\sigma \in \mathfrak{S}_N} T_\sigma , \tag{3}$$

then

$$Q_\lambda(x_1, \ldots, x_N; q^{-1}) = q^{-\binom{N}{2}} \frac{(1-q^{-1})^{\ell(\lambda)}}{[m_0]_{q^{-1}}!} S^{(N)}(x^\lambda) . \tag{4}$$

The normalization factor $1/[m_0]_q!$ is here to ensure stability with respect to the adjunction of variables, and if we denote by X the infinite set $X = \{x_1, x_2, \ldots, \}$ then $Q_\lambda(X; q) = \lim_{N\to\infty} Q_\lambda(x_1, \ldots, x_N; q)$.

The P-functions are defined by

$$P_\lambda(X; q) = \frac{1}{(1-q)^{\ell(\lambda)}[m_1]_q! \cdots [m_n]_q!} Q_\lambda(X; q)$$

where m_i is the multiplicity of the part i in λ.

We consider these functions as elements of the algebra $Sym = Sym(X)$ of symmetric functions with coefficients in $\mathbb{C}(q)$. In this paper, the scalar product $\langle\, , \rangle$ on Sym will always be the standard one, for which the Schur functions s_λ form an orthonormal basis.

We denote by $(Q'_\mu(X;q))$ the adjoint basis of $(P_\lambda(X;q))$ for this scalar product. It is easy to see that $Q'_\mu(X;q)$ is the image of $Q_\mu(X;q)$ by the ring homomorphism $p_k \mapsto (1-q^k)^{-1}p_k$ (in λ-ring notation, $Q'_\mu(X;q) = Q(X/(1-q);q))$. In the Schur basis,

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda\mu}(q)s_\lambda(X) \tag{5}$$

where the $K_{\lambda\mu}(q)$ are the Kostka-Foulkes polynomials. The polynomial $K_{\lambda\mu}(q)$ is the generating function of a statistic c called *charge* on the set $\mathrm{Tab}\,(\lambda,\mu)$ of Young tableaux of shape λ and weight μ ([21, 22], see also [14, 4])

$$K_{\lambda\mu}(q) = \sum_{\mathbf{t}\in \mathrm{Tab}\,(\lambda,\mu)} q^{c(\mathbf{t})}\ . \tag{6}$$

We shall also need the $\tilde{Q}'$-functions, defined by

$$\tilde{Q}'_\mu(X;q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q)s_\lambda(X) = q^{n(\mu)}Q'_\mu(X;q^{-1})\ , \tag{7}$$

where $n(\mu) = \sum_{i\ge 1}(i-1)\mu_i$. The polynomial $\tilde{K}_{\lambda\mu}(q)$ is the generating function of the complementary statistic $\tilde{c}(\mathbf{t}) = n(\mu) - c(\mathbf{t})$, which is called *cocharge*.

When the parameter q is interpreted as the cardinality of a finite field $\mathbb{F}_q$, it is known that $\tilde{K}_{\lambda\mu}(q)$ is equal to the value $\chi^\lambda(u)$ of the unipotent character χ^λ of $G = GL(n,\mathbb{F}_q)$ on a unipotent element u with Jordan canonical form specified by the partition μ ([7] see also [31]).

In this specialization, the coefficients

$$\tilde{G}_{\nu\mu}(q) = \langle h_\nu\,,\, \tilde{Q}'_\mu\rangle \tag{8}$$

of the $\tilde{Q}'$-functions on the basis of monomial symmetric functions are also the values of certain characters of G on unipotent classes. Let $\mathcal{P}_\nu$ denote a parabolic subgroup of type ν of G, for example the group of upper block triangular matrices with diagonal blocks of sizes $\nu_1,\ldots,\nu_r$, and consider the permutation representation of G over $\mathbb{C}[G/\mathcal{P}_\nu]$. The value $\xi^\nu(g)$ of the character ξ^ν of this representation on an element $g\in G$ is equal to the number of fixed points of g on $G/\mathcal{P}_\nu$. It can be shown that, for a unipotent u of type μ,

$$\xi^\nu(u) = \tilde{G}_{\nu\mu}(q)\ . \tag{9}$$

The factor set $G/\mathcal{P}_\nu$ can be identified with the variety $\mathcal{F}_\nu$ of ν-flags in $V = \mathbb{F}_q^n$

$$V_{\nu_1} \subset V_{\nu_1+\nu_2} \subset \ldots \subset V_{\nu_1+\ldots+\nu_r} = V$$

where $\dim V_i = i$. Thus, $\tilde{G}_{\nu\mu}(q)$ is equal to the number of $\mathbb{F}_q$-rational points of the algebraic variety $\mathcal{F}_\nu^u$ of fixed points of u in $\mathcal{F}_\nu$.

III Specializations at roots of unity

As recalled in the preceding section, the Hall-Littlewood functions with parameter specialized to the cardinality q of a finite field $\mathbb{F}_q$ provide information about the complex characters of the linear group $GL(n, \mathbb{F}_q)$ over this field. It turns out that when the parameter is specialized to a complex root of unity, one obtains information about representations of $GL(n, \mathbb{C})$ (or $\mathrm{U}(n)$), that is, a combinatorial decomposition of certain plethysms [8, 9]. We give now a brief review of these results.

The first one is a factorization property of the functions $Q'_\lambda(X, q)$ when q is specialized to a primitive root of unity. This is to be seen as a generalization of the fact that when q is specialized to 1 the function $Q'_\lambda(X; q)$ reduces to $h_\lambda(X) = \prod_i h_{\lambda_i}(X)$.

Theorem III.1 [8] *Let $\lambda = (1^{m_1} 2^{m_2} \ldots n^{m_n})$ be a partition written multiplicatively. Set $m_i = kq_i + r_i$ with $0 \le r_i < k$, and $\mu = (1^{r_1} 2^{r_2} \ldots n^{r_n})$. Then, ζ being a primitive k-th root of unity,*

$$Q'_\lambda(X; \zeta) = Q'_\mu(X; \zeta) \prod_{i \ge 1} [Q'_{(i^k)}(X; \zeta)]^{q_i} \ . \tag{10}$$

The functions $Q'_{(i^k)}(X; \zeta)$ appearing in the right-hand side of (10) can be expressed as plethysms.

Theorem III.2 [8] *Let $p_k \circ h_n$ denote the plethysm of the complete function h_n by the power-sum p_k, which is defined by the generating series*

$$\sum_n p_k \circ h_n(X)\, z^n = \prod_{x \in X} (1 - zx^k)^{-1} \ .$$

Then, if ζ is as above a primitive k-th root of unity, one has

$$Q'_{(n^k)}(X; \zeta) = (-1)^{(k-1)n} p_k \circ h_n(X).$$

For example, with $k=3$ ($\zeta = e^{2i\pi/3}$), we have

$$Q'_{444433311}(X;\zeta) = Q'_{411}(X;\zeta)\, Q'_{333}(X;\zeta)\, Q'_{444}(X;\zeta) = Q'_{411}(X;\zeta)\, p_4 \circ h_{43} \ .$$

Let V be a polynomial representation of $GL(n,\mathbb{C})$, with character the symmetric function f. Let γ be the cyclic shift operator on $V^{\otimes k}$, that is,

$$\gamma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_2 \otimes v_3 \otimes \cdots \otimes v_k \otimes v_1 \ .$$

Let $\zeta = \exp(2\mathrm{i}\pi/k)$, and denote by $E^{(r)}$ the eigenspace of γ in $V^{\otimes k}$ associated with the eigenvalue ζ^r. As γ commutes with the action of $GL(n,\mathbb{C})$, these eigenspaces are representations of $GL(n,\mathbb{C})$, and their characters are given by the plethysms $\ell_k^{(r)} \circ f$ of the character f of V by certain symmetric functions $\ell_k^{(r)}$ that we shall now describe.

For $k,n \in \mathbb{N}$, the *Ramanujan* or *Von Sterneck sum* $c(k,n)$ (also denoted $\Phi(k,n)$) is the sum of the k-th powers of the *primitive* n-th roots of unity. Its value is given by *Hölder's formula*: if $(k,n)=d$ and $n=md$, then $c(k,n) = \mu(m)\phi(n)/\phi(m)$, where μ is the Moebius function and ϕ is the Euler totient function (see *e.g.* [32]).

The symmetric functions $\ell_k^{(r)}$ are given by the formula

$$\ell_k^{(r)} = \frac{1}{k}\sum_{d|k} c(r,d) p_d^{k/d} \ . \tag{11}$$

These functions are the Frobenius characteristics of the representations of the symmetric group induced by irreducible representations of a transitive cyclic subgroup [33]. A combinatorial interpretation of the multiplicity $\langle s_\lambda\, ,\, \ell_n^{(k)}\rangle$ has been given by Kraskiewicz and Weyman [34]. This result is equivalent to the congruence

$$Q'_{(1^n)}(X;q) \equiv \sum_{0\le k\le n-1} q^k \ell_n^{(k)} \ (\ \mathrm{mod}\ 1-q^n)\ .$$

Another proof can be found in [35]. Now, if V is a product of exterior powers of the fundamental representation $\mathbb{C}^n$

$$V = \Lambda^{\nu_1}\mathbb{C}^n \otimes \Lambda^{\nu_2}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{\nu_m}\mathbb{C}^n$$

the character of $E^{(r)}$ is $\ell_k^{(r)} \circ e_\nu$, and similarly if V is a product of symmetric powers with character h_ν, the character of $E^{(r)}$ is $\ell_k^{(r)} \circ h_\nu$.

Given two partitions λ and μ, we denote by $\lambda \vee \mu$ the partition obtained by reordering the concatenation of λ and ν, *e.g.* $(2,\,2,\,1) \vee (5,\,2,\,1,\,1) = (5,\,2^3,\,1^3)$. We write $\mu^k = \mu \vee \mu \vee \cdots \vee \mu$ (k factors). If $\mu = (\mu_1,\,\ldots,\,\mu_r)$, we set $k\mu = (k\mu_1,\,\ldots,\,k\mu_r)$.

Taking into account Theorems III.1 and III.2 and following the method of [35], one arrives at the following combinatorial formula for the decomposition of $E^{(r)}$ into irreducibles:

Theorem III.3 [9] *Let e_i be the i-th elementary symmetric function, and for $\lambda = (\lambda_1, \ldots, \lambda_m)$, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r}$. Then, the multiplicity $\langle s_\mu\, ,\, \ell_k^{(r)} \circ e_\lambda \rangle$ of the Schur function s_μ in the plethysm $\ell_k^{(r)} \circ e_\lambda$ is equal to the number of Young tableaux of shape μ' (conjugate partition) and weight λ^k whose charge is congruent to r modulo k.*

This gives as well the plethysms with product of complete functions, since

$$\langle s_{\mu'}\, ,\, \ell_k^{(r)} \circ e_\lambda \rangle = \begin{cases} \langle s_\mu\, ,\, \ell_k^{(r)} \circ h_\lambda \rangle & \text{if } |\lambda| \text{ is even} \\ \langle s_\mu\, ,\, \tilde{\ell}_k^{(r)} \circ h_\lambda \rangle & \text{if } |\lambda| \text{ is odd} \end{cases}$$

where $\tilde{\ell}_k^{(r)} = \omega(\ell_k^{(r)}) = \ell_k^{(s)}$ with $s = k(k-1)/2 - r$.

For example, $\langle s_{732}\, ,\, \ell_4^{(2)} \circ e_{21} \rangle = 5$ is the number of tableaux with shape $(3, 3, 2, 1, 1, 1, 1)$, weight $(2, 2, 2, 2, 1, 1, 1, 1)$ and charge $\equiv 2 \pmod 4$.

Another combinatorial formulation of Theorems III.1 and III.2 can be presented using the notion of *ribbon tableau*, which will also provide the clue for their generalization.

IV Ribbon tableaux

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is represented graphically by its *Young diagram* Y_λ, having λ_i square cells on its ith row. If μ is another partition such that $Y_\mu \subset Y_\lambda$, one defines the *skew Young diagram* $Y_{\lambda/\mu} = Y_\lambda - Y_\mu$. We call *ribbon* a connected skew Young diagram of width 1, *i.e.* which does not contain any 2×2 square. A k-ribbon is a ribbon made of k square cells.

To a partition λ is associated a k-core $\lambda_{(k)}$ and a k-quotient $\lambda^{(k)}$ [36]. The k-core is the unique partition obtained by successively removing k-ribbons from λ. The different possible ways of doing so can be distinguished from one another by labelling 1 the last ribbon removed, 2 the penultimate, and so on. Thus Figure 1 shows two different ways of reaching the 3-core $\lambda_{(3)} = (2, 1^2)$ of $\lambda = (8, 7^2, 4, 1^5)$. These pictures represent two 3-ribbon tableaux T_1, T_2 of shape $\lambda/\lambda_{(3)}$ and weight $\mu = (1^9)$.

Figure 1

To define k-ribbon tableaux of general weight and shape, we need some terminology. The *initial cell* of a k-ribbon R is its rightmost and bottommost cell. Let $\theta = \beta/\alpha$ be a skew shape, and set $\alpha_+ = (\beta_1) \vee \alpha$, so that α_+/α is the horizontal strip made of the bottom cells of the columns of θ. We say that θ is a *horizontal k-ribbon strip* of weight m, if it can be tiled by m k-ribbons the initial cells of which lie in α_+/α. (One can check that if such a tiling exists, it is unique).

Now, a *k-ribbon tableau* T of shape λ/ν and weight $\mu = (\mu_1, \ldots, \mu_r)$ is defined as a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal k-ribbon strip of weight μ_i. Graphically, T may be described by numbering each k-ribbon of α^i/α^{i-1} with the number i. We denote by $\mathrm{Tab}_k(\lambda/\nu, \mu)$ the set of k-ribbon tableaux of shape λ/ν and weight μ, and we set

$$K^{(k)}_{\lambda/\nu,\mu} = |\mathrm{Tab}_k(\lambda/\nu, \mu)| \ .$$

Finally we recall the definition of the k-sign $\epsilon_k(\lambda/\nu)$. Define the sign of a ribbon R as $(-1)^{h-1}$, where h is the height of R. The k-sign $\epsilon_k(\lambda/\nu)$ is the product of the signs of all the ribbons of a k-ribbon tableau of shape λ/ν (this does not depend on the particular tableau chosen, but only on the shape).

The origin of these combinatorial definitions is best understood by analyzing carefully the operation of multiplying a Schur function s_ν by a plethysm of the form $\psi^k(h_\mu) = p_k \circ h_\mu$. Equivalently, thanks to the involution ω, one may rather consider a product of the type $s_\nu\,[p_k \circ e_\mu]$. To this end, since

$$p_k \circ e_\mu = (e_{\mu_1} \circ p_k) \cdots (e_{\mu_n} \circ p_k) = m_{k^{\mu_1}} \cdots m_{k^{\mu_n}}$$

one needs only to apply repeatedly the following multiplication rule due to Muir (see [4]):

$$s_\nu\, m_\alpha = \sum_\beta s_{\nu+\beta} \ ,$$

sum over all distinct permutations β of $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0, \ldots)$. Here the Schur functions $s_{\nu+\beta}$ are not necessarily indexed by partitions and have therefore to be put in standard form, this reduction yielding only a finite number of nonzero summands. For example,

$$s_{31}\, m_3 = s_{61} + s_{313} + s_{31003} = s_{61} - s_{322} + s_{314} \ .$$

Other terms such as s_{34} or s_{3103} reduce to 0. It is easy to deduce from this rule that the multiplicity

$$\langle s_\nu\, m_{k^{\mu_i}} \ , \ s_\lambda \rangle$$

is nonzero iff λ'/μ' is a horizontal k-ribbon strip of weight μ_i, in which case it is equal to $\epsilon_k(\lambda/\nu)$. Hence, applying ω we arrive at the expansion

$$s_\nu\,[p_k \circ h_\mu] = \sum_\lambda \epsilon_k(\lambda/\mu)\, K^{(k)}_{\lambda/\nu\,,\mu}\, s_\lambda \tag{12}$$

from which we deduce by III.1, III.2 that

$$K^{(k)}_{\lambda\mu} = (-1)^{(k-1)|\mu|}\, \epsilon_k(\lambda)\, K_{\lambda\,\mu^k}(\zeta)$$

and more generally, defining as in [24] the skew Kostka-Foulkes polynomial $K_{\lambda/\nu\,,\alpha}(q)$ by

$$K_{\lambda/\nu\,,\alpha}(q) = \langle s_{\lambda/\nu}\,,\, Q'_\alpha(q)\rangle$$

(or as the generating function of the charge statistic on skew tableaux of shape λ/μ and weight α), we can write

$$K^{(k)}_{\lambda/\nu\,,\mu} = (-1)^{(k-1)|\mu|}\, \epsilon_k(\lambda/\nu)\, K_{\lambda/\nu\,,\mu^k}(\zeta)\ .$$

It turns out that enumerating k-ribbon tableaux is equivalent to enumerating k-tuples of ordinary Young tableaux, as shown by the correspondence to be described now. This bijection was first introduced by Stanton and White [1] in the case of ribbon tableaux of right shape λ (without k-core) and standard weight $\mu = (1^n)$ (see also [37]). In the sequel we shall refer to it as the *Stanton-White correspondence*. We need some additional definitions.

Let R be a k-ribbon of a k-ribbon tableau. R contains a unique cell with coordinates $(x,\, y)$ such that $y - x \equiv 0 \pmod k$. We decide to write in this cell the number attached to R, and we define the *type* $i \in \{1,\, \ldots,\, k\}$ of R as the position of this cell inside R, the position of the initial cell being equal to 1. For example, the 3-ribbons of T_1 are divided up into three classes (see Fig. 1):

- 4, 6, 8, of type 1;
- 1, 2, 7, 9, of type 2;
- 3, 5, of type 3.

Define the *diagonals* of a k-ribbon tableau as the sequences of integers read along the straight lines $D_i\, :\, y - x = ki$. Thus T_1 has the sequence of diagonals

$$((8),\,(4),\,(2,\,3,\,6),\,(1,\,5,\,9),\,(7))\ .$$

This definition applies in particular to 1-ribbon tableaux, *i.e.* ordinary Young tableaux. It is obvious that a Young tableau is uniquely determined by its sequence of diagonals. Hence, we can associate to a given k-ribbon tableau T of shape λ/ν a k-tuple $(\mathbf{t}_1,\, \ldots,\, \mathbf{t}_k)$ of Young tableaux defined as follows; the diagonals of $\mathbf{t}_i$ are obtained by erasing in the diagonals of T the labels of all the ribbons of type $\neq i$. For instance, if $T = T_1$ the first ribbon tableau of Figure 1, the sequence of diagonals of $\mathbf{t}_2$ is $((2),\,(1,\,9),\,(7))$, and

$$\mathbf{t}_2 = \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array}$$

The complete triple $(\mathbf{t}_1,\, \mathbf{t}_2,\, \mathbf{t}_3)$ of Young tableaux associated to T_1 is

$$\tau^1 = \left(\begin{array}{|c|c}\cline{1-1} 8 & \\ \hline 4 & \multicolumn{1}{c|}{6} \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 2 & 9 \\ \hline 1 & 7 \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array} \right)$$

whereas that corresponding to T_2 is

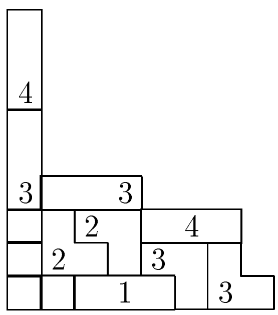
$$\tau^2 = \left(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{8} \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 6 & 9 \\ \hline 4 & 5 \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 2 & 7 \\ \hline \end{array} \right)$$

One can show that if $\nu = \lambda_{(k)}$, the k-core of λ, the k-tuple of shapes $(\lambda^1,\, \ldots,\, \lambda^k)$ of $(\mathbf{t}_1,\, \ldots,\, \mathbf{t}_k)$ depends only on the shape λ of T, and is equal to the k-quotient $\lambda^{(k)}$ of λ. Moreover the correspondence $T \longrightarrow (\mathbf{t}_1,\, \ldots,\, \mathbf{t}_k)$ establishes a bijection between the set of k-ribbon tableaux of shape $\lambda/\lambda_{(k)}$ and weight μ, and the set of k-tuples of Young tableaux of shapes $(\lambda^1,\, \ldots,\, \lambda^k)$ and weights $(\mu^1,\, \ldots,\, \mu^k)$ with $\mu_i = \sum_j \mu_i^j$.

For example, keeping $\lambda = (8,\, 7^2,\, 4,\, 1^5)$, the triple

$$\tau = \left(\begin{array}{|c|c}\cline{1-1} 4 & \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \right)$$

with weights $((0,\, 0,\, 2,\, 1),\, (1,\, 1,\, 1,\, 1),\, (0,\, 1,\, 1,\, 0))$ corresponds to the 3-ribbon tableau

$T =$ 

of weight $\mu = (1,\, 2,\, 4,\, 2)$.

As before, the significance of this combinatorial construction becomes clearer once interpreted in terms of symmetric functions. Recall the definition of φ_k, the adjoint of the linear operator $\psi^k :\ F \mapsto p_k \circ F$ acting on the space of symmetric functions. In other words, φ_k is characterized by

$$\langle \varphi_k(F)\,,\, G\rangle = \langle F\,,\, p_k \circ G\rangle\ , \quad F,\, G \in Sym\ .$$

Littlewood has shown [38] that if λ is a partition whose k-core $\lambda_{(k)}$ is empty, then

$$\varphi_k(s_\lambda) = \epsilon_k(\lambda)\, s_{\lambda^1} \cdots s_{\lambda^k} \tag{13}$$

where $\lambda^{(k)} = (\lambda^1, \ldots, \lambda^k)$ is the k-quotient. Therefore,

$$K^{(k)}_{\lambda\,\mu} = \epsilon_k(\lambda)\, \langle p_k \circ h_\mu \,,\, s_\lambda \rangle = \epsilon_k(\lambda)\, \langle \varphi_k(s_\lambda) \,,\, h_\mu \rangle = \langle s_{\lambda^1} \cdots s_{\lambda^k} \,,\, h_\mu \rangle$$

is the multiplicity of the weight μ in the product of Schur functions $s_{\lambda^1} \cdots s_{\lambda^k}$, that is, is equal to the number of k-tuples of Young tableaux of shapes $(\lambda^1, \ldots, \lambda^k)$ and weights $(\mu^1, \ldots, \mu^k)$ with $\mu_i = \sum_j \mu_i^j$. Thus, the bijection described above gives a combinatorial proof of (13).

More generally, if λ is replaced by a skew partition λ/ν, (13) becomes [39]

$$\varphi_k(s_{\lambda/\nu}) = \epsilon_k(\lambda/\nu)\, s_{\lambda^1/\nu^1} \cdots s_{\lambda^k/\nu^k}$$

if $\lambda_{(k)} = \nu_{(k)}$, and 0 otherwise. This can also be deduced from the previous combinatorial correspondence, but we shall not go into further details.

Returning to Kostka polynomials, we may summarize this discussion by stating Theorems III.1 and III.2 in the following way:

Theorem IV.1 *Let λ and ν be partitions and set $\nu = \mu^k \vee \alpha$ with $m_i(\alpha) < k$. Denoting by ζ a primitive kth root of unity, one has*

$$K_{\lambda,\nu}(\zeta) = (-1)^{(k-1)|\mu|} \sum_\beta \epsilon_k(\lambda/\beta)\, K^{(k)}_{\lambda/\beta,\,\mu}\, K_{\beta,\alpha}(\zeta)\ . \tag{14}$$

Example IV.2 We take $\lambda = (4^2, 3)$, $\nu = (2^2, 1^7)$ and $k = 3$ ($\zeta = e^{2i\pi/3}$). In this case, $\nu = \mu^k \vee \alpha$ with $\mu = (1^2)$ and $\alpha = (2^2, 1)$. The summands of (14) are parametrized by the 3-ribbon tableaux of external shape λ and weight μ. Here we have three such tableaux:

Figure 2

so that

$$K_{443,\,221111111}(\zeta) = 2K_{41,\,221}(\zeta) - K_{32,\,221}(\zeta) = 2(\zeta^2 + \zeta^3) - (\zeta + \zeta^2) = 2\zeta^2 + 3\ .$$

When $|\alpha| \le |\lambda_{(k)}|$, (14) becomes simpler. For if $|\alpha| < |\lambda_{(k)}|$ then $K_{\lambda,\nu}(\zeta) = 0$, and otherwise the sum reduces to one single term

$$K_{\lambda,\nu}(\zeta) = (-1)^{(k-1)|\mu|} \epsilon_k(\lambda/\lambda_{(k)})\, K^{(k)}_{\lambda/\lambda_{(k)},\,\mu}\, K_{\lambda_{(k)},\alpha}(\zeta)\ .$$

In particular, for $\nu = (1^n)$, one recovers the expression of $K_{\lambda,(1^n)}(\zeta)$ given by Morris and Sultana [11].

Finally, let us observe that the notion of k-sign of a partition can be lifted to a statistic on ribbon tableaux, which for technical reasons that will appear in Section VI, takes values in $\frac{1}{2}\mathbb{N}$, and will be called *spin*.

Let R be a k-ribbon, and denote by $h(R)$ its *heigth*:

Figure 3

The *spin* of R, denoted by $s(R)$, is defined as

$$s(R) = \frac{h(R) - 1}{2} \tag{15}$$

and the spin of a ribbon tableau T is by definition the sum of the spins of its ribbons. For example, the ribbon tableau

Figure 4

has spin 6.

We define the spin $s(\lambda/\mu)$ of a horizontal k-ribbon strip λ/μ of weight m as the spin of the unique k-ribbon tableau of shape λ/μ and weight (m). This number can be computed from the inversion number of a certain permutation. Set $n = \ell(\lambda')$ and let $\nu = \mu'$ considered as an element of $\mathbb{N}^n$. Let α be the vector of $\mathbb{N}^n$ such that $\alpha_j = k$ if one of the ribbons of the horizontal strip λ/μ has its initial cell in the jth column of λ, and $\alpha_j = 0$ otherwise. Let $\rho = (n-1, n-2, \ldots, 1, 0)$. The fact that λ/μ is a horizontal k-ribbon strip is equivalent to the property that the components of the vector $\gamma = \nu + \alpha + \rho$ are pairwise distinct. Let $I(\gamma)$ be the length of the minimal permutation sorting γ in decreasing order. Then

$$s(\lambda/\mu) = \frac{1}{2}\left((k-1)m - I(\gamma)\right) . \tag{16}$$

Note that the k-sign of a partition λ is equal to $(-1)^{2s(T)}$, for any ribbon tableau T of shape λ. For example, we can rewrite the particular case $\nu = (0)$ of formula (12) as

$$\psi^k(h_\mu) = p_k \circ h_\mu = \sum_{T \in \mathrm{Tab}_k(\,\cdot\,,\mu)} (-1)^{2s(T)} s_T \tag{17}$$

where $\mathrm{Tab}_k(\,\cdot\,,\mu)$ is the set of k-ribbon tableaux of weight μ, and $s_T = s_\lambda$ if λ is the shape of T. We shall see in the next section that this formula leads to a simple construction of a basis of highest weight vectors in the Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_k)$.

V The Fock representation of $U_q(\widehat{\mathfrak{sl}}_n)$

The affine Lie algebra $\widehat{\mathfrak{sl}}_n = A_{n-1}^{(1)}$ has a natural action on the space *Sym* of symmetric functions, called the bosonic Fock space representation. This representation is equivalent to the infinite wedge, or fermionic Fock space representation, and the isomorphism can be realized by means of vertex operators (see *e.g.* [40]).

Let us recall briefly the fermionic version. Let V be the vector space $\mathbb{C}^{(\mathbb{Z})}$ and $(u_i)_{i\in\mathbb{Z}}$ be its canonical basis. The fermionic Fock space $\bigwedge^\infty V$ is defined as the vector space spanned by the infinite exterior products $u_{i_1} \wedge u_{i_2} \wedge \cdots \wedge u_{i_n} \wedge \cdots$ satisfying $i_1 > i_2 > i_3 > \ldots > i_n > \ldots$ and $i_k - i_{k+1} = 1$ for $k >> 0$. The wedge product is as usual alternating, and linear in each of its factors. We denote by $\mathcal{F}$ the subspace of $\bigwedge^\infty V$ spanned by the elements such that $i_k = -k+1$ for $k >> 0$ (usually this subspace is denoted by $\mathcal{F}^{(0)}$, but as we shall not need the other sectors $\mathcal{F}^{(m)}$, we drop the superscript).

The Lie algebra $\mathfrak{gl}_\infty$ of $\mathbb{Z}\times\mathbb{Z}$ complex matrices $A = (a_{ij})$ with finitely many nonzero entries acts on $\bigwedge^\infty V$ by

$$A \cdot u_{i_1} \wedge u_{i_2} \wedge \cdots = (Au_{i_1}) \wedge u_{i_2} \wedge \cdots + u_{i_1} \wedge (Au_{i_2}) \wedge \cdots + \cdots$$

the sum having only a finite number of nonzero terms.

Let E_{ij} be the infinite matrix $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. The subalgebra $\mathfrak{sl}_\infty$ of $\mathfrak{gl}_\infty$ consisting of the matrices with zero trace has for Chevalley generators $e_i^\infty = E_{i,i+1}$, $f_i^\infty = E_{i+1,i}$. The infinite sums

$$e_i = \sum_{j\equiv i \bmod n} e_j^\infty \ , \qquad f_i = \sum_{j\equiv i \bmod n} f_j^\infty \ , \qquad (0 \le i \le n-1)$$

do not belong to $\mathfrak{sl}_\infty$, but they have a well-defined action on $\bigwedge^\infty V$, and it can be checked that they generate a representation of $\widehat{\mathfrak{sl}}'_n$ with central charge $c = 1$. The remaining generator D of $\widehat{\mathfrak{sl}}_n = \widehat{\mathfrak{sl}}'_n \oplus \mathbb{C}\, D$ can be implemented by

$$D := -\sum_{i\in\mathbb{Z}} \left[\frac{i}{n}\right] (E_{ii} - \theta(-i)) \ ,$$

where $\theta(x) = 1$ for $x \ge 0$ and $\theta(x) = 0$ otherwise (*cf.* [41]).

The basis vectors of $\mathcal{F}$ can be labelled by partitions, by setting

$$|\lambda\rangle = u_{\lambda_1} \wedge u_{\lambda_2 - 1} \wedge u_{\lambda_3 - 2} \wedge \cdots \ .$$

With this indexation, the action of the Chevalley generators of $\widehat{\mathfrak{sl}}_n$ can be described as follows [41].

To each node (i,j) of a Young diagram, one can associate its *residue* $\rho_{i,j} = j - i \bmod n \in \{0,\ldots,n-1\}$. Then,

$$e_r|\lambda\rangle = \sum |\mu\rangle \ , \qquad f_r|\lambda\rangle = \sum |\nu\rangle \tag{18}$$

where μ (*resp.* ν) runs over all diagrams obtained from λ by removing (*resp.* adding) a node of n-residue r.

In this picture, one can observe that e_r and f_r are exactly the r-restricting and r-inducing operators introduced by G. de B. Robinson in the context of the modular representation theory of the symmetric group [2].

The natural way of interpreting the basis vectors $|\lambda\rangle$ as symmetric functions is to put $|\lambda\rangle = s_\lambda$. This is imposed by the boson-fermion correspondence, and it is also compatible with the modular representation interpretation.

In this realization, it can be shown that the image $U(\widehat{\mathfrak{sl}}_n)|0\rangle$ of the constant $|0\rangle = s_0 = 1$, which is the basic representation $M(\Lambda_0)$, is equal to the subalgebra

$$\mathcal{T}^{(n)} = \mathbb{C}\,[p_i \,|\, i \not\equiv 0 \bmod n]$$

generated by the power-sums p_i such that $n \not| i$.

The bosonic operators

$$b_k \ : \ f \longmapsto D_{p_{kn}} f = kn\frac{\partial}{\partial p_{kn}} f \quad \text{and} \quad b_{-k} \ : \ f \longmapsto p_{kn} f \qquad (k \geq 1)$$

commute with the action of $\widehat{\mathfrak{sl}}_n$. They generate a Heisenberg algebra $\mathcal{H}$, and the irreducible $\mathcal{H}$-module $U(\mathcal{H})|0\rangle$ is exactly the space $\mathcal{S}^{(n)}$ of highest weight vectors of the Fock space, viewed as an $\widehat{\mathfrak{sl}}_n$-module. Thus, these highest weight vectors are exactly the plethysms $\psi^n(f), f \in Sym$. Natural bases of $\mathcal{S}^{(n)}$ are therefore $\psi^n(p_\mu) = p_{n\mu}$, $\psi^n(s_\mu)$ or $\psi^n(h_\mu)$. We know from Section IV that this last one admits a simple combinatorial description in terms of ribbon tableaux:

$$\psi^n(h_\mu) = \sum_\lambda \epsilon_n(\lambda/\mu) K^{(n)}_{\lambda\mu} s_\lambda = \sum_{T\in \mathrm{Tab}\,_n(\,\cdot\,,\mu)} (-1)^{2s(T)} s_T \ . \tag{19}$$

This formula is especially meaningful in the quantized version, that we shall now describe.

We first recall the definition of $U_q(\widehat{\mathfrak{sl}}_n)$ (*cf.* [42] and references therein). Let $\mathfrak{h}$ be a $(n+1)$-dimensional vector space over $\mathbb{Q}$ with basis $\{h_0, h_1, \ldots, h_{n-1}, D\}$. We denote by $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}, \delta\}$ the dual basis of $\mathfrak{h}^*$, that is,

$$\langle \Lambda_i, h_j\rangle = \delta_{ij}, \quad \langle \Lambda_i, D\rangle = 0, \quad \langle \delta, h_i\rangle = 0, \quad \langle \delta, D\rangle = 1,$$

and we set $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i0}\,\delta$ for $i = 0, 1, \ldots, n-1$. (Here and in the sequel, the indices of the fundamental weights Λ_i are understood modulo n). The $n \times n$ matrix $[\langle \alpha_i, h_j \rangle]$ is the generalized Cartan matrix associated to $\widehat{\mathfrak{sl}}_n$. The weight lattice is $P = (\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i) \bigoplus \mathbb{Z}\delta$, its dual is $P^\vee = (\bigoplus_{i=0}^{n-1} \mathbb{Z}\, h_i) \bigoplus \mathbb{Z}\, D$, and the root lattice is $Q = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i$. One defines $U_q(\widehat{\mathfrak{sl}}_n)$ as the associative algebra with 1 over $\mathbb{Q}(q)$ generated by the symbols $e_i,\ f_i,\ 0 \le i \le n-1$, and $q^h,\ h \in P^\vee$, subject to the relations

$$q^h\, q^{h'} = q^{h+h'}, \quad q^0 = 1,$$

$$q^h e_j q^{-h} = q^{\langle \alpha_j, h\rangle} e_j,$$

$$q^h f_j q^{-h} = q^{-\langle \alpha_j, h\rangle} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j \rangle \\ k \end{bmatrix} e_i^{1-\langle \alpha_i, h_j\rangle - k} e_j e_i^k = 0 \quad (i \ne j),$$

$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j \rangle \\ k \end{bmatrix} f_i^{1-\langle \alpha_i, h_j\rangle - k} f_j f_i^k = 0 \quad (i \ne j).$$

Here the q-integers, q-factorials and q-binomial coefficients are the symmetric ones:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [k]\,[k-1]\,\cdots[1], \quad \begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]!\,[k]!}\,.$$

We now recall some definitions relative to $U_q(\widehat{\mathfrak{sl}}_n)$-modules. Let M be a $U_q(\widehat{\mathfrak{sl}}_n)$-module and $\Lambda \in P$ a weight. The subspace

$$M_\Lambda = \{ v \in M \mid q^h\, v = q^{\langle \Lambda, h\rangle}\, v,\ h \in P^\vee \}$$

is called the weight space of weight Λ of M and its elements are called the weight vectors of weight Λ. The module M is said to be integrable if

(i) $M = \bigoplus_{\Lambda \in P} M_\Lambda$,

(ii) $\dim M_\Lambda < \infty$ for $\Lambda \in P$,

(iii) for $i = 0, 1, \ldots, n-1$, M decomposes into a direct sum of finite dimensional U_i-modules, where U_i denotes the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $e_i,\ f_i,\ q^{h_i},\ q^{-h_i}$.

A highest weight vector $v \in M$ is a vector annihilated by all raising operators e_i. The module M is said to be a highest weight module if there exists a highest weight vector v such that $M = U_q(\widehat{\mathfrak{sl}}_n)\, v$. The weight of v is called the highest weight of M.

By the representation theory of $U_q(\widehat{\mathfrak{sl}}_n)$, there exists for each dominant integral weight Λ (*i.e.* $\langle \Lambda, h_i\rangle \in \mathbb{Z}_+$ for $i = 0, 1, \ldots, n-1$) a unique integrable highest weight module $M_q(\Lambda)$ with highest weight Λ.

A q-analogue of the Fock representation of $\widehat{\mathfrak{sl}}_n$ can be realized in the $\mathbb{Q}(q)$-vector space $\mathcal{F}_q$ spanned by all partitions:

$$\mathcal{F}_q = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q}(q)\, |\lambda\rangle$$

the action being defined in combinatorial terms.

Let us say that a point (a, b) of $\mathbb{Z}_+ \times \mathbb{Z}_+$ is an indent i-node of a Young diagram λ if a box of residue $i = a - b \bmod n$ can be added to λ at position (a, b), in such a way that the new diagram still corresponds to a partition. Similarly, a node of λ of residue i which can be removed will be called a removable i-node.

Let $i \in \{0, 1, \ldots, n-1\}$ and let λ, ν be two partitions such that ν is obtained from λ by filling an indent i-node γ We set:

$N_i(\lambda) = \sharp\{$ indent i-nodes of λ $\} - \sharp\{$ removable i-nodes of λ $\}$,

$N_i^l(\lambda, \nu) = \sharp\{$ indent i-nodes of λ situated to the *left* of γ (not counting γ) $\}$ $-\sharp\{$ removable i-nodes of λ situated to the *left* of γ $\}$,

$N_i^r(\lambda, \nu) = \sharp\{$ indent i-nodes of λ situated to the *right* of γ (not counting γ) $\} - \sharp\{$ removable i-nodes of λ situated to the *right* of γ $\}$,

$N^0(\lambda) = \sharp\{$ 0-nodes of λ $\}$.

Figure 5

The following result is due to Hayashi [43], and the formulation that we use has been given by Misra and Miwa [42] (with a slight change, namely, conjugation of partitions and $q \to 1/q$).

Theorem V.1 *The algebra $U_q(\widehat{\mathfrak{sl}}_n)$ acts on $\mathcal{F}_q$ by*

$q^{h_i}\, |\lambda\rangle = q^{N_i(\lambda)}\, |\lambda\rangle$,

$q^D\, |\lambda\rangle = q^{-N^0(\lambda)}\, |\lambda\rangle$,

$f_i |\lambda\rangle = \sum_\nu q^{N_i^r(\lambda,\nu)}\, |\nu\rangle$, *sum over all partitions ν such that ν/λ is a i-node,*

$e_i |\nu\rangle = \sum_\lambda q^{-N_i^l(\lambda,\nu)}\, |\lambda\rangle$, *sum over all partitions λ such that ν/λ is a i-node.*

It is easy to see that $\mathcal{F}_q$ is an integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module. It is not irreducible. Actually it decomposes as

$$\mathcal{F}_q \cong \bigoplus_{k\geq 0} M_q(\Lambda_0 - k\delta)^{\oplus p(k)} ,$$

where $p(k)$ is the number of partitions of k. Obviously, the empty partition $|0\rangle$ is a highest weight vector of weight Λ_0. The submodule $U_q(\widehat{\mathfrak{sl}}_n)\,|0\rangle$ is isomorphic to $M_q(\Lambda_0)$, also called the *basic representation* of $U_q(\widehat{\mathfrak{sl}}_n)$. Again, one can identify $\mathcal{F}_q$ with *Sym* (with coefficients in $\mathbb{Q}(q)$) and interpret $|\lambda\rangle$ as s_λ. Then, a natural q-analogue of (19) gives a basis of highest weight vectors for $U_q(\widehat{\mathfrak{sl}}_n)$ in $\mathcal{F}_q$:

Proposition V.2 *Define a linear operator ψ_q^n on Sym by*

$$\psi_q^n(h_\mu) = \sum_{T\in \mathrm{Tab}_n(\,\cdot\,,\mu)} (-q)^{-2s(T)} s_T \ .$$

Then, its image $\psi_q^n(Sym)$ is the space $\mathcal{S}_q^{(n)}$ of highest weight vectors of $U_q(\widehat{\mathfrak{sl}}_n)$ in Sym.

Example V.3 The plethysm $\psi^2(h_{21})$ is given by the following domino tableaux

Figure 6

and the corresponding highest weight vector of $U_q(\widehat{\mathfrak{sl}}_2)$ is

$$\psi_q^2(h_{21}) = s_6 - q^{-1}s_{51} + (1+q^{-2})s_{42} - q^{-1}s_{411}$$

$$-(q^{-1}+q^{-3})s_{33} + q^{-2}s_{3111} + q^{-2}s_{222} - q^{-3}s_{2211}$$

The proposition is a consequence of the following more precise statement.

Theorem V.4 *Let U_k, V_k ($k\geq 1$) be the linear operators defined by*

$$V_k s_\lambda = \sum_\mu (-q)^{-2s(\mu/\lambda)} s_\mu \ ,$$

where μ runs over all partitions such that μ/λ is a horizontal n-ribbon strip of weight (k) (see Section IV*) and*

$$U_k s_\lambda = \sum_\nu (-q)^{-2s(\lambda/\nu)} s_\nu \ ,$$

where ν runs over all partitions such that λ/ν is a horizontal n-ribbon strip of weight (k). Thus, V_k is a q-analogue of $f\mapsto \psi^n(h_k)f$, and U_k is its adjoint. Then, U_k and V_k commute with the action of $U_q(\widehat{\mathfrak{sl}}_n)$. In particular, each $\psi_q^n(h_\mu) = V_{\mu_r}\cdots V_{\mu_1}|0\rangle$ is a highest weight vector.

This result can be obtained by a direct verification, using formula (12). However, a more illuminating approach comes from comparison with a recent construction of Stern [5] and Kashiwara, Miwa and Stern [6]. These authors construct the q-analogue of the Fock representation by means of a q-deformation of the wedge product, defined in terms of an action of the affine Hecke algebra $\widehat{H}_N(q^{-2})$ on a tensor product $V(z)^{\otimes N}$ of evaluation modules.

Here, $V(z)$ is $\mathbb{C}^{(\mathbb{Z})}$ realized as

$$\left(\bigoplus_{i=1}^{n} \mathbb{C}\, v_i\right) \otimes \mathbb{C}\,[z, z^{-1}]\ ,$$

where $z^i v_j$ is identified with u_{j-ni}. The space $V(z)$ is endowed with a left action of $U_q(\widehat{\mathfrak{sl}}_n)$ defined by requiring that u_k is a weight vector of weight $\Lambda_k - \Lambda_{k+1}$ and

$$f_i\, u_k = \delta_{k\equiv i \bmod n}\, u_{k+1}\,, \qquad e_i\, u_k = \delta_{k\equiv i+1 \bmod n}\, u_{k-1}\,.$$

Writing $z^{r_1}v_{m_1}\otimes\cdots\otimes z^{r_N}v_{m_N}$ as $v_{\mathbf{m}}z^{\mathbf{r}} = v_{m_1}\otimes\cdots\otimes v_{m_N}\cdot z_1^{r_1}\cdots z_N^{r_N}$, the right action of $\widehat{H}_N(q^{-2})$ on $V(z)^{\otimes N}$ is described by the following formulas [44, 5, 6]. The generator y_i acts as multiplication by z_i^{-1} and

$$(v_{\mathbf{m}}\cdot z^{\mathbf{r}})T_i = \begin{cases} -q^{-1}v_{\mathbf{m}\sigma_i}\cdot\sigma_i(z^{\mathbf{r}}) + (q^{-2}-1)v_{\mathbf{m}}\cdot\partial_i(z_iz^{\mathbf{r}}) \text{ if } m_i < m_{i+1} \\ -v_{\mathbf{m}}\cdot\sigma_i(z^{\mathbf{r}}) + (q^{-2}-1)v_{\mathbf{m}}\cdot z_i\partial_i(z^{\mathbf{r}}) \text{ if } m_i = m_{i+1} \\ -q^{-1}v_{\mathbf{m}\sigma_i}\cdot\sigma_i(z^{\mathbf{r}}) + (q^{-2}-1)v_{\mathbf{m}}\cdot z_i\partial_i(z^{\mathbf{r}}) \text{ if } m_i > m_{i+1} \end{cases} \tag{20}$$

where $\mathbf{m}\sigma = (m_{\sigma(1)},\ldots,m_{\sigma(N)})$ and ∂_i is the divided difference operator $f(z)\mapsto (f - \sigma_i(f))/(z_i - z_{i+1})$. This action can be regarded as a generalization of the one given in [29], which would correspond to the degenerate case $n=1$.

There is a $U_q(\widehat{\mathfrak{sl}}_n)$-action on $V(z)^{\otimes N}$ via the iterated coproduct

$$\Delta^N f_i = f_i\otimes 1\otimes\cdots\otimes 1 + q^{h_i}\otimes f_i\otimes 1\otimes\cdots\otimes 1 + \cdots + q^{h_i}\otimes\cdots\otimes q^{h_i}\otimes f_i\,,$$

$$\Delta^N e_i = e_i\otimes q^{-h_i}\otimes\cdots\otimes q^{-h_i} + 1\otimes e_i\otimes q^{-h_i}\otimes\cdots\otimes q^{-h_i} + \cdots + 1\otimes\cdots\otimes 1\otimes e_i\,.$$

The important point is that the right action of $\widehat{H}_N(q^{-2})$ commutes with the left action of $U_q(\widehat{\mathfrak{sl}}_n)$. Let $A^{(N)} = \sum_{\sigma\in\mathfrak{S}_N} T_\sigma$. This is a q-analogue of the total antisymmetrizer of $\mathfrak{S}_N$, since signs have been incorporated in formulas (20) in such a way that T_i acts as a q-analogue of $-\sigma_i$. Kashiwara, Miwa and Stern define then the q-exterior powers by $\bigwedge_q^N V(z) = V(z)^{\otimes N}/\ker A^{(N)}$, and denote by $u_{i_1}\wedge_q\cdots\wedge_q u_{i_N}$ the image of $u_{i_1}\otimes\cdots\otimes u_{i_N}$ in the quotient. A basis of $\bigwedge_q^N V(z)$ is formed by the normally ordered products $u_{i_1}\wedge_q\cdots\wedge_q u_{i_N}$, where $i_1 > i_2 > \ldots > i_N$, and any q-wedge product can be expressed on this basis, by means of the following relations iteratively applied to consecutive factors. Suppose that $\ell < m$ and that $\ell - m \bmod n = i$. Set $t = q^{-1}$. Then, if $i=0$ one has $u_\ell \wedge_q u_m = -u_m \wedge_q u_\ell$,

otherwise, $u_\ell \wedge_q u_m = -t u_m \wedge_q u_\ell + (t^2-1)(u_{m-i} \wedge_q u_{\ell+i} - t u_{m-n} \wedge_q u_{m+n} + t^2 u_{m-n-i} \wedge_q u_{\ell+n+i} + \cdots)$,

where the only terms to be taken into account in this last expression are the normally ordered ones.

There is then a well-defined action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the "thermodynamic limit"

$$\bigwedge_q^{\infty} V(z) = \lim_{N\to\infty} \bigwedge_q^{N} V(z)\,,$$

which provides another realization of the q-Fock representation. As above, we denote by $\mathcal{F}_q$ the subspace spanned by the q-wedges such that $i_k = -k+1$ for $k >> 0$. The action of Theorem V.1 can be recovered via the isomorphism

$$|\lambda\rangle \longrightarrow u_{\lambda_1} \wedge_q u_{\lambda_2 - 1} \wedge_q u_{\lambda_3 - 2} \wedge_q \cdots .$$

The center of the affine Hecke algebra $\widehat{H}_N(q^{-2})$ acts on $\bigwedge_q^N V(z)$. This center is generated by the power sums

$$p_k(Y) = \sum_{i=1}^{N} y_i^k \qquad k = \pm 1, \pm 2, \ldots$$

and in the thermodynamic limit, the operators

$$B_k = \sum_{i=1}^{\infty} y_i^{-k}$$

are shown in [6] to generate an action of a Heisenberg algebra on $\mathcal{F}_q$, with

$$[B_k, B_\ell] = k \frac{1-q^{-2nk}}{1-q^{-2k}} \cdot \delta_{k,-\ell}\,. \tag{21}$$

If one interprets the infinite q-wedges as Schur functions, by the same rule as in the classical case, one sees that B_{-k} ($k \geq 1$) is a q-analogue of the multiplication operator $f \mapsto p_{nk} f$, and that B_k corresponds to its ajoint $D_{p_{nk}}$.

To connect this construction with the preceding one, take for generators of the center of the affine Hecke algebra the elementary symmetric functions in the y_i and y_i^{-1} instead of the power sums, and define operators on $\mathcal{F}_q$ by

$$\tilde{U}_k = e_k(y_1, y_2, \ldots), \qquad \tilde{V}_k = e_k(y_1^{-1}, y_2^{-1}, \ldots)\,.$$

These operators commute with $U_q(\widehat{\mathfrak{sl}}_n)$, and their action can be described in terms of ribbon tableaux:

Lemma V.5

$$\tilde{U}_k|\lambda\rangle = \sum_{\nu}(-q)^{2s(\lambda'/\nu')-k(n-1)}|\nu\rangle \tag{22}$$

where the summation runs over all partitions ν such that λ'/ν' is a horizontal n-ribbon strip of weight (k). Similarly,

$$\tilde{V}_k|\lambda\rangle = \sum_{\mu}(-q)^{2s(\mu'/\lambda')-k(n-1)}|\mu\rangle \tag{23}$$

where the summation runs over all partitions μ such that μ'/λ' is a horizontal n-ribbon strip of weight (k).

Proof — It is sufficient to work with $\bigwedge_q^N V(z)$ for N sufficiently large. Then,

$$\tilde{V}_k u_{i_1}\wedge_q\cdots\wedge_q u_{i_N} = \sum_J u_{i_1+j_1}\wedge_q\cdots\wedge_q u_{i_N+j_N}$$

where $J = (j_1,\ldots,j_N)$ runs through the distinct permutations of the integer vector $(0^{N-k}n^k)$. The only reorderings needed to express a term of this sum in standard form are due to the appearance of factors of the form $u_i\wedge_q u_{j+n}$ with $j+n>i>j$. In this case, $u_i\wedge_q u_{j+n} = -tu_{j+n}\wedge_q u_i$ since the other terms $(t^2-1)(u_{j+n-a}\wedge_q u_{i+a} - tu_j\wedge_q u_{i+n}+\cdots)$ vanish, the residue $a = j+n-i \bmod n$ being actually equal to $j+n-i$. The first case of the straightening rule is never encountered because $j+n-i\equiv 0 \bmod n$ would imply $i-j=bn$ with $b>0$, so that $j+n\not> i$.

Thus,

$$u_{i_1+j_1}\wedge_q\cdots\wedge_q u_{i_N+j_N} = (-t)^{\ell(\sigma)}u_{i_{\sigma(1)}+j_{\sigma(1)}}\wedge_q\cdots\wedge_q u_{i_{\sigma(N)}+j_{\sigma(N)}}$$

where σ is the shortest permutation such that the result is normally ordered. In view of formula (16), this gives the result for $\tilde{V}_k$. The argument for $\tilde{U}_k$ is similar.

Corollary V.6 *The operators U_k, V_k of Theorem* V.4 *act on $\mathcal{F}_q$ as $h_k(y_1,y_2,\ldots)$ and $h_k(y_1^{-1},y_2^{-1},\ldots)$ respectively. In particular, $[U_i,U_j]=[V_i,V_j]=0$.*

VI H-functions

Let λ be a partition without k-core, and with k-quotient $(\lambda^0,\ldots,\lambda^{k-1})$. For a ribbon tableau T of weight μ, let $x^T = x_1^{\mu_1}x_2^{\mu_2}\cdots x_r^{\mu_r}$. Then, the correspondence between k-ribbon tableaux and k-tuples of ordinary tableaux shows that the generating function

$$\mathcal{G}_\lambda^{(k)} = \sum_{T\in\mathrm{Tab}\,_k(\lambda,\,\cdot\,)} x^T = \prod_{i=0}^{k-1}\sum_{\mathbf{t}_i\in\mathrm{Tab}\,(\lambda^i,\,\cdot\,)} x^{\mathbf{t}_i} = \prod_{i=0}^{k-1} s_{\lambda^i} \tag{24}$$

is a product of Schur functions. Introducing in this equation an appropriate statistic on ribbon tableaux, one can therefore obtain q-analogues of products of Schur functions. The statistic spin leads to q-analogues with interesting properties.

For a partition λ without k-core, let

$$s_k^*(\lambda) = \max\{s(T) \mid T \in \mathrm{Tab}_k(\lambda, \cdot)\} . \tag{25}$$

The *cospin* $\tilde{s}(T)$ of a k-ribbon tableau T of shape λ is then

$$\tilde{s}(T) = s_k^*(\lambda) - s(T) . \tag{26}$$

Although $s(T)$ can be a half-integer, it is easily seen that $\tilde{s}(T)$ is always an integer. Also, there is one important case where $s(T)$ is an integer. This is when the shape λ of T is of the form $k\mu = (k\mu_1, k\mu_2, \ldots, k\mu_r)$. In this case, the partitions constituting the k-quotient of λ are formed by parts of μ, grouped according to the class modulo k of their indices. More precisely, $\lambda^i = \{\mu_r \mid r \equiv -i \bmod k\}$

We can now define

$$\tilde{G}_\lambda^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(\lambda, \cdot)} q^{\tilde{s}(T)} x^T \tag{27}$$

$$\tilde{H}_\mu^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(k\mu, \cdot)} q^{\tilde{s}(T)} x^T = \tilde{G}_{k\mu}^{(k)}(X;q) \tag{28}$$

$$H_\mu^{(k)}(X;q) = \sum_{T \in \mathrm{Tab}_k(k\mu, \cdot)} q^{s(T)} x^T = q^{s_k^*(k\mu)} \tilde{H}_\mu^{(k)}(X;1/q) . \tag{29}$$

Theorem VI.1 (symmetry) *$\tilde{G}_\lambda^{(k)}$, $\tilde{H}_\mu^{(k)}$ and $H_\mu^{(k)}$ are symmetric functions of $X = \{x_1, x_2, \ldots\}$.*

This property follows from Corollary V.6. Indeed, the commutation relation $[V_i, V_j] = 0$ proves that if α is a rearrangement of a partition μ,

$$V_{\alpha_r} \cdots V_{\alpha_1} |0\rangle = V_{\mu_r} \cdots V_{\mu_1} |0\rangle = \psi_q^k(h_\mu)$$

which shows that for any partition λ, the sets $\mathrm{Tab}_k(\lambda, \mu)$ and $\mathrm{Tab}_k(\lambda, \alpha)$ have the same spin polynomials.

The parameter k will be called the *level* of the corresponding symmetric functions.

Remark VI.2 If one defines the linear operator φ_q^k as the adjoint of ψ_q^k for the standard scalar product, the H-functions can also be defined by the equation

$$H_\lambda^{(k)}(X; q^{-2}) = \varphi_q^k(s_{k\lambda}) . \tag{30}$$

There is strong experimental evidence for the following conjectures.

Conjecture VI.3 (positivity) *The coefficients of $\tilde{G}_\lambda^{(k)}, \tilde{H}_\lambda^{(k)}$ and $H_\lambda^{(k)}$ on the basis of Schur functions are polynomials in q with nonnegative integer coefficients.*

Conjecture VI.4 (monotonicity) $H_\mu^{(k+1)} - H_\mu^{(k)}$ *is positive on the Schur basis, that is, the coefficients are in $\mathbb{N}[q]$.*

Conjecture VI.5 (plethysm) *When $\mu = \nu^k$, for ζ a primitive k-th root of unity,*

$$H_{\nu^k}^{(k)}(\zeta) = (-1)^{(k-1)|\nu|}\ p_k \circ s_\nu$$

and more generally, when $d|k$ and ζ is a primitive d-th root of unity,

$$H_{\nu^k}^{(k)}(\zeta) = (-1)^{(d-1)|\nu|k/d} p_d^{k/d} \circ s_\nu\ .$$

Equivalently,

$$H_{\nu^k}^{(k)}(q) \bmod 1 - q^k = \sum_{i=0}^{k-1} q^k \ell_k^{(i)} \circ s_\nu$$

The following statements will be proved in the forthcoming sections.

Theorem VI.6 *For $k \geq \ell(\mu)$, $H_\mu^{(k)}$ is equal to the Hall-Littlewood function Q'_μ.*

Theorem VI.7 *The difference $Q'_\mu - H_\mu^{(2)}$ is nonnegative on the Schur basis.*

Taking into account the results of [8, 9] and [27], this is sufficient to establish the conjectures for $k = 2$ and $k \geq \ell(\mu)$.

Example VI.8 (i) The 3-quotient of $\lambda = (3, 3, 3, 2, 1)$ is $((1), (1, 1), (1))$ and

$$\begin{aligned}\tilde{G}_{33321}(q) &= m_{31} + (1+q)m_{22} + (2+2q+q^2)m_{211} \\ &\quad + (3+5q+3q^2+q^3)m_{1111} \\ &= s_{31} + qs_{22} + (q+q^2)s_{211} + q^3 s_{1111}\end{aligned}$$

is a q-analogue of the product

$$s_1 s_{11} s_1 = s_{31} + s_{22} + 2s_{211} + s_{1111}\ .$$

(ii) The H-functions associated to the partition $\lambda = (3, 2, 1, 1)$ are

$$\begin{aligned}
H^{(2)}_{3211} &= s_{3211} + q\, s_{322} + q\, s_{331} + q\, s_{4111} \\
&\quad + (q + q^2)\, s_{421} + q^2\, s_{43} + q^2\, s_{511} + q^3\, s_{52} \\
H^{(3)}_{3211} &= s_{3211} + q\, s_{322} + (q + q^2)\, s_{331} + q\, s_{4111} \\
&\quad + (q + 2q^2)\, s_{421} + (q^2 + q^3)\, s_{43} + (q^2 + q^3)\, s_{511} \\
&\quad + 2q^3\, s_{52} + q^4\, s_{61} \\
H^{(4)}_{3211} &= s_{3211} + q\, s_{322} + (q + q^2)\, s_{331} + q\, s_{4111} \\
&\quad + (q + 2q^2 + q^3)\, s_{421} + (q^2 + q^3 + q^4)\, s_{43} + (q^2 + q^3 + q^4)\, s_{511} \\
&\quad + (2q^3 + q^4 + q^5)\, s_{52} + (q^4 + q^5 + q^6)\, s_{61} + q^7\, s_7 \\
&= Q'_{3211}
\end{aligned}$$

and we see that $s_{3211} < H^{(2)}_{3211} < H^{(3)}_{3211} < H^{(4)}_{3211} = Q'_{3211}$.

(iii) The plethysms of s_{21} with the cyclic characters $\ell_3^{(i)}$ are given by the reduction modulo $1 - q^3$ of

$$\begin{aligned}
H^{(3)}_{222111} &= q^9 s_{63} + (q+1)q^7 s_{621} + q^6 s_{6111} + (q+1)q^7 s_{54} + (q^3 + 2q^2 + 2q + 1)q^5 s_{531} \\
&\quad + (q^2 + 2q + 1)q^5 s_{522} + (q^3 + 2q^2 + 2q + 1)q^4 s_{5211} + (q+1)q^4 s_{51111} \\
&\quad + (q^2 + 2q + 1)q^5 s_{441} + (q^3 + 2q^2 + 3q + 2)q^4 s_{432} + (2q^3 + 3q^2 + 3q + 1)q^3 s_{4311} \\
&\quad + (q^3 + 3q^2 + 3q + 2)q^3 s_{4221} + (q^3 + 2q^2 + 2q + 1)q^2 s_{42111} + q^3 s_{411111} + (q^3 + 1)q^3 s_{333} \\
&\quad + (2q^3 + 3q^2 + 2q + 1)q^2 s_{3321} + (q^2 + 2q + 1)q^2 s_{33111} + (q^2 + 2q + 1)q^2 s_{3222} \\
&\quad + (q^3 + 2q^2 + 2q + 1)q s_{32211} + (q+1)q s_{321111} + (q+1)q s_{22221} + s_{222111}
\end{aligned}$$

Indeed,

$$\begin{aligned}
H^{(3)}_{222111} \bmod 1 - q^3 &= (2s_{5211} + s_{22221} + s_{321111} + 3s_{4311} \\
&\quad + 2s_{32211} + s_{522} + 3s_{432} + 3s_{3321} + s_{33111} + s_{3222} + s_{51111} \\
&\quad + 3s_{4221} + 2s_{531} + 2s_{42111} + s_{54} + s_{621} + s_{441})q^2 \\
&\quad + (2s_{5211} + s_{22221} + s_{321111} + 3s_{4311} + 2s_{32211} + s_{522} \\
&\quad + 3s_{432} + 3s_{3321} + s_{33111} + s_{3222} + s_{51111} \\
&\quad + 3s_{4221} + 2s_{531} + 2s_{42111} + s_{54} + s_{621} + s_{441})q \\
&\quad + 2s_{33111} + s_{63} + s_{6111} + 2s_{531} + 2s_{522} + 2s_{5211} + 2s_{441} \\
&\quad + 2s_{432} + 3s_{4311} + 3s_{4221} + 2s_{42111} + s_{411111} + 2s_{333}
\end{aligned}$$

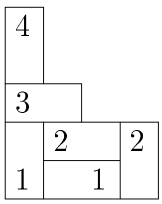
$$
\begin{aligned}
&+2s_{3321}+2s_{3222}+s_{222111}+2s_{32211}\\
=\ & q^2\ell_3^{(2)}\circ s_{21}+q\ell_3^{(1)}\circ s_{21}+\ell_3^{(0)}\circ s_{21}\ .
\end{aligned}
$$

VII The case of dominoes

For $k=2$, the conjectures can be established by means of the combinatorial constructions of [27] and [28]. In this case, conjectures VI.1, VI.3 and VI.5 follow directly from the results of [27], and the only point remaining to be proved is Theorem VI.7.

The important special feature of domino tableaux is that there exits a natural notion of *Yamanouchi domino tableau.* These tableaux correspond to highest weight vectors in tensor products of two irreducible (polynomial) representations of $GL(n,\mathbb{C})$, in the same way as ordinary Yamanouchi tableaux are the natural labels for highest weight vectors of irreducible representations.

The *column reading* of a domino tableau T is the word obtained by reading the successive columns of T from top to bottom and left to right. Horizontal dominoes, which belong to two succesive columns i and $i+1$ are read only once, when reading column i. For example, the column reading of the domino tableau



is $\mathrm{col}\,(T)=431212$.

A *Yamanouchi word* is a word $w=x_1x_2\cdots x_n$ such that each right factor $v=x_i\cdots x_n$ of w satisfies $|v|_j\ge|v|_{j+1}$ for each j, where $|v|_j$ denotes the number of occurences of the letter j in v.

A *Yamanouchi domino tableau* is a domino tableau whose column reading is a Yamanouchi word. We denote by $\mathrm{Yam}_2(\lambda,\mu)$ the set of Yamanouchi domino tableaux of shape λ and weight μ.

It follows from the results of [27], Section 7, that the Schur expansions of the H-functions of level 2 are given by

$$
H_\lambda^{(2)}=\sum_\mu\sum_{T\in\mathrm{Yam}_2(2\lambda,\mu)}q^{s(T)}s_\mu\ . \qquad (31)
$$

On the other hand,

$$Q'_\lambda = \sum_\mu \sum_{\mathbf{t}\in \mathrm{Tab}\,(\mu,\lambda)} q^{c(\mathbf{t})} s_\mu \ . \tag{32}$$

To prove Theorem VI.7, it is thus sufficient to exhibit an injection

$$\eta : \quad \mathrm{Yam}_2(2\lambda,\mu) \longrightarrow \mathrm{Tab}\,(\mu,\lambda)$$

satisfying

$$c(\eta(T)) = s(T) \ .$$

To achieve this, we shall make use of a bijection described in [45], and extended in [28], which sends a domino tableau $T \in \mathrm{Tab}_{\,2}(\alpha,\mu)$ over the alphabet $X = \{1,\ldots,n\}$, to an ordinary tableau $\mathbf{t} = \phi(T) \in \mathrm{Tab}\,(\alpha,\bar\mu\mu)$ over the alphabet $\bar X \cup X = \{\bar n < \ldots < \bar 1 < 1 < \ldots < n\}$. The weight $\bar\mu\mu$ means that $\mathbf{t}$ contains μ_i occurences of i and of $\bar\imath$. The tableau $\phi(T)$ is invariant under Schützenberger's involution Ω, and the spin of T can be recovered from $\mathbf{t}$ by the following procedure [46].

Let $\alpha = 2\lambda$, $\beta = \alpha'$, $\beta_{\mathrm{odd}} = (\beta_1,\beta_3,\ldots)$ and $\beta_{\mathrm{even}} = (\beta_2,\beta_4,\ldots)$. Then, there exists a unique factorisation $\mathbf{t} = \tau_1\tau_2$ in the plactic monoid $\mathrm{Pl}\,(X\cup\bar X)$, such that τ_1 is a contretableau of shape $\alpha^1 = (\beta_{\mathrm{even}})'$ and τ_2 is a tableau of shape $\alpha^2 = (\beta_{\mathrm{odd}})'$. The spin of $T = \phi^{-1}(\mathbf{t})$ is then equal to the number $|\tau_1|_+$ of positive letters in τ_1, which is also equal to the number $|\tau_2|_-$ of negative letters in τ_2. Moreover, $\tau_2 = \Omega(\tau_1)$.

Example VII.1 With the following tableau T of shape $(4,4,2,2)$, one finds

$$T = \begin{array}{|c|c|cc|} \hline & 3 & & \\ 2 & & & \\ & 1 & & 2 \\ 1 & & 1 & \\ \hline \end{array} \qquad \xrightarrow{\quad\phi\quad} \qquad \mathbf{t} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & & \\ \bar 1 & 2 & & \\ \bar 2 & \bar 1 & 1 & 2 \\ \bar 3 & \bar 2 & \bar 1 & 1 \\ \hline \end{array}$$

By *jeu de taquin*, we find that in the plactic monoid

$$\mathbf{t} \ = \ \begin{array}{cc} \bar 1 & 1 \\ \bar 2 & \bar 1 \\ & \bar 2 \\ & \bar 3 \end{array} \ \begin{array}{cc} 3 & \\ 2 & \\ 1 & 2 \\ \bar 1 & 1 \end{array} \ = \ \tau_1\tau_2 \ .$$

The number of positive letters of τ_1 and the number of negative letters of τ_2 are both equal to 1, which is the spin of T.

This correspondence still works in the general case (α need not be of the form 2λ) and the invariant tableau associated to a domino tableau T admits a similar factorisation $\mathbf{t} = \tau_1\tau_2$, but in general $\tau_2 \neq \Omega(\tau_1)$ and the formula for the spin is $s(T) = \frac{1}{2}(|\tau_1|_+ + |\tau_2|_-)$.

The map $\eta : \mathrm{Yam}_2(2\lambda, \mu) \longrightarrow \mathrm{Tab}\,(\mu, \lambda)$ is given by the following algorithm: to compute $\eta(T)$,

1. construct the invariant tableau $\mathbf{t} = \phi(T)$
2. apply the *jeu de taquin* algorithm to $\mathbf{t}$ to obtain the plactic factorization $\mathbf{t} = \tau_1\tau_2$, and keep only τ_2.
3. Apply the evacuation algorithm to the *negative* letters of τ_2, keeping track of the successive stages. After all the negative letters have been evacuated, one is left with a Yamanouchi tableau τ in positive letters.
4. Complete the tableau τ to obtain the tableau $\mathbf{t}' = \eta(T)$ using the following rule: suppose that at some stage of the evacuation, the box of τ_2 which disappeared after the elimination of $\bar{i}$ was in row j of τ_2. Then add a box numbered j to row i of τ.

Theorem VII.2 *The above algorithm defines an injection*

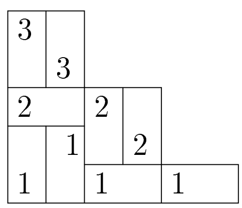
$$\eta : \mathrm{Yam}_2(2\lambda, \mu) \longrightarrow \mathrm{Tab}\,(\mu, \lambda)$$

satisfying $c \circ \eta = s$.

The proof follows from the constructions of [46].

Corollary VII.3 $H_\lambda^{(2)} \leq Q'_\lambda$

Example VII.4 Let T be the following Yamanouchi domino tableau, which is of shape $2\lambda = (6, 4, 4, 2, 2)$, of weight $\mu = (4, 3, 2)$ and has spin $s(T) = 3$

$T =$ 

Then,

$$\phi(T) \;=\; \begin{array}{llllll} 3 & 3 & & & & \\ 1 & 2 & & & & \\ \bar{1} & \bar{1} & 2 & 2 & & \\ \bar{2} & \bar{2} & \bar{1} & 1 & & \\ \bar{3} & \bar{3} & \bar{2} & \bar{1} & 1 & 1 \end{array} \qquad \equiv \qquad \begin{array}{lll} \bar{1} & 1 & 3 \\ & \bar{1} & 2 \\ & \bar{2} & \bar{1} \\ & & \bar{2} \\ & & \bar{3} \end{array} \; \begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & \\ \bar{2} & 1 & \\ \bar{3} & \bar{1} & 1 \end{array}$$

and the succesive stages of the evacuation process are

$$\begin{array}{lll} 3 & & \\ 2 & & \\ 1 & 2 & \\ \bar{2} & 1 & \\ \bar{3} & \bar{1} & 1 \end{array} \;\underset{\substack{\bar{i}=\bar{3}\\ j=5}}{\longrightarrow}\; \begin{array}{lll} \times & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & \\ \bar{2} & \bar{1} & 1 \end{array} \;\underset{\substack{\bar{i}=\bar{2}\\ j=3}}{\longrightarrow}\; \begin{array}{lll} 3 & & \\ 2 & \times & \\ 1 & 2 & \\ \bar{1} & 1 & 1 \end{array} \;\underset{\substack{\bar{i}=\bar{1}\\ j=4}}{\longrightarrow}\; \begin{array}{lll} \times & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array}$$

so that we find

$$\eta(T) = \begin{array}{llll} 3 & 5 & & \\ 2 & 2 & 3 & \\ 1 & 1 & 1 & 4 \end{array}$$

a tableau of shape $\mu = (4,3,2)$, weight $\lambda = (3,2,2,1,1)$ and charge $c(\mathbf{t}') = 3$.

VIII The stable case

As the Q'-functions are known to verify all the conjectured properties of H-functions, the stable case of the conjectures will be a consequence of Theorem VI.6. This result will be proved by means of Shimomura's cell decompositions of unipotent varieties.

A Unipotent varieties

Let $u \in GL(n, \mathbb{C})$ be a unipotent element,and let $\mathcal{F}_\nu^u[\mathbb{C}]$ be the variety of ν-flags of $\mathbb{C}^n$ which are fixed by u.

It has been shown by N. Shimomura ([18], see also [12]) that the variety $\mathcal{F}_\nu^u[\mathbb{C}]$ admits a cell decomposition, involving only cells of even real dimensions. More precisely, this cell decomposition is a partition in locally closed subvarieties, each being algebraically isomorphic to an affine space. Thus, the odd-dimensional homology groups are zero, and if

$$\Pi_{\nu\mu}(t^2) = \sum_i t^{2i} \dim H_{2i}(\mathcal{F}_\nu^u, \mathbb{Z})$$

is the Poincaré polynomial of $\mathcal{F}_\nu^u[\mathbb{C}]$, one has $|\mathcal{F}_\nu^u[\mathbb{F}_q]| = \Pi_{\nu\mu}(q)$. But, by Section 2, this is also equal to $\tilde{G}_{\nu\mu}(q)$, and as this is true for an infinite set of values of q, one has

$\Pi_{\nu\mu}(z) = \tilde{G}_{\nu\mu}(z)$ as polynomials. That is, the coefficient of $\tilde{Q}'_\mu$ on the monomial function m_ν is the Poincaré polynomial of $\mathcal{F}^u_\nu$, for a unipotent u of type μ.

Writing

$$\tilde{Q}'_\mu = \sum_{\lambda,\nu} \tilde{K}_{\lambda\mu(q)} K_{\lambda\nu}\, m_\nu \ , \tag{33}$$

one sees that

$$\tilde{G}_{\nu\mu}(q) = \sum_{(\mathbf{t}_1,\mathbf{t}_2)\in \mathrm{Tab}\,(\lambda,\mu)\times \mathrm{Tab}\,(\nu,\mu)} q^{\tilde{c}(\mathbf{t}_1)} \ . \tag{34}$$

Knuth's extension of the Robinson-Schensted correspondence [47] is a bijection between the set

$$\coprod_\lambda \mathrm{Tab}\,(\lambda,\mu) \times \mathrm{Tab}\,(\lambda,\nu)$$

of pairs of tableaux with the same shape, and the double coset space $\mathfrak{S}_\mu \backslash \mathfrak{S}_n / \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$ modulo two parabolic subgroups. Double cosets can be encoded by two-line arrays, integer matrices with prescribed row and column sums, or by *tabloids.*

Let ν and μ be arbitrary compositions of the same integer n. A μ-tabloid of shape ν is a filling of the diagram of boxes with row lengths $\nu_1, \nu_2, \dots, \nu_r$, the lowest row being numbered 1 (French convention for tableaux), such that the number i occurs μ_i times, and such that each row is nondecreasing. For example,

3		
1	1	1
1	1	3
2	3	

is a $(5,1,3)$-tabloid of shape $(2,3,3,1)$.

We denote by $L(\nu,\mu)$ the set of tabloids of shape ν and weight μ. A tabloid will be identified with the word obtained by reading it from left to right and top to bottom. Recall that two words u, v are plactically equivalent ($u \equiv v$) if and only if they are mapped to the same tableau by Schensted's algorithm [20]. Then, writing

$$\tilde{G}_{\nu\,\mu}(q) = \langle\, h_\nu\,,\, \tilde{Q}'_\mu(q)\,\rangle = \sum_\lambda \langle\, h_\nu\,,\, s_\lambda\,\rangle\langle\, s_\lambda\,,\, \tilde{Q}'_\mu(q)\,\rangle\,,$$

interpreting $\langle\, h_\nu\,,\, s_\lambda\,\rangle$ as the number of tabloids of shape ν which, as words, are plactically equivalent to a tableau of shape λ, and using the fact that cocharge is compatible with the plactic congruence (*i.e.* $w \equiv w'$ implies $\tilde{c}(w) = \tilde{c}(w')$ [20]) we see that

$$\tilde{G}_{\nu\mu}(q) = \sum_{T\in L(\nu,\mu)} q^{\tilde{c}(T)} \ . \tag{35}$$

Example VIII.1 To compute $\tilde{G}_{42,321}(q)$ one lists the elements of $L((4,2),(3,2,1))$, which are

$$\begin{array}{|c|c|cc}\hline 2&3&\multicolumn{2}{c}{}\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{2}\\ \hline\end{array}\qquad \begin{array}{|c|c|cc}\hline 2&2&\multicolumn{2}{c}{}\\ \hline 1&1&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{3}\\ \hline\end{array}\qquad \begin{array}{|c|c|cc}\hline 1&3&\multicolumn{2}{c}{}\\ \hline 1&1&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{2}\\ \hline\end{array}\qquad \begin{array}{|c|c|cc}\hline 1&2&\multicolumn{2}{c}{}\\ \hline 1&1&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}\\ \hline\end{array}\qquad \begin{array}{|c|c|cc}\hline 1&1&\multicolumn{2}{c}{}\\ \hline 1&2&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}\\ \hline\end{array}$$

Reading them as prescribed, we obtain the words

$$231112 \qquad 221113 \qquad 131122 \qquad 121123 \qquad 111223$$

whose respective charges are $2,1,3,2,4$. The cocharge polynomial is thus $\tilde{G}_{42,321}(q) = 1 + q + 2q^2 + q^3$.

In Shimomura's decomposition of the fixed point variety $\mathcal{F}_\mu^u$ of a unipotent of type ν, the cells are indexed by tabloids of shape ν and weight μ. The dimension $d(T)$ of the cell c_T indexed by $T \in L(\nu,\mu)$ is computed by an algorithm described below, and gives another combinatorial interpretation of the polynomial $\tilde{G}_{\mu\nu}(q)$, exchanging the rôles of shape and weight:

$$\tilde{G}_{\mu\nu}(q) = \sum_{T\in L(\mu,\nu)} q^{\tilde{c}(T)} = \sum_{T\in L(\nu,\mu)} q^{d(T)} \ . \tag{36}$$

The dimensions $d(T)$ are given by the following algorithm.

1. If $T \in L(\nu,(n))$ then $d(T) = 0$;

2. If $\mu = (\mu_1,\mu_2)$ has exactly two parts, and $T \in L(\nu,\mu)$, then $d(T)$ is computed as follows. A box α of T is said to be *special* if it contains the rightmost 1 of its row. For a box β of T, put $d(\beta)$=0 if β does not contain a 2, and if β contains a 2, set $d(\beta)$ equal to the number of nonspecial 1's lying in the column of β, plus the number of special 1's lying in the same column, but in a lower position. Then
$$d(T) = \sum_\beta d(\beta) \ .$$

3. Let $\mu = (\mu_1,\ldots,\mu_k)$ and $\mu^* = (\mu_1,\ldots,\mu_{k-1})$. For $T \in L(\nu,\mu)$, let T_1 be the tabloid obtained by changing the entries k into 2 and all the other ones into 1. Let T_2 be the tabloid obtained by erasing all the entries k, *and rearranging the rows in the appropriate order.* Then,
$$d(T) = d(T_1) + d(T_2) \ . \tag{37}$$

Example VIII.2 With $T = \begin{array}{|c|c|c|}\cline{1-2} 1 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array} \in L(332, 4211)$, one has

$$T_1 := \begin{array}{|c|c|c|}\cline{1-2} 1 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & \mathbf{1} \\ \hline 1 & 1 & \mathbf{1} \\ \hline \end{array} \qquad T_2 = \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array} \qquad T_{21} = \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \hline 1 & \mathbf{1} & 2 \\ \hline 1 & 1 & \mathbf{1} \\ \hline \end{array} \qquad T_{22} = \begin{array}{|c|c|c|}\cline{1-1} 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \mathbf{1} & 2 & \multicolumn{1}{c}{} \\ \hline 1 & \mathbf{1} & 2 \\ \hline \end{array}$$

where the special entries are printed in boldtype. Thus, $d(T) = t(T_1) + d(T_2) = 2 + d(T_{21}) + d(T_{22}) = 4$.

We shall need a variant of this construction, in which the shape ν is allowed to be an arbitrary composition, and where in step 3, the rearranging of the rows is supressed. Such a variant has already been used by Terada [19] in the case of complete flags.

That is, we associate to a tabloid $T \in L(\nu, \mu)$ an integer $e(T)$, defined by

1. For $T \in L(\nu, (n))$, $e(T) = d(T) = 0$;
2. For $T \in L(\nu, (\mu_1, \mu_2))$, $e(T) = d(T)$;
3. Otherwise $e(T) = e(T_1) + e(T_2)$ where T_1 is defined as above, but this time T_2 is obtained from T by erasing the entries k, without reordering.

Lemma VIII.3 *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition, and let $\nu = \lambda \cdot \sigma = (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(r)})$, $\sigma \in \mathfrak{S}_r$. Then, the distribution of e on $L(\nu, \mu)$ is the same as the distribution of d on $L(\lambda, \mu)$. That is,*

$$D_{\lambda\mu}(q) = \sum_{T \in L(\lambda,\mu)} q^{d(T)} = E_{\nu\mu}(q) = \sum_{T \in L(\nu,\mu)} q^{e(T)} \ .$$

In particular, $D_{\lambda\mu}(q) = E_{\lambda\mu}(q)$.

Proof — This could be proved by repeating word for word the geometric argument of [18]. We give here a short combinatorial argument. As the two statistics coincide on tabloids whose shape is a partition and whose weight has at most two parts, the only thing to prove, thanks to the recurrence formula, is that e has the same distribution on $L(\beta, (\mu_1, \mu_2))$ as on $L(\alpha, (\mu_1, \mu_2))$ when β is a permutation of α. The symmetric group being generated by the elementary transpositions $\sigma_i = (i, i+1)$, one may assume that $\beta = \alpha\sigma_i$. We define the image $T\sigma_i$ of a tabloid $T \in L(\alpha, (\mu_1, \mu_2))$ by distinguishing among the following configurations for rows i and $i+1$:

1.
$$\begin{array}{ccccc} x_1 & \dots & x_k & 2 & 2^r \\ 1 & \dots & 1 & \mathbf{1} & 2^s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{ccccc} x_1 & \dots & x_k & 2 & 2^s \\ 1 & \dots & 1 & \mathbf{1} & 2^r \end{array}$$

2.
$$\begin{array}{ccccc} 1 & \dots & 1 & \mathbf{1} & 2^r \\ x_1 & \dots & x_k & 2 & 2^s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{ccccc} 1 & \dots & 1 & \mathbf{1} & 2^s \\ x_1 & \dots & x_k & 2 & 2^r \end{array}$$

3. In all other cases, the two rows are exchanged:

$$\begin{array}{cccc} x_1 & \dots & x_r & \\ y_1 & \dots & & y_s \end{array} \quad \xrightarrow{\sigma_i} \quad \begin{array}{cccc} y_1 & \dots & & y_s \\ x_1 & \dots & x_r & \end{array}$$

From this definition, it is clear that $e(T\sigma_i) = e(T)$. Moreover, it is not difficult to check that this defines an e-preserving action of the symmetric group $\mathfrak{S}_m$ on the set of μ-tabloids with m rows, such that $L(\alpha, \mu)\sigma = L(\alpha\sigma, \mu)$ (the only point needing a verification is the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$).

Thus, for a partition λ and a two-part weight $\mu = (\mu_1, \mu_2)$, d and e coincide on $L(\lambda, \mu)$, and for $\sigma \in \mathfrak{S}_m$, $E_{\lambda\sigma,\mu}(q) = D_{\lambda\mu}(q)$. Now, by induction, for $\mu = (\mu_1, \dots, \mu_k)$,

$$D_{\lambda\mu}(q) = \sum_{T \in L(\lambda,\mu)} q^{d(T_1)} q^{d(T_2)}$$

$$= \sum_{\bar\lambda = \text{shape}\,(T_1)} q^{d(T_1)} D_{\bar\lambda,\mu^*}(q) = \sum_{\bar\lambda = \text{shape}\,(T_1)} e^{e(T_1)} E_{\bar\lambda,\mu^*}(q) = E_{\lambda\mu}(q) \ .$$

□

Example VIII.4 Take $\lambda = (3, 2, 1)$, $\mu = (4, 2)$ and $\nu = \lambda\sigma_1\sigma_2 = (3, 1, 2)$. The μ-tabloids of shape λ are

$$\begin{array}{cccccc} T & \begin{array}{lll} 2 & & \\ 1 & 2 & \\ 1 & 1 & 1 \end{array} & \begin{array}{lll} 2 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{array} & \begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 2 & 2 \end{array} & \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} & \begin{array}{lll} 1 & & \\ 1 & 2 & \\ 1 & 1 & 2 \end{array} \\ d(T) & 3 & 2 & 0 & 2 & 1 \end{array}$$

The ν-tabloids of shape λ are

$$\begin{array}{cccccc} T & \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 2 & 2 & \end{array} & \begin{array}{lll} 1 & 1 & 1 \\ 2 & & \\ 1 & 2 & \end{array} & \begin{array}{lll} 1 & 1 & 2 \\ 1 & & \\ 1 & 2 & \end{array} & \begin{array}{lll} 1 & 1 & 2 \\ 2 & & \\ 1 & 1 & \end{array} & \begin{array}{lll} 1 & 2 & 2 \\ 1 & & \\ 1 & 1 & \end{array} \\ e(T) & 2 & 3 & 0 & 2 & 1 \end{array}$$

Thus, $D_{\lambda\mu}(q) = E_{\nu\mu}(q) = 1 + q + 2q^2 + q^3 = \tilde{G}_{\mu\lambda}(q)$. The tabloids contributing a term q^2 are paired in the following way:

$$\begin{array}{lll} 2 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{array} \longrightarrow \begin{array}{lll} 1 & 1 & 2 \\ 2 & & \\ 1 & 1 & \end{array} \qquad \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \longrightarrow \begin{array}{lll} 1 & 1 & 1 \\ 1 & & \\ 2 & 2 & \end{array}$$

Remark VIII.5 The only property that we shall need in the sequel is the equality $D_{\lambda\mu}(q) = E_{\lambda\mu}(q)$. However, it is possible to be more explicit by constructing a bijection exchanging d and e. The above action of $\mathfrak{S}_m$ can be extended to tabloids with arbitrary weight, still preserving e. Suppose for example that we want to apply σ_i to a tabloid T whose restriction to rows $i, i+1$ is

1	1	2	3	7	7	9			
1	1	1	2	6	6	6	8	8	9

One first determines the positions of the greatest entries, which are the 9's, in $T\sigma_i$. Starting with an empty diagram of the permuted shape $(10, 7)$, one constructs T_1 as above by converting all the entries 9 of T into 2 and the remaining ones into 1. Then we apply σ_i to T_1, and the positions of the 2 in $T_1\sigma_i$ give the positions of the 9 in $T\sigma_i$. Then, the entries 9 are removed from T ad the procedure is iterated until one reaches a tabloid whose rows i and $i+1$ are of equal lengths. This tabloid is then copied (without permutation) in the remaining part of the result. On the example, this gives

Figure 7

B Labelling of cells by ribbon tableaux

A tabloid $\mathbf{t}$ of shape $\nu = (\nu_1, \ldots, \nu_k)$ can be identified with a k-tuple $(w_1, \ldots, w_k)$ of words, w_i being a row tableau of length ν_i. The Stanton-White correspondence Ψ associates to such a k-tuple of tableaux a unique k-ribbon tableau with empty k-core $T = \Psi(\mathbf{t})$ (*cf.* Sect IV). Thus, the cells of a unipotent variety $\mathcal{F}_\mu^u$ (where u is of type ν) are labelled by k-ribbon tableaux of a special kind. The following theorem, which implies the stable case of the conjectures, shows that this labelling is natural from a geometrical point of view.

Theorem VIII.6 *The Stanton-White correspondence Ψ sends a tabloid $\mathbf{t} \in L(\nu, \mu)$ onto a ribbon tableau $T = \Psi(\mathbf{t})$ whose cospin is equal to the dimension of the cell $c_\mathbf{t}$ of $\mathcal{F}_\mu^u$ labelled by $\mathbf{t}$, when one uses the modified indexation for which the dimension of $c_\mathbf{t}$ is $e(\mathbf{t})$. That is,*

$$\tilde{s}(\Psi(\mathbf{t})) = e(\mathbf{t}) \ .$$

Example VIII.7 Let $\mathbf{t} \in L((2,3,2,1), (2,3,1,1,1))$ be the following tabloid (the number under a letter y is the number of e-inversions of the form (y, x)):

$$\mathbf{t} \quad = \quad \left(\begin{array}{cc} \boxed{2}\boxed{3} \\ 1\ \ 1 \end{array} \ , \ \begin{array}{c} \boxed{1}\boxed{1}\boxed{2} \\ 0\ \ 0\ \ 0 \end{array} \ , \ \begin{array}{c} \boxed{4}\boxed{5} \\ 3\ \ 1 \end{array} \ , \ \begin{array}{c} \boxed{2} \\ 1 \end{array} \right)$$

so that $e(\mathbf{t}) = 7$. Its image under the SW-correspondence is the 4-ribbon tableau

Figure 8

whose cospin is equal to 7.

Proof — We shall first observe, following [19], that the e-statistic can be given a non-recursive definition, as a kind of inversion number. Let $\mathbf{t} = (w_1, \ldots, w_k)$ be a tabloid, identified with a k-tuple of row tableaux. Let y be the r-th letter of w_i and x be the s-th letter of w_j, and suppose that $x < y$. Then, the pair (y, x) is said to be a *generalized inversion* if either

(a) $i < j$ and $s = r$,

or

(b) $i > j$ and $s = r + 1$.

Then $e(\mathbf{t})$ is equal to the number of inversions (y, x) in $\mathbf{t}$.

Let us now show that the cospin of the k-ribbon tableau $\Psi(\mathbf{t})$ is also equal to the number of generalized inversions of $\mathbf{t}$. To do this, we shall make use of a combinatorial description different from that of Section IV.

As explained in Section V, partitions λ are the natural labels of the Fock space basis in its bosonic realization, while strictly decreasing sequences form an appropriate set of indices in the fermionic realization. At the combinatorial level the boson-fermion correspondence becomes

$$\lambda = (\lambda_i)_{i \geq 1} \quad \longrightarrow \quad \beta = (\beta_i)_{i \geq 1} = (\lambda_i - i + 1)_{i \geq 1} .$$

Here we regard both λ and β as infinite with the tail conditions $\lambda_i = 0$ and $\beta_i = -i + 1$ for i large enough. Graphically, the sequence β can be pictured as a set of beads on an infinite runner. For example, $\beta = (5, 4, -1, -3, -4, -5, \ldots)$ is represented by

Figure 9

When the action of $\widehat{\mathfrak{sl}}_k$ on the Fock space is considered, it is convenient to display the β-numbers on k different runners, thus getting a k-abacus in the terminology of James [36]. For instance, when $k = 3$, the same β-set as above becomes

Figure 10

It is easily checked that moving a bead one step to the right (*resp.* to the left) on the ith runner amounts to remove (*resp.* to add) a k-ribbon of type i to the Young diagram of λ. Hence, λ has empty k-core if and only if by moving all the beads to the right as far as possible, one gets the k-abacus of the empty partition. Also, the k-quotient of λ is obtained by reading the partitions encoded in the successive runners of its abacus. In particular, if the k-quotient of λ is a sequence of row shapes $(nu\_1, \dots, \nu_k)$, and its k-core is empty, then the corresponding abacus has the following simple form:

Figure 11

where there are ν_i holes on the ith runner. The spin is also easily read on this representation, namely, moving the bead containing the β-number b one step to the left, that is replacing this β-number by $b+k$, will add a k-ribbon with spin $j/2$, where j is the number of β_i between b and $b+k$.

Consider now a tabloid $\mathbf{t} = (w_1, \cdots, w_k)$. The k-ribbon tableau $\Psi(\mathbf{t})$ is by definition (*cf.* Section IV) a chain of partitions $\alpha^0 \subset \alpha^1 \subset \cdots \alpha^r$, where α^i/α^{i-1} consists of the ribbons numbered i. If this chain is mapped to an abacus, and the holes corresponding to α^i/α^{i-1} are numbered i, then one gets

Figure 12

where w_j^1, ..., $w_j^{s_j}$ are the successive letters of the word w_j. Now taking into account the above description of the spin created by moving one bead to the left, we see that the variation of cospin coming from the addition of a letter $y = w_j^i$ is equal to the number of generalized inversions of the form (y, x). □

C Atoms

The H-functions indexed by columns can also be completely described in terms of Hall-Littlewood functions.

Proposition VIII.8 *Let $n = sk + r$ with $0 \le r < k$, and set $\lambda = ((s+1)^r, s^{k-r})$. Then,*

$$H_{(1^n)}^{(k)} = \omega\left(\tilde{Q}'_\lambda\right)$$

where ω is the involution $s_\lambda \mapsto s_{\lambda'}$.

The k-quotient of (k^n) is $(1^s,\dots,1^s,1^{s+1},\dots,1^{s+1})$, where the partition (1^s) is repeated $k-r$ times. Thus, a k-ribbon tableau is mapped by the Stanton-White correspondence to a k-tuple of columns, which can be interpreted as a tabloid, and the result follows again from Shimomura's decomposition.

The partitions arising in Proposition VIII.8 have the property that, if $\le$ denotes the natural order on partitions,

$$\alpha \le \lambda \qquad \Longleftrightarrow \qquad \ell(\alpha) \le \ell(\mu) \ .$$

There are canonical injections

$$\iota_{\alpha\beta} : \mathrm{Tab}\,(\,\cdot\,,\alpha) \longrightarrow \mathrm{Tab}\,(\,\cdot\,,\beta)$$

when $\alpha \le \beta$ (*cf.* [21, 14]). The *atom* $\mathcal{A}(\mu)$ is defined as the set of tableaux in $\mathrm{Tab}\,(\,\cdot\,,\mu)$ which are not in the image of any $\iota_{\alpha\mu}$. Define the symmetric functions (*cocharge atoms*)

$$\tilde{A}_\mu(X;q) = \sum_\lambda \left(\sum_{\mathbf{t}\in\mathcal{A}(\mu)\cap\mathrm{Tab}\,(\lambda,\mu)} q^{\tilde{c}(\mathbf{t})} \right) s_\lambda(X) \ . \qquad (38)$$

Proposition VIII.8 can then be rephrased as

$$H^{(k)}_{(1^n)} = \omega \left(\sum_{\ell(\mu)\le k} \tilde{A}_\mu \right) \ . \qquad (39)$$

It seems that the difference between the stable H-functions and the immediately lower level can also be described in terms of atoms. For $\ell(\lambda)=r$, set

$$\tilde{D}_\lambda(q) = \tilde{H}^{(r)}_\lambda - \tilde{H}^{(r-1)}_\lambda \ .$$

These functions seem to be sums of cocharge atoms over certain intervals in the lattice of partitions.

Conjecture VIII.9 *For any partition λ, there exists a partition $f(\lambda)$ such that*

$$\tilde{D}_\lambda = \sum_{\mu\le f(\lambda)} \tilde{A}_\mu \ .$$

Example VIII.10 In weight 6, the partition $f(\lambda)$ is given by the following table:

λ	(51)	(42)	(411)	(33)	(321)	(3111)	(222)	(2211)	(21111)	(111111)
$f(\lambda)$	(6)	(51)	(51)	(42)	(42)	(411)	(321)	(321)	(3111)	(21111)

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Figure 1

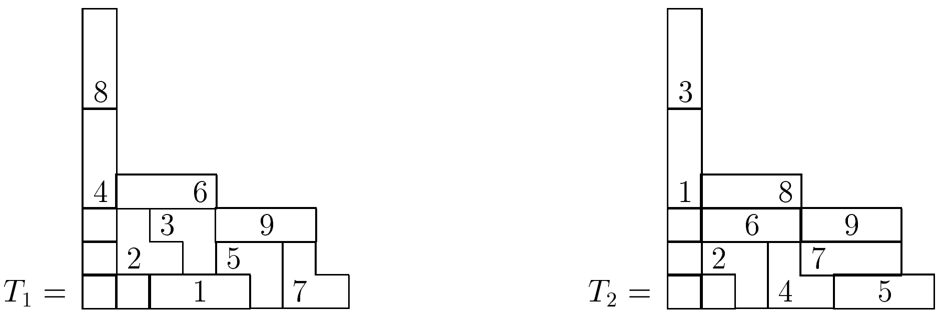


Figure 1:

Figure 2

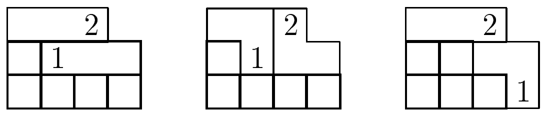
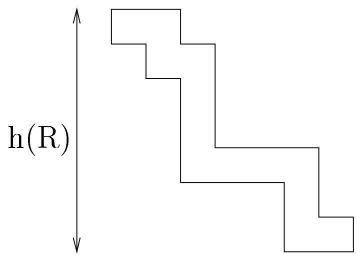


Figure 3



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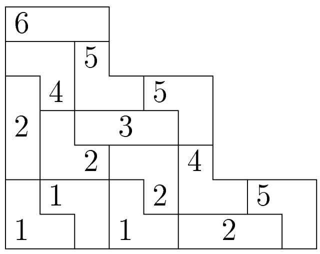

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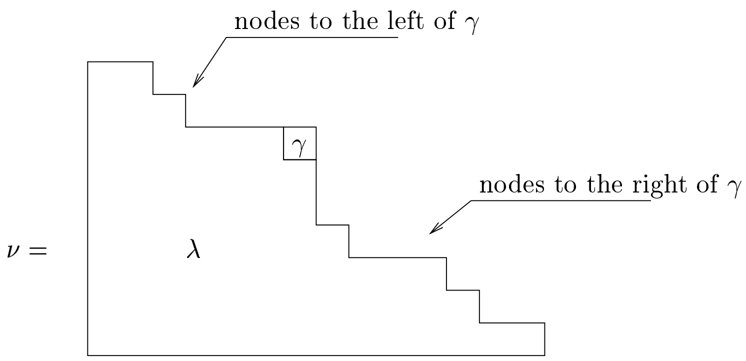
nodes to the left of γ
γ
nodes to the right of γ
$\nu =$
λ

Figure 6

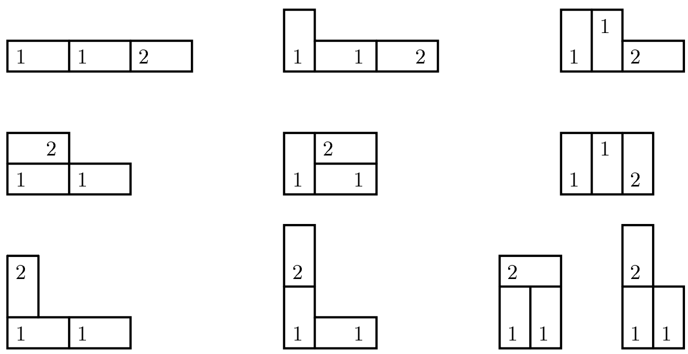

Figure 7

$$\begin{array}{llllllllll} 1&1&1&1&1&1&2&&& \\ 1&1&1&1&1&1&1&1&1&2 \end{array} \longrightarrow \begin{array}{llllllllll} 1&1&1&1&1&1&1&1&1&2 \\ 1&1&1&1&1&1&2&&& \end{array} \qquad \begin{array}{llllllllll} \square&\square&\square&\square&\square&\square&\square&\square&\square&9 \\ \square&\square&\square&\square&\square&\square&9&&& \end{array}$$

$$\begin{array}{lllllllll} 1&1&1&1&1&1&&& \\ 1&1&1&1&1&1&1&2&2 \end{array} \longrightarrow \begin{array}{lllllllll} 1&1&1&1&1&1&1&2&2 \\ 1&1&1&1&1&1&&& \end{array} \qquad \begin{array}{llllllllll} \square&\square&\square&\square&\square&\square&\square&8&8&9 \\ \square&\square&\square&\square&\square&\square&9&&& \end{array}$$

$$\begin{array}{lllllll} 1&1&1&1&2&2& \\ 1&1&1&1&1&1&1 \end{array} \longrightarrow \begin{array}{lllllll} 1&1&1&1&1&1&1 \\ 1&1&1&1&2&2& \end{array} \qquad \begin{array}{llllllllll} \square&\square&\square&\square&\square&\square&\square&8&8&9 \\ \square&\square&\square&\square&7&7&9&&& \end{array}$$

$$\begin{array}{lllllll} 1&1&1&1&&& \\ 1&1&1&1&2&2&2 \end{array} \longrightarrow \begin{array}{lllllll} 1&1&1&1&2&2&2 \\ 1&1&1&1&&& \end{array} \qquad \begin{array}{llllllllll} \square&\square&\square&\square&6&6&6&8&8&9 \\ \square&\square&\square&\square&7&7&9&&& \end{array}$$

$$\begin{array}{llll} 1&1&1&2 \\ 1&1&1&1 \end{array} \longrightarrow \begin{array}{llll} 1&1&1&2 \\ 1&1&1&1 \end{array} \qquad \begin{array}{llllllllll} 1&1&2&3&6&6&6&8&8&9 \\ 1&1&1&2&7&7&9&&& \end{array}$$

Figure 8

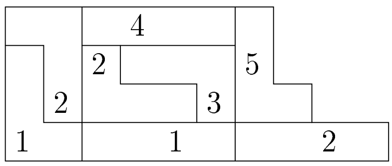

Figure 9

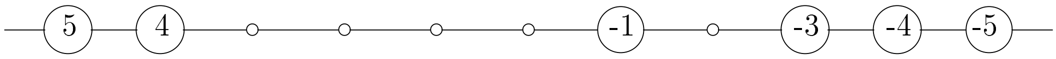

Figure 10

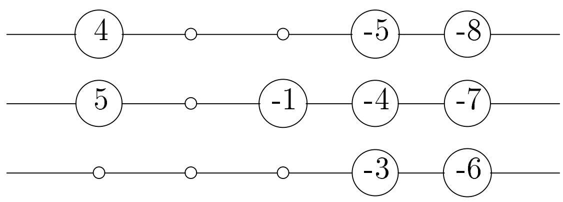

Figure 11

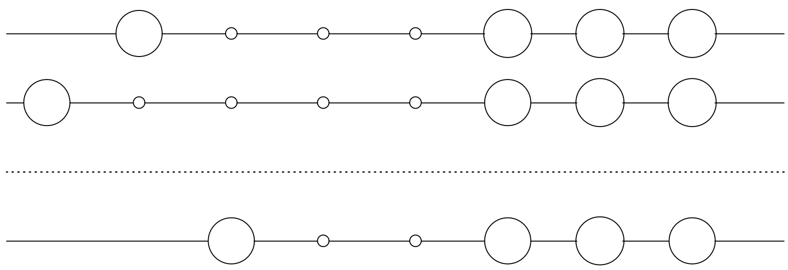

Figure 12