Zelevinsky's involution at roots of unity

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Abstract

We give a combinatorial algorithm for computing Zelevinsky's involution of the set of isomorphism classes of irreducible representations of the affine Hecke algebra $\widehat{H}_m(t)$ when t is a primitive nth root of 1. We show that the same map can also be interpreted in terms of aperiodic nilpotent orbits of $\mathbf{Z}/n\mathbf{Z}$-graded vector spaces.

1 Introduction

In [32], Zelevinsky has introduced an involution of the Grothendieck group of the category of complex smooth representations of finite length of $G = GL(m, F)$, where F is a $\mathfrak{p}$-adic field, and conjectured that this involution permutes the classes of irreducible representations ([32], 9.17). In [33], Zelevinsky further conjectured a geometric description of this involution in terms of the graded nilpotent orbits that parametrize the simple G-modules. Both conjectures have been proved by Moeglin and Waldspurger [22] for the category of admissible representations of G generated by their space of I-fixed vectors, where I is an Iwahori subgroup of G. In this case, by a theorem of Bernstein, Borel and Matsumoto, the conjectures can be reformulated in terms of the Hecke algebra $\widehat{H}_m(q)$ of G with respect to I. This is an affine Hecke algebra of type A with parameter q equal to the cardinality of the residue field of F. As shown in [22], the problem then becomes that of describing how the isomorphism classes of simple $\widehat{H}_m(q)$-modules are permuted when the action is twisted by a certain involutive automorphism τ of $\widehat{H}_m(q)$. In fact the answer does not depend on q, and is the same for any complex parameter t of infinite multiplicative order.

This is no longer the case if one considers the Hecke algebra $\widehat{H}_m(t)$ with t a primitive nth root of 1. According to a conjecture of Vigneras [31], the solution of the problem in this case would be relevant for the l-modular representation theory of G, where l is a prime different from the characteristic p of the residue field, and n is the multiplicative order of q in the field with l elements.

In this paper, we obtain results similar to those of Moeglin and Waldspurger in the root of unity case. Namely, (i) given a simple $\widehat{H}_m(t)$-module L, we describe in Section 5.2

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a simple combinatorial algorithm for computing the isomorphism class $[L^\tau]$ of the simple module obtained by twisting with τ, and (ii) we prove in Section 5.3 that the geometric description conjectured by Zelevinsky still holds in this case, this time in terms of the $\mathbf{Z}/n\mathbf{Z}$-graded nilpotent orbits parametrizing the simple $\widehat{H}_m(t)$-modules.

However, our methods are different from those of [22] (and thus we obtain new proofs of these results in the non root of unity case). Our main tool is a recent theorem of Ariki [1] which relates the Grothendieck groups of the algebras $\widehat{H}_m(t)$ with the enveloping algebra $U^-(\mathfrak{g})$, where $\mathfrak{g}$ is the Kac-Moody algebra of type $A^{(1)}_{n-1}$ if t has order n, and A_∞ if t has infinite order. Using this result, we show that our problem is equivalent to describing the natural 2-fold symmetry of Kashiwara's crystal graph of $U_v^-(\mathfrak{g})$, induced by the root diagram automorphism exchanging the simple roots α_i and α_{-i}. Then the combinatorial algorithm for $[L^\tau]$ follows immediately from the explicit description of this graph (Theorem 4.1), and the equivalent formulation in terms of nilpotent orbits is deduced from a geometric construction of the crystal basis conjectured by Lusztig [18] and recently verified by Kashiwara and Saito [13].

We note that Procter [25] and Aubert [2] have established independently that the first conjecture of Zelevinsky holds for the whole category of complex smooth representations of finite length of G. Assuming this hypothesis, Moeglin and Walspurger had already shown in [22] that their geometric and combinatorial descriptions of Zelevinsky's duality were also valid for this larger class of representations.

The paper is structured as follows. In Section 2 we recall the definition of Zelevinsky's involution, and the parametrization of simple $\widehat{H}_m(t)$-modules by multisegments. Then we can formulate our problem in a more precise way. In Section 3 we review the Hall-Ringel algebra $\mathcal{H}$ associated with a quiver and the canonical basis of its subalgebra $\mathcal{C}$. In Section 4, using the isomorphism $\mathcal{C} \cong U_v^-(\mathfrak{g})$, we compute the action of the Chevalley generators of $U_v^-(\mathfrak{g})$ on the PBW-basis of $\mathcal{H}$. This allows us to describe the crystal graph of $U_v^-(\mathfrak{g})$ in Lusztig's parametrization by nilpotent orbits, and to derive the combinatorial description of τ (5.2). Finally, in 5.3 we briefly recall the results of Lusztig, Kashiwara and Saito on the construction of the crystal basis in terms of irreducible components of a certain Lagrangian variety, and we deduce from them the geometric description of τ.

2 Affine Hecke algebras

2.1 Let t be a nonzero complex number. The affine Hecke algebra $\widehat{H}_m(t)$ associated to $GL(m)$ is the associative $\mathbf{C}$-algebra with invertible generators $T_1, \ldots, T_{m-1}$ and $y_1, \ldots, y_m$ subject to the relations

$$\begin{aligned}
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \le i \le m-2,\\
&T_iT_j = T_jT_i, \qquad |i-j| > 1,\\
&(T_i - t)(T_i + 1) = 0, \quad 1 \le i \le m-1,\\
&y_iy_j = y_jy_i, \qquad 1 \le i, j \le m,\\
&y_jT_i = T_iy_j, \qquad \text{for } j \ne i, i+1,\\
&T_iy_iT_i = t\, y_{i+1}, \qquad 1 \le i \le m-1.
\end{aligned}$$

The algebra $\widehat{H}_m(t)$ admits involutive automorphisms τ, $\flat$, $\sharp$, given on the generators by

$$T_i^\tau = -tT_{m-i}^{-1}, \qquad y_j^\tau = y_{m+1-j},$$

$$T_i^\flat = T_{m-i}, \qquad y_j^\flat = y_{m+1-j}^{-1},$$
$$T_i^\sharp = -tT_i^{-1}, \qquad y_j^\sharp = y_j^{-1}.$$

One checks that τ and $\flat$ commute and that $x^\sharp = (x^\flat)^\tau$. The involution $\sharp$ was defined by Iwahori and Matsumoto in [8], p. 280, and τ is usually called Zelevinsky's involution (see [22], p. 146), because of its relation with the involution on representations of $\mathfrak{p}$-adic $GL(m)$ introduced by Zelevinsky in [32, 33].

2.2 For $\mu = (\mu_1, \ldots, \mu_r)$ a composition of m, set $D(\mu) = \{\mu_1 + \cdots + \mu_k,\ 1 \le k \le r-1\} = \{d_1, \ldots, d_{r-1}\}$ and denote by $\widehat{H}_\mu$ the subalgebra of $\widehat{H}_m(t)$ generated by $y_1, \ldots, y_m$ and $\{T_i \mid i \notin D(\mu)\}$. For $a \in \mathbf{Z}^r$, let $\mathbf{C}_{\mu,a}$ be the 1-dimensional representation of $\widehat{H}_\mu$ defined by $T_i \mapsto t$, $i \notin D(\mu)$, and $y_{d_{i-1}+1} \mapsto t^{a_i}$, $i = 1, \ldots, r$, where $d_0 = 0$. We denote by $M_{\mu,a}$ the induced $\widehat{H}_m(t)$-module

$$M_{\mu,a} = \widehat{H}_m(t) \otimes_{\widehat{H}_\mu} \mathbf{C}_{\mu,a}\,.$$

Let $R(\widehat{H}_m(t))$ be the complexified Grothendieck group of the category of finite dimensional $\widehat{H}_m(t)$-modules, and let $R_m(t)$ be the linear span of the classes of the composition factors of the various $M_{\mu,a}$ in $R(\widehat{H}_m(t))$. We set

$$R(t) = \bigoplus_{m \ge 0} R_m(t)\,,$$

where we have put for convenience $R_0(t) = \mathbf{C}$. When (ν, b) is a permutation of (μ, a), *i.e.* $\nu_i = \mu_{\sigma(i)}$ and $b_i = a_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_r$, the induced modules $M_{\mu,a}$ and $M_{\nu,b}$ are in general non isomorphic, but their classes in $R(t)$ are equal [32] (see [30], p. 455). In the sequel, we shall always work in $R(t)$ and therefore shall only consider pairs (μ, a) up to permutation. In fact, such unordered pairs (μ, a) are naturally identified with certain graded nilpotent orbits O (see below Section 3), which gives a canonical labelling of the classes of the induced modules by these orbits : $[M_{\mu,a}] = [M_O]$. There is a partial order on orbits given by $O \trianglelefteq P$ if $O \subset \overline{P}$.

When t is not a root of unity, it is known [32] that for all O, there is a unique simple module whose class occurs in the expansion of $[M_O]$ but does not occur in any $[M_P]$ with $O \triangleleft P$. Let L_O denote this simple module. Then the L_O are pairwise non isomorphic, and one has in $R(t)$

$$[M_O] = [L_O] + \sum_{O \triangleleft P} K_{O,P}\,[L_P]$$

for some positive integers $K_{O,P}$. (The $K_{O,P}$ are in fact values at 1 of Kazhdan-Lusztig polynomials of type A, as conjectured by Zelevinsky [33, 34] and proved by Ginzburg [5], Theorem 8.6.23.)

If t is a primitive nth root of unity, we may consider that in a pair (μ, a), a belongs to $(\mathbf{Z}/n\mathbf{Z})^r$. Let us say that (μ, a) is aperiodic if for each $\ell \in \mathbf{N}^*$, the set $\{a_i \in \mathbf{Z}/n\mathbf{Z} \mid \mu_i = \ell\}$ has at most $n-1$ elements. These are the labels associated to the so-called aperiodic graded nilpotent orbits (see [19], 15.3). It follows from Ariki's theorem [1] that for all aperiodic O, there is a unique simple module L_O whose class occurs in the expansion of $[M_O]$ but does not occur in any $[M_P]$ with $O \triangleleft P$. Moreover, $\{[L_O] \mid O \text{ aperiodic }\}$ is a basis of $R(t)$. Finally, the multiplicities of the $[L_P]$ in the $[M_O]$ are given by certain Kazhdan-Lusztig polynomials of affine type $\tilde{A}$. (For the general type a similar result was previously announced by Grojnowski [7]).

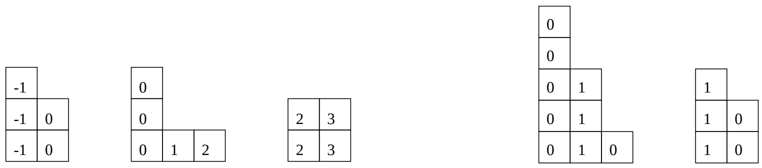

Figure 1: The diagrams of $(\mu, a) = ((2,2,3,1,1,2,2,1),(2,2,0,0,0,-1,-1,-1))$ and its reduction modulo 2

2.3 When t is not a root of unity, the pairs (μ, a) can be identified with Zelevinsky's multisegments [32]. We recall that a segment is an interval $[i,j]$ in $\mathbf{Z}$, and that a multisegment is a formal finite unordered sum $\mathbf{m} = \sum_{i\le j} m_{ij}[i,j]$. (Here m_{ij} stands for the multiplicity of the segment $[i,j]$ in $\mathbf{m}$.) The multisegment corresponding to (μ, a) is then

$$\mathbf{m} = \sum_{i=1}^{r} [a_i, a_i + \mu_i - 1]\,.$$

A multisegment can be conveniently regarded as a coloured multipartition, *i.e.* as a sequence $(\lambda^{(i)})_{i\in\mathbf{Z}}$ of partitions, such that the parts of $\lambda^{(i)}$ are the lengths of the segments $[i,j]$, and each cell of the kth column of the Young diagram of $\lambda^{(i)}$ contains the integer $i+k-1$.

In the case where t is a primitive nth root of unity, we shall also identify the labels (μ, a) with multisegments, this time over $\mathbf{Z}/n\mathbf{Z}$, where we regard $\mathbf{Z}/n\mathbf{Z}$ as a set of n points on the unit circle. For $\ell \in \mathbf{N}^*$ and $i \in \mathbf{Z}/n\mathbf{Z}$ we define the segment of length ℓ and origin i on this circle (or more appropriately the loop) as:

$$[i;\ell) := [i, i+1, \ldots, i+\ell-1]\,.$$

Sometimes we shall also need the dual notation

$$(\ell; i] := [i-\ell+1, i-\ell+2, \ldots, i]\,.$$

A multisegment over $\mathbf{Z}/n\mathbf{Z}$ is now a formal finite sum

$$\mathbf{m} = \sum_{i\in\mathbf{Z}/n\mathbf{Z},\ \ell\in\mathbf{N}^*} m_{[i;\ell)}\,[i;\ell) = \sum_{j\in\mathbf{Z}/n\mathbf{Z},\ k\in\mathbf{N}^*} m_{(k;j]}\,(k;j]\,.$$

The cyclic multisegment corresponding to the label (μ, a) is $\mathbf{m} = \sum_i [a_i;\mu_i)$. These cyclic multisegments may also be represented by n-tuples of partitions $(\lambda^{(i)})_{i\in\mathbf{Z}/n\mathbf{Z}}$, with cells labelled by integers modulo n. A label (μ, a) is aperiodic if for each ℓ, there is at least one component $\lambda^{(i)}$ which does not contain the part ℓ (this is, up to conjugation of partitions, the labelling used by Ringel in [29] for a basis of the composition algebra of the cyclic quiver). These definitions are illustrated in Figure 1.

Let $I = \mathbf{Z}$ (*resp.* $\mathbf{Z}/n\mathbf{Z}$). The set of all multisegments over I will be denoted by $\mathcal{M}(I)$, or simply $\mathcal{M}$ if no confusion can arise. Thus we have a parametrization of the simple $\widehat{H}_m(t)$-modules $L_{\mathbf{m}}$ by $\mathbf{m} \in \mathcal{M}(I)$ with $\sum_{i,\ell} \ell\, m_{[i;\ell)} = m$ (and $\mathbf{m}$ aperiodic for $I = \mathbf{Z}/n\mathbf{Z}$). We introduce a $\mathbf{N}^{(I)}$-grading of $R(t)$ by putting $\deg\,[L_{\mathbf{m}}] = (d_i)_{i\in I}$, where d_i is the number of cells of the diagram of $\mathbf{m}$ which contain i.

2.4 The three involutions τ, $\flat$ and $\sharp$ induce involutions on the set of simple $\widehat{H}_m(t)$-modules $L_{\mathbf{m}}$, and hence on the set of multisegments $\mathbf{m}$ that parametrize them. It is easy to see that $L_{\mathbf{m}}^\flat = L_{\mathbf{m}^\flat}$ where if $\mathbf{m} = \sum_{i\in I,\ \ell\in\mathbf{N}^*} m_{[i;\ell)}\,[i;\ell)$, then $\mathbf{m}^\flat = \sum_{i\in I,\ \ell\in\mathbf{N}^*} m_{[i;\ell)}\,(\ell;-i]$. Therefore, it is equivalent to describe either τ or $\sharp$. For generic t, a geometric description of τ in terms of graded nilpotent orbits has been conjectured by Zelevinsky [33]. This conjecture was proved by Moeglin and Waldspurger [22], who also gave a combinatorial algorithm for computing $\mathbf{m}\mapsto\mathbf{m}^\tau$. An explicit formula for $\mathbf{m}^\tau$ has also been found by Knight and Zelevinsky [15].

The finite Hecke algebra $H_m(t)$ is the quotient of $\widehat{H}_m(t)$ by the relation $y_1 = 1$. Hence, $\sharp$ induces an involution on simple $H_m(t)$-modules. When t is not a root of unity, these modules are parametrized by partitions of m, and $\sharp$ is just the conjugation of partitions. When t is an nth root of unity, the simple $H_m(t)$-modules are labelled by n-regular partitions, *i.e.* partitions with no part repeated more than $n-1$ times, and it was recently proved that $\sharp$ coincides with a bijection previously defined by Mullineux [23]. One proof follows from [16] p. 229 and Ariki's theorem, and a second proof was obtained by Brundan [4] by "quantizing" the results of Kleshchev on modular representations of symmetric groups.

The aim of this paper is to describe the maps τ and $\sharp$ in the affine case when t is a primitive nth root of unity. Proposition 5.1 below can be regarded as a common generalization of Mullineux's bijection as described by Kleshchev [14], and of the algorithm of Moeglin and Waldspurger.

3 Graded nilpotent orbits and Hall-Ringel algebras

3.1 Let Γ denote the quiver of type A_{n-1} (*resp.* A_∞ or $A_{n-1}^{(1)}$). The set of vertices of Γ is $I = [1, n-1]$ (*resp.* $\mathbf{Z}$ or $\mathbf{Z}/n\mathbf{Z}$), and the set of arrows is $\Omega = \{i \to i-1 \mid i, i-1 \in I\}$. A nilpotent representation of Γ over the field $\mathbf{k}$ is a pair (V, x), where $V = \oplus_{i\in I} V_i$ is an I-graded finite-dimensional vector space, and $x = (x_a)_{a\in\Omega}$ is a degree -1 nilpotent linear operator on V. In other words, x is an element of $N_{V,\Omega}$, the subset of nilpotent elements of

$$E_{V,\Omega} = \bigoplus_{i\to j\in\Omega} \mathrm{Hom}\,(V_i, V_j)\,.$$

The vector $\mathbf{d} = \mathbf{dim}\, V = (\dim V_i)_{i\in I} \in \mathbf{N}^{(I)}$ is the (graded) dimension of the representation. The group $G_V = \prod_{i\in I} GL(V_i)$ acts on $E_{V,\Omega}$ and $N_{V,\Omega}$ by conjugation. Two representations (V,x) and (V,x') are equivalent if x and x' lie in the same orbit. Thus, the set of isomorphism classes of nilpotent Γ-representations is naturally identified with $\mathcal{O} = \sqcup_{\mathbf{d}\in\mathbf{N}^{(I)}} \mathcal{O}_{\mathbf{d}}$, where $\mathcal{O}_{\mathbf{d}}$ is the set of G_V-orbits in $N_{V,\Omega}$ for any graded $\mathbf{k}$-vector space V of dimension $\mathbf{d}$.

The isomorphism classes of nilpotent Γ-representations are parametrized by $\mathcal{M}(I)$ in the following way. For each vertex $i \in I$, let $\mathbf{k}[i]$ denote the simple Γ-module for which $V = V_i = \mathbf{k}$ and $x = 0$. Given a positive integer ℓ, there is a unique (up to isomorphism) indecomposable Γ-module $\mathbf{k}[\ell; i]$ of length ℓ with head $\mathbf{k}[i]$, and all indecomposable modules are of this type. Therefore, any nilpotent Γ-module M is isomorphic to

$$\mathbf{k}[\mathbf{m}] := \bigoplus_{i\in I,\ell\in\mathbf{N}^*} \mathbf{k}[\ell; i]^{\oplus m_{(\ell;i]}}$$

for a unique multisegment $\mathbf{m}$. We denote by $O_{\mathbf{m}}$ the G_V-orbit of the module $\mathbf{k}[\mathbf{m}]$. For simplicity, we also denote by O_i the orbit of the simple module $\mathbf{k}[i]$.

3.2 We recall the definition of the twisted Hall algebra associated with the quiver Γ (see [27, 28, 21]). The classification of nilpotent Γ-modules is independent of the ground field. Take $\mathbf{k} = \mathbf{F}_q$, the field with q elements, and let O, P, Q be orbits in $N_{V,\Omega}, N_{W,\Omega}, N_{U,\Omega}$ respectively, where $\mathbf{dim}\, U = \mathbf{dim}\, V + \mathbf{dim}\, W$. Let M be a Γ-module lying on the orbit Q. Then it is was proved by Ringel [27, 29] that the number of submodules of M with type P and cotype O is a polynomial in q with integer coefficients, independent of the choice of M on Q. This is the Hall polynomial $F_{O,P}^{Q}(q)$.

Define a bilinear form m on $\mathbf{N}^{(I)}$ by

$$m(\mathbf{a}, \mathbf{b}) = \sum_{i \to j \in \Omega} a_i b_j + \sum_{i \in I} a_i b_i \,. \tag{1}$$

Then the generic twisted Hall algebra associated with Γ is the $\mathbf{C}(v)$-algebra $\mathcal{H}$ with basis $\{u_O \mid O \in \mathcal{O}\}$ and multiplication

$$u_O \circ u_P = v^{m(\mathbf{dim}\, V, \mathbf{dim}\, W)} \sum_Q F_{O,P}^{Q}(v^{-2})\, u_Q. \tag{2}$$

(In the setting of [21], u_O stands for the product of characteristic functions of the sets of $\mathbf{F}_{p^{2m}}$-rational points of O for a fixed prime p.)

3.3 Let O be a G_V-orbit in $N_{V,\Omega}$. One sets

$$\langle\, O \,\rangle = v^{\dim O}\, u_O \,, \tag{3}$$

where $\dim O$ denotes the dimension of the orbit O (not to be confused with the dimension $\dim V$ of the representation of Γ corresponding to a point of O). The basis $\{\langle\, O \,\rangle\}$ is called the Poincaré-Birkhoff-Witt basis of $\mathcal{H}$. If P is a G_W-orbit in $N_{W,\Omega}$, it follows from (2) that

$$\langle\, O \,\rangle \circ \langle\, P \,\rangle = \sum_Q v^{\alpha(O,P,Q)}\, F_{O,P}^{Q}(v^{-2})\, \langle\, Q \,\rangle, \tag{4}$$

where

$$\alpha(O, P, Q) = \dim O + \dim P - \dim Q + m(\mathbf{dim}\, V, \mathbf{dim}\, W)\,. \tag{5}$$

In the sequel, we shall use another expression for $\alpha(O, P, Q)$. For $\mathbf{a}, \mathbf{b} \in \mathbf{N}^{(I)}$, set

$$r(\mathbf{a}, \mathbf{b}) = -\sum_{i \to j \in \Omega} a_i b_j + \sum_{i \in I} a_i b_i \,, \tag{6}$$

and denote by $\varepsilon(O)$ the dimension of the space of endomorphisms of a module M lying in O. Then

$$\alpha(O, P, Q) = -\varepsilon(O) - \varepsilon(P) + \varepsilon(Q) - r(\mathbf{dim}\, V, \mathbf{dim}\, W)\,. \tag{7}$$

Indeed,

$$\dim O = \dim G_V - \dim(\mathrm{Stab}\, M)\,,$$

and

$$\dim(\mathrm{Stab}\, M) = \dim(\mathrm{Aut}\, M) = \dim(\mathrm{End}\, M) = \varepsilon(O)\,.$$

Thus, if we set $\mathbf{dim}\, V = (d_{i,V})_{i \in I}$ and $\mathbf{dim}\, W = (d_{i,W})_{i \in I}$, we have

$$\begin{aligned}\alpha(O, P, Q) &= -\varepsilon(O) - \varepsilon(P) + \varepsilon(Q) + \sum_i d_{i,V}^2 + \sum_i d_{i,W}^2 - \sum_i (d_{i,V} + d_{i,W})^2 + m(\mathbf{dim}\, V, \mathbf{dim}\, W) \\ &= -\varepsilon(O) - \varepsilon(P) + \varepsilon(Q) - r(\mathbf{dim}\, V, \mathbf{dim}\, W)\,.\end{aligned}$$

3.4 Let $\mathcal{C}$ be the twisted composition algebra of Γ, *i.e.* the subalgebra of $\mathcal{H}$ generated by the characteristic functions $u_i = \langle O_i \rangle$ of the orbits of the simple modules $\mathbf{k}[i]$. When Γ is of type A_{n-1} or A_∞, one has $\mathcal{C} = \mathcal{H}$, but $\mathcal{C}$ is strictly contained in $\mathcal{H}$ for the type $A_{n-1}^{(1)}$. The vector space $\mathcal{H}$ has a basis $\{b_O \mid O \in \mathcal{O}\}$ given by

$$b_O = \sum_{i,O'} v^{-i+\dim O - \dim O'} \dim \mathcal{H}^i_{O'}(IC_O) \langle O' \rangle \tag{8}$$

where $\mathcal{H}^i_{O'}(IC_O)$ is the stalk at a point of O' of the ith intersection cohomology sheaf of the closure $\overline{O}$ of O.

For Γ of type A_∞, these sheaves have been first considered by Zelevinsky in [33], where he made a conjecture about their connection with representations of $\mathfrak{p}$-adic $GL(m)$ known as the $\mathfrak{p}$-adic analogue of the Kazhdan-Lusztig conjecture. In [34], Zelevinsky further proved that the Poincaré polynomials $\sum_i t^i \dim \mathcal{H}^i_{O'}(IC_O)$ are equal to certain Kazhdan-Lusztig polynomials $P_{w(O')w(O)}(t^2)$ of type A.

For Γ of type $A_{n-1}^{(1)}$, Lusztig has shown that the Poincaré polynomials are also Kazhdan-Lusztig polynomials, this time of affine type $\tilde{A}$ ([17],11). In [19], he introduced the subfamily of aperiodic nilpotent orbits. These are the orbits $O_{\mathbf{m}}$ parametrized by aperiodic multisegments (see 2.3). It is proved in [20], Section 5 that $\{b_O \mid O \text{ aperiodic}\}$ is a basis of $\mathcal{C}$. Note that non aperiodic orbits O' may occur in the right-hand side of (8) for an aperiodic orbit O.

Let $x \mapsto \bar{x}$ be the ring automorphism of $\mathcal{C}$ defined by $\bar{u}_i = u_i$ and $\bar{v} = v^{-1}$. Let $\mathcal{L}$ be the $\mathbf{Z}[v]$-submodule of $\mathcal{H}$ spanned by the PBW basis $\langle O_{\mathbf{m}} \rangle$. Let $O \in \mathcal{O}$ and suppose that O is aperiodic if Γ is of type $A_{n-1}^{(1)}$. Then b_O is characterized as the unique element of $\mathcal{L} \cap \mathcal{C}$ such that $\overline{b_O} = b_O$ and $b_O \equiv \langle O \rangle \mod v\mathcal{L}$.

4 Affine Lie algebras and quantum affine algebras

4.1 Let $\mathfrak{g}$ be the Kac-Moody Lie algebra associated with the quiver Γ of type A_{n-1} (*resp.* A_∞ or $A_{n-1}^{(1)}$), namely, $\mathfrak{g} = \mathfrak{sl}_n$ (*resp.* $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\widehat{\mathfrak{sl}}_n$). Let $U_v(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra, with generators $e_i, f_i, v^{h_i}, (i \in I)$. The subalgebra generated by $f_i, (i \in I)$ is denoted by $U_v^-(\mathfrak{g})$. This algebra is isomorphic to the twisted composition algebra $\mathcal{C}$ of Γ, the isomorphism being given by $f_i \mapsto \langle O_i \rangle$. It follows that the twisted Hall algebra $\mathcal{H}$ can be regarded as a left $U_v^-(\mathfrak{g})$-module. The basis $\{b_O\}$ of $\mathcal{C}$ becomes via this isomorphism a basis of $U_v^-(\mathfrak{g})$: this is Lusztig's canonical basis. It is known that it coincides with Kashiwara's lower global basis of $U_v^-(\mathfrak{g})$. By taking $v = 1$ one obtains the canonical basis of $U^-(\mathfrak{g})$, that we shall still denote by $\{b_O\}$. The aim of this section is to describe Kashiwara's crystal graph of $\mathcal{H}$. To do so, we shall first obtain explicit formulas for the action of the Chevalley generators f_i and their adjoint e'_i on $\mathcal{H}$. We believe that these formulas are of independent interest. For example they yield an algorithm similar to that of [16] for computing the canonical basis in its expansion on the PBW-basis.

4.2 We describe the action of the f_i on $\mathcal{H}$ in the PBW-basis. We assume from now on that Γ is of type $A_{n-1}^{(1)}$. (The other two cases are readily deduced from this one, since $U_v(\mathfrak{sl}_n)$ is a natural subalgebra of $U_v(\widehat{\mathfrak{sl}}_n)$, and $U_v(\mathfrak{sl}_\infty)$ is the "limit $n \to \infty$ of $U_v(\mathfrak{sl}_n)$").

As explained above, the PBW-basis is naturally labelled by multisegments $\mathbf{m}$ over $\mathbf{Z}/n\mathbf{Z}$. For simplicity, we abuse notation and write $\langle\, \mathbf{m}\,\rangle := \langle\, O_{\mathbf{m}}\,\rangle$ for the corresponding element of the PBW-basis. Given $i\in \mathbf{Z}/n\mathbf{Z}$, $\ell \in \mathbf{N}^*$, and a multisegment $\mathbf{m}$ such that $m_{(\ell-1;i-1]}\neq 0$ if $\ell>1$, we define a new multisegment

$$\mathbf{m}^+_{\ell,i} := \left|\begin{array}{cl} \mathbf{m}+(1;i] & \text{if } \ell=1, \\ \mathbf{m}+(\ell;i]-(\ell-1;i-1] & \text{if } \ell>1. \end{array}\right.$$

In case $m_{(\ell-1;i-1]}=0$, we put $\langle\, \mathbf{m}^+_{\ell,i}\,\rangle := 0$.

Proposition 4.1 *The Chevalley generators of $U_v^-(\widehat{\mathfrak{sl}}_n)$ act on the PBW-basis of $\mathcal{H}$ by*

$$f_i\,\langle\, \mathbf{m}\,\rangle = \sum_{\ell\in\mathbf{N}^*} v^{\sum_{k>\ell}(m_{(k-1;i-1]}-m_{(k;i]})}\,[m_{(\ell;i]}+1]\,\langle\, \mathbf{m}^+_{\ell,i}\,\rangle\,,$$

where for $a\in\mathbf{Z}$, $[a]=(v^a-v^{-a})/(v-v^{-1})$ is the usual v-integer.

Proof — We have to apply formulas (4) and (7) with $O=O_i$ and $P=O_{\mathbf{m}}$. We see easily that here, $\varepsilon(O)=1$, and

$$r(\mathbf{dim}\,V,\mathbf{dim}\,W) = \sum_{k\in\mathbf{N}^*} m_{[i;k)} - \sum_{k\in\mathbf{N}^*} m_{(k;i-1]}\,.$$

Also, it is clear that in this case, the orbits Q giving a nonzero contribution to the right-hand side of (4) are those parametrized by the multisegments of the form $\mathbf{m}^+_{\ell,i}$ for some $\ell\in\mathbf{N}^*$. We distinguish two cases depending on whether $\ell=1$ or $\ell>1$.

In the first case we have

$$\begin{aligned} \varepsilon(Q)-\varepsilon(P) &= \dim(\mathrm{End}\,(\mathbf{k}[i])) + \dim(\mathrm{Hom}\,(\mathbf{k}[i],\mathbf{k}[\mathbf{m}])) + \dim(\mathrm{Hom}\,(\mathbf{k}[\mathbf{m}],\mathbf{k}[i])) \\ &= 1+\sum_{k\in\mathbf{N}^*} m_{[i;k)} + \sum_{k\in\mathbf{N}^*} m_{(k;i]}\,. \end{aligned}$$

To calculate the Hall polynomial $F^Q_{O,P}$, let us count the number of submodules (V,x) of $(V',x')=\mathbf{F}_q[\mathbf{m}^+_{1,i}]$ isomorphic to $\mathbf{F}_q[\mathbf{m}]$. For $j\neq i$ we must have $V_j=V'_j$, and V_i must be a hyperplane in V'_i which (i) contains $x(V'_{i+1})$ (so that V be stable under x') and (ii) does not contain $\mathrm{Ker}\,x'|_{V'_i}$ (so that (V,x) be isomorphic to $\mathbf{F}_q[\mathbf{m}]$). The number of hyperplanes satisfying (i) is clearly $[\![1+\sum_{k\in\mathbf{N}^*} m_{(k;i]}]\!]$ and the number of those satisfying (i) but not (ii) is $[\![\sum_{k\geq 2} m_{(k;i]}]\!]$. (Here we have used the notation $[\![a]\!]=(q^a-1)/(q-1)$.) Therefore,

$$F^Q_{O,P}(q) = [\![1+\sum_{k\in\mathbf{N}^*} m_{(k;i]}]\!] - [\![\sum_{k\geq 2} m_{(k;i]}]\!] = q^{\sum_{k\geq 2} m_{(k;i]}}\,[\![1+m_{(1;i]}]\!]\,,$$

and finally, the coefficient of $\langle\, \mathbf{m}^+_{1,i}\,\rangle$ in $f_i\,\langle\, \mathbf{m}\,\rangle$ is equal to

$$\begin{aligned} & v^{-1+1+\sum_{k\in\mathbf{N}^*} m_{[i;k)}+\sum_{k\in\mathbf{N}^*} m_{(k;i]}-\sum_{k\in\mathbf{N}^*} m_{[i;k)}+\sum_{k\in\mathbf{N}^*} m_{(k;i-1]}-2\sum_{k\geq 2} m_{(k;i]}-m_{(1;i]}}[m_{(1;i]}+1] \\ &\qquad = v^{\sum_{k>1}(m_{(k-1;i-1]}-m_{(k;i]})}\,[m_{(1,i]}+1]\,. \end{aligned}$$

In the case $\ell>1$, the calculation is similar and one gets

$$\begin{aligned} \varepsilon(Q)-\varepsilon(P) &= 1+\sum_{k\in\mathbf{N}^*} m_{[i;k)} + \sum_{k\geq\ell} m_{(k;i]} - \sum_{k\leq\ell-1} m_{(k;i-1]}\,, \\ F^Q_{O,P}(q) &= [\![1+\sum_{k\geq\ell} m_{(k;i]}]\!] - [\![\sum_{k\geq\ell+1} m_{(k;i]}]\!] = q^{\sum_{k\geq\ell+1} m_{(k;i]}}\,[\![1+m_{(\ell;i]}]\!]\,, \end{aligned}$$

so that the coefficient of $\langle\, \mathbf{m}^+_{\ell,i}\,\rangle$ in $f_i\,\langle\, \mathbf{m}\,\rangle$ is equal to

$$v^{-1+1+\sum_{k\in\mathbf{N}^*} m_{[i;k)}+\sum_{k\geq\ell} m_{(k;i]}-\sum_{k\leq\ell-1} m_{(k;i-1]}-\sum_{k\in\mathbf{N}^*} m_{[i;k)}+\sum_{k\in\mathbf{N}^*} m_{(k;i-1]}}$$

$$\times\; v^{-2\sum_{k\geq\ell+1} m_{(k;i]}}\, v^{-m_{(\ell;i]}}\,[m_{(\ell;i]}+1]$$

$$=\; v^{\sum_{k>\ell}(m_{(k-1;i-1]}-m_{(k;i]})}\,[m_{(\ell,i]}+1]\,.$$

□

Remark The specialization $v=1$ of Proposition 4.1 appears in [1] Lemma 4.2.

4.3 To define the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$ of the $U_v^-(\widehat{\mathfrak{sl}}_n)$-module $\mathcal{H}$, one needs to introduce endomorphisms e'_i of the space $\mathcal{H}$ (see [11], 3.4). First one defines a scalar product on $\mathcal{H}$ for which the basis $\{\langle\, \mathbf{m}\,\rangle\}$ is orthogonal by

$$(\langle\, \mathbf{m}\,\rangle\,,\,\langle\, \mathbf{m}'\,\rangle) = v^{-2\varepsilon(\mathbf{m})}\,\frac{(1-v^2)^{\dim \mathbf{k}[\mathbf{m}]}}{|\mathrm{Aut}\,\mathbf{F}_q[\mathbf{m}]|_{q=v^{-2}}}\,\delta_{\mathbf{m}\,,\,\mathbf{m}'}\,. \tag{9}$$

Here we have written for short $\varepsilon(\mathbf{m})=\varepsilon(O_{\mathbf{m}})$. This scalar product is the one introduced by Green [6], multiplied by a normalization factor $(v^{-2}-1)^{\dim \mathbf{k}[\mathbf{m}]}$ so that it induces Kashiwara's scalar product [11] on $\mathcal{C}\cong U_v^-(\widehat{\mathfrak{sl}}_n)$.

Lemma 4.1 *Put $\varphi_m(t)=(1-t^m)(1-t^{m-1})\cdots(1-t)$. We have*

$$|\mathrm{Aut}\,\mathbf{F}_q[\mathbf{m}]| = |\mathrm{End}\,\mathbf{F}_q[\mathbf{m}]|\,\prod_{i,\ell}\frac{\left|GL(m_{(\ell;i]},\mathbf{F}_q)\right|}{\left|\mathrm{Mat}(m_{(\ell;i]},\mathbf{F}_q)\right|} = q^{\varepsilon(\mathbf{m})}\,\prod_{i,\ell}\varphi_{m_{(\ell;i]}}(q^{-1})\,.$$

Proof — Let (V,x) be a Γ-module isomorphic to $\mathbf{F}_q[\mathbf{m}]$, and let $W_{\ell,i}$ denote the subspace of V_i consisting of all the generators of an indecomposable summand of (V,x) of type $(\ell;i]$. Thus, $\dim W_{\ell,i}=m_{(\ell;i]}$.

An endomorphism φ of (V,x) is completely determined by its restriction $\varphi_{|W}$ to $W=\oplus_{\ell,i}W_{\ell,i}$. Let U_i be a complement of $\oplus_\ell W_{\ell,i}$ in V_i, and $p_{\ell,i}$ be the corresponding projection of V_i onto $W_{\ell,i}$. Then, $x\circ\varphi=\varphi\circ x$ implies that $p_{\ell,i}\circ\varphi_{|W_{k,i}}=0$ for $\ell>k$. On the other hand, $p_{\ell,i}\circ\varphi_{|W_{\ell,i}}$ is not submitted to any condition and may be an arbitrary element of $\mathrm{End}\,W_{\ell,i}\cong\mathrm{Mat}(m_{(\ell;i]},\mathbf{F}_q)$.

Now, φ is an automorphism of (V,x) if and only if $p_W\circ\varphi_{|W}$ is in $GL(W)$, where $p_W=\sum_{\ell,i}p_{\ell,i}$. Hence φ is an automorphism if and only if $p_{\ell,i}\circ\varphi_{|W_{\ell,i}}$ belongs to $GL(W_{\ell,i})\cong GL(m_{(\ell;i]},\mathbf{F}_q)$, and the lemma follows. □

Using Lemma 4.1 and the formula $\dim\mathbf{k}[\mathbf{m}]=\sum_{i,\ell}\ell\, m_{(\ell;i]}$, we rewrite (9) as

$$(\langle\, \mathbf{m}\,\rangle\,,\,\langle\, \mathbf{m}'\,\rangle) = \prod_{i,\ell} v^{-\binom{m_{(\ell;i]}}{2}}\,\frac{(1-v^2)^{(\ell-1)m_{(\ell;i]}}}{[m_{(\ell;i]}]!}\,\delta_{\mathbf{m}\,,\,\mathbf{m}'}\,. \tag{10}$$

Next we define $e'_i\in\mathrm{End}_{\mathbf{C}(v)}\mathcal{H}$ as the adjoint operator of f_i with respect to this scalar product. The following proposition describes the action of e'_i on the PBW-basis. Given $i\in\mathbf{Z}/n\mathbf{Z}$, $\ell\in\mathbf{N}^*$, and a multisegment $\mathbf{m}$ such that $m_{(\ell;i]}\neq 0$, we put

$$\mathbf{m}^-_{\ell,i} := \left|\begin{array}{cl} \mathbf{m}-(1;i] & \text{if } \ell=1,\\ \mathbf{m}-(\ell;i]+(\ell-1;i-1] & \text{if } \ell>1.\end{array}\right.$$

In case $m_{(\ell;i]}=0$, we put $\langle\, \mathbf{m}^-_{\ell,i}\,\rangle := 0$.

Proposition 4.2 *The endomorphisms e'_i act on the PBW-basis of $\mathcal{H}$ by*

$$e'_i \langle \mathbf{m} \rangle = v^{\sum_{k>1}(m_{(k-1;i-1]}-m_{(k;i]})}\, v^{-m_{(1;i]}+1} \langle \mathbf{m}^-_{1,i} \rangle$$

$$+ \sum_{\ell \geq 2} v^{\sum_{k>\ell}(m_{(k-1;i-1]}-m_{(k;i]})}\, v^{-m_{(\ell;i]}+1} \left(1 - v^{2(m_{(\ell-1;i-1]}+1)}\right) \langle \mathbf{m}^-_{\ell,i} \rangle .$$

Proof — By definition of e'_i and using Proposition 4.1, the coefficient of $\langle \mathbf{m}^-_{\ell,i} \rangle$ in $e'_i \langle \mathbf{m} \rangle$ is equal to

$$\frac{(\langle \mathbf{m} \rangle, f_i \langle \mathbf{m}^-_{\ell,i} \rangle)}{(\langle \mathbf{m}^-_{\ell,i} \rangle, \langle \mathbf{m}^-_{\ell,i} \rangle)} = \frac{(\langle \mathbf{m} \rangle, \langle \mathbf{m} \rangle)}{(\langle \mathbf{m}^-_{\ell,i} \rangle, \langle \mathbf{m}^-_{\ell,i} \rangle)}\, v^{\sum_{k>\ell}(m_{(k-1;i-1]}-m_{(k;i]})}\, [m_{(\ell;i]}] .$$

Now by (10), if $\ell > 1$

$$\frac{(\langle \mathbf{m} \rangle, \langle \mathbf{m} \rangle)}{(\langle \mathbf{m}^-_{\ell,i} \rangle, \langle \mathbf{m}^-_{\ell,i} \rangle)} = (1 - v^2)\, v^{-m_{(\ell;i]}+m_{(\ell-1;i-1]}+1}\, \frac{[m_{(\ell-1;i-1]}+1]}{[m_{(\ell;i]}]} ,$$

and if $\ell = 1$

$$\frac{(\langle \mathbf{m} \rangle, \langle \mathbf{m} \rangle)}{(\langle \mathbf{m}^-_{1,i} \rangle, \langle \mathbf{m}^-_{1,i} \rangle)} = \frac{v^{-m_{(1;i]}+1}}{[m_{(1;i]}]} ,$$

which gives the required result. □

Remark Propositions 4.1, 4.2 should be compared to the formulas of [9] for the action of the Chevalley generators of $U_v(\widehat{\mathfrak{sl}}_n)$ on the level l Fock spaces. Actually, our formulas can be regarded as some appropriate limit $l \to \infty$ of those of [9].

4.4 Let $\widehat{B}(\infty)$ be Kashiwara's crystal graph of $\mathcal{H}$ for type $A^{(1)}_{n-1}$ in Lusztig's geometric parametrization. That is, the vertices of this graph are the multisegments $\mathbf{m}$ (or the corresponding orbits $O_{\mathbf{m}}$), and there is an arrow $\mathbf{m} \xrightarrow{i} \mathbf{m}'$ in the graph if and only if $\tilde{f}_i \langle O_{\mathbf{m}} \rangle \equiv \langle O_{\mathbf{m}'} \rangle \bmod v\mathcal{L}$, where $\tilde{f}_i$ denotes as usual the Kashiwara operator [11]. In particular, the connected component $B(\infty)$ of this graph containing the empty multisegment gives the crystal graph of $\mathcal{C} \cong U_v^-(\widehat{\mathfrak{sl}}_n)$ parametrized by the set of aperiodic orbits.

There are several descriptions of the crystal graph of $U_v^-(\mathfrak{g})$ available in the literature (see [12], [10], [24]), but the only ones using Lusztig's parametrization are given by Reineke [26] and Kashiwara-Saito [13]. Unfortunately, Reineke's description is restricted to finite dimensional $\mathfrak{g}$, while Kashiwara-Saito's description is in terms of the geometry of orbits, and does not seem to yield a combinatorial algorithm (see Section 5.3). Our description of $\widehat{B}(\infty)$ will be an extension to type $A^{(1)}_{n-1}$ of Reineke's result for type A_{n-1}.

Theorem 4.1 *Let $\mathbf{m}$ be a multisegment over $\mathbf{Z}/n\mathbf{Z}$, and fix $i \in \mathbf{Z}/n\mathbf{Z}$. For $k \in \mathbf{N}$, put $S_{k,i} = \sum_{\ell \geq k}(m_{(\ell;i-1]} - m_{(\ell;i]})$ and let k_0 be minimal such that $S_{k_0,i} = \min_k S_{k,i}$. Then there is an arrow $\mathbf{m} \xrightarrow{i} \mathbf{m}^+_{k_0,i}$ in the crystal graph $\widehat{B}(\infty)$ of $\mathcal{H}$.*

Proof — Our strategy is to reduce the problem to Reineke's description of the crystal graph of $U_v^-(\mathfrak{sl}_n)$. We first note that one can let n tend to infinity in Reineke's work and get the crystal graph of $U_v^-(\mathfrak{sl}_\infty)$.

Let us identify $i \in \mathbf{Z}/n\mathbf{Z}$ with its representative in $\{0,1,\ldots,n-1\}$. Choose n integers $j_0, j_1, \ldots, j_{n-1}$ such that $j_r \equiv r \bmod n$ and $j_{i-1} = i-1,\ j_i = i$. For example $\{j_r\} = \{0,1,\ldots,n-1\}$ if $i \neq 0$, and $\{j_r\} = \{-1,0,1,\ldots,n-2\}$ if $i=0$. Define an embedding ϕ_i of the set of $\mathbf{Z}/n\mathbf{Z}$-multisegments in the set of $\mathbf{Z}$-multisegments by

$$\phi_i\left(\sum_{\ell\in\mathbf{N}^*}\sum_{r\in\mathbf{Z}/n\mathbf{Z}} m_{(\ell;r]}\,(\ell;r]\right) = \sum_{\ell\in\mathbf{N}^*}\sum_{r\in\mathbf{Z}/n\mathbf{Z}} m_{(\ell;r]}\,(\ell;j_r]\,.$$

Then define a $\mathbf{C}(v)$-linear embedding $\Phi_i\ :\ \mathcal{H} \longrightarrow U_v^-(\mathfrak{sl}_\infty)$ by

$$\Phi_i(\langle\, O_{\mathbf{m}}\,\rangle) = \langle\, O_{\phi_i(\mathbf{m})}\,\rangle\,.$$

Let f_i^∞ denote the Chevalley element in $U_v^-(\mathfrak{sl}_\infty)$ and $e_i^{'\infty}$ its adjoint. Proposition 4.1 and 4.2 readily imply

Lemma 4.2 (i) *The subspace $\Phi_i(\mathcal{H})$ of $U_v^-(\mathfrak{sl}_\infty)$ is stable under f_i^∞ and $e_i^{'\infty}$.*
(ii) $\quad \Phi_i\circ f_i = f_i^\infty\circ\Phi_i\,, \qquad \Phi_i\circ e_i' = e_i^{'\infty}\circ\Phi_i\,.$

Let γ_i (*resp.* γ_i^∞) be the subgraph of the crystal of $\mathcal{H}$ (*resp.* $U_v^-(\mathfrak{sl}_\infty)$) obtained by erasing all arrows labelled by $j \neq i$. Lemma 4.2 shows that

$$\Phi_i\circ\tilde f_i = \tilde f_i^\infty\circ\Phi_i\,, \qquad \Phi_i\circ\tilde e_i = \tilde e_i^\infty\circ\Phi_i\,.$$

Thus, ϕ_i induces an isomorphism of graphs between γ_i and the full subgraph of γ_i^∞ with vertex set $\{\phi_i(\mathbf{m})\}$, where $\mathbf{m}$ runs over all $\mathbf{Z}/n\mathbf{Z}$-multisegments. The description of this last graph follows from [26] and this finishes the proof of Theorem 4.1. $\Box$

Theorem 4.1 is illustrated in Figure 2 in the case $n=3$.

Remarks 1 It follows easily from Theorem 4.1 that the crystal $\widehat{B}(\infty)$ of $\mathcal{H}$ decomposes into an infinite number of connected components isomorphic to $B(\infty)$, the highest weight vertices of these components being the periodic multisegments, that is, the multisegments $\mathbf{m}$ for which $m_{(\ell,r]} = m_{(\ell,s]}$ for all $r,s\in\mathbf{Z}/n\mathbf{Z}$. This is in agreement with [11], Remarks 3.4.10, 3.5.1.

2 The crystal graphs of the level l integrable representations of $\widehat{\mathfrak{sl}}_n$ as described by [9] embed in a natural way into the crystal graph of $\mathcal{H}$ (*cf.* 4.3 Remark). Each l-tuple of partitions $\boldsymbol{\lambda} = (\lambda^1,\ldots,\lambda^l)$ is mapped to the multisegment obtained by taking the formal sum of the rows of the λ^i.

5 Ariki's theorem and Zelevinsky's involution

5.1 Let $K(t)$ denote the graded dual of the $\mathbf{N}^{(I)}$-graded space $R(t)$. This is made into an associative algebra by taking as product the dual of the comultiplication

$$R_k(t) \to \bigoplus_{k=k'+k''} R_{k'}(t)\otimes R_{k''}(t)$$

coming from the restriction maps

$$\widehat{H}_k(t) \to \widehat{H}_{k'}(t)\otimes\widehat{H}_{k''}(t) \quad (k=k'+k'').$$

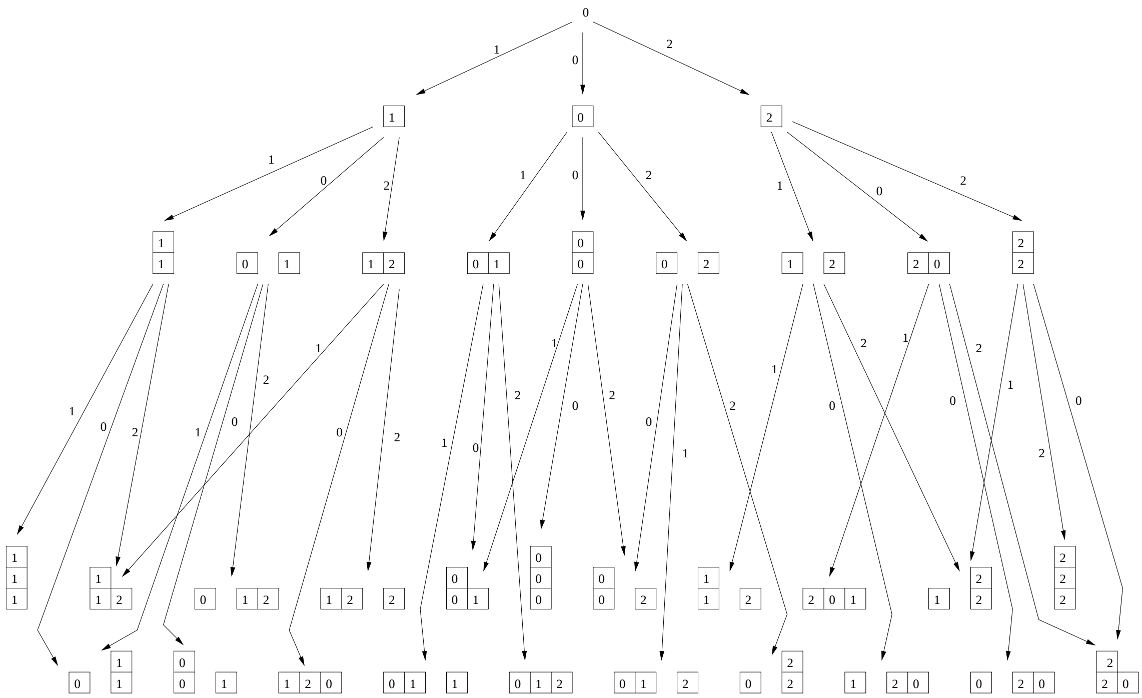

Figure 2: The crystal graph of $U_v^-(\widehat{\mathfrak{sl}}_3)$ up to degree 3 in its labelling by aperiodic orbits

For any $i \in \mathbf{Z}$ let $\theta_i \in R_1(t)$ be the one-dimensional module such that $\theta_i(y_1) = t^i$. If t is generic (*resp.* $t = e^{2i\pi/n}$) then $\{\theta_i \,|\, i \in \mathbf{Z}\}$ (*resp.* $\{\theta_i \,|\, i = 0, 1, ..., n-1\}$) is a basis of $R_1(t)$. The dual basis vectors are denoted by θ^i. Recall that if M is a simple $\widehat{H}_k(t)$-module then $c_k := y_1 + y_2 + \cdots + y_k$ acts as a scalar, say z_M, on M. Then the i-restriction of the class $[M]$ is defined in [1] as the class i-res $[M]$ of the $\widehat{H}_{k-1}(t)$-submodule

$$M_i = \mathrm{Ker}(c_{k-1} - z_M + t^i)^l \subseteq M, \quad l >> 1\,,$$

and i-res is extended to $R(t)$ by linearity. It is straightforward to check that for $M \in R_k(t)$ and $f \in R_{k-1}(t)^*$, one has $(\theta^i \cdot f)(M) = f(i\text{–res}\,(M))$.

Lemma 5.1 (i) $\sharp$ *induces an automorphism of* $K(t)$ *such that* $(\theta^i)^\sharp = \theta^{-i}$.
(ii) τ *induces an anti-automorphism of* $K(t)$ *such that* $(\theta^i)^\tau = \theta^i$.

Proof — Claim (i) is obvious since $\sharp$ commutes with the natural embedding

$$i: \ \widehat{H}_{k'}(t) \otimes \widehat{H}_{k''}(t) \hookrightarrow \widehat{H}_k(t)\,.$$

Claim (ii) follows from the identity $\tau \circ i(a \otimes b) = j(\tau(b) \otimes \tau(a))$ where $a \in \widehat{H}_{k'}(t)$, $b \in \widehat{H}_{k''}(t)$, and j stands for the natural embedding of $\widehat{H}_{k''}(t) \otimes \widehat{H}_{k'}(t)$ into $\widehat{H}_k(t)$. $\square$

5.2 It was proved by Ariki that the map $f_i \mapsto \theta^i$ is an isomorphism of algebras $U^-(\widehat{\mathfrak{sl}}_n) \to K(t)$ if $t = e^{2i\pi/n}$ (*resp.* $U^-(\mathfrak{sl}_\infty) \to K(t)$ if t is generic) ([1], Prop. 4.3; see also the Appendix). Recall from 2.2 that $\{[L_O]\}$ is a basis of $R(t)$. Ariki's theorem states that under the previous isomorphism, one has $b_O \mapsto [L_O]^*$, that is, the canonical basis of $U^-(\widehat{\mathfrak{sl}}_n)$ (*resp.* $U^-(\mathfrak{sl}_\infty)$) is mapped to the basis of $K(t)$ dual to $\{[L_O]\}$.

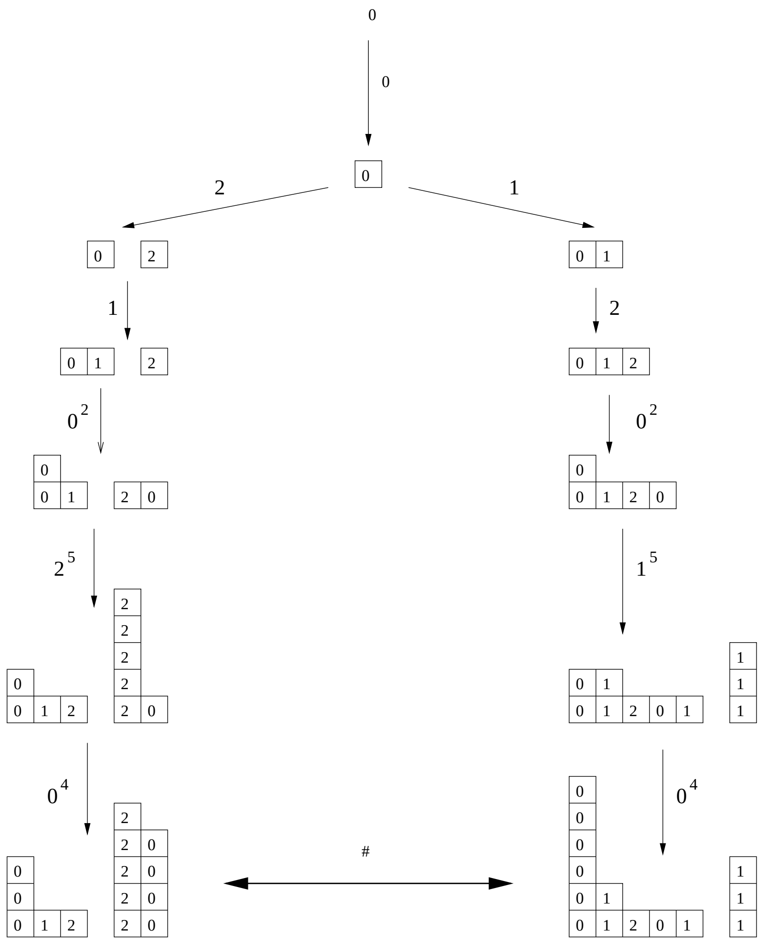

Figure 3: Computing the dual of a multisegment at a 3rd root of 1

Proposition 5.1 *Let* $\mathbf{m}$ *be a* $\mathbf{Z}$*-multisegment* (resp. *an aperiodic* $\mathbf{Z}/n\mathbf{Z}$*-multisegment*) *and let* $t \in \mathbf{C}^*$ *be generic* (resp. $t = e^{2i\pi/n}$). *Let* $L_{O_\mathbf{m}} = L_\mathbf{m}$ *be the corresponding simple* $\widehat{H}_m(t)$*-module. Then, the twisted module* $L_\mathbf{m}^\sharp$ *is isomorphic to* $L_{\mathbf{m}^\sharp}$*, where* $\mathbf{m}^\sharp$ *is the vertex of the crystal graph of* $U_v^-(\mathfrak{sl}_\infty)$ (resp. $U_v^-(\widehat{\mathfrak{sl}}_n)$) *obtained from the vertex* $\mathbf{m}$ *by the 2-fold symmetry* $i \longleftrightarrow -i$ *of the graph.*

Proof — Let $\sharp$ denote the automorphism of $U^-(\mathfrak{g})$ induced by the 2-fold symmetry $i \longleftrightarrow -i$ of the Dynkin diagram of $\mathfrak{g}$. Repeating the argument of [16] 7.3, 7.4, we see that $\sharp$ preserves the canonical basis $\{b_\mathbf{m}\}$ and maps $b_\mathbf{m}$ to $b_{\mathbf{m}^\sharp}$, where $\mathbf{m}^\sharp$ is obtained from $\mathbf{m}$ by the symmetry $i \longleftrightarrow -i$ of the crystal graph. The result then follows from Lemma 5.1 (i) and Ariki's theorem. $\square$

Using Theorem 4.1, it is then easy to describe a simple algorithm for computing $\mathbf{m}^\sharp$. Take any path $\emptyset \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdot \ \cdots \ \cdot \xrightarrow{i_k} \mathbf{m}$ from the empty multisegment to $\mathbf{m}$ in the crystal graph. Then $\emptyset \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \ \cdots \ \cdot \xrightarrow{-i_k} \mathbf{m}^\sharp$ is a path to $\mathbf{m}^\sharp$. This is illustrated in Figure 3 in the case $n = 3$.

Recall that Zelevinsky's involution τ is related to $\sharp$ by $\tau = \sharp \circ \flat$. As explained in 2.4, $\flat$ acts on a multisegment $\mathbf{m}$ by simply changing all segments $(\ell; i]$ into $[-i; \ell)$. Thus Proposition 5.1 gives as well an algorithm for τ.

Remarks 1 Consider the crystal of $U_v^-(\mathfrak{sl}_\infty)$. Among all paths joining a given $\mathbf{m}$ to $\emptyset$, there is a distinguished one obtained as follows. Let

$$S := \{(k,l) \in \mathbf{Z} \times \mathbf{N}^* \mid m_{(l,k]} \neq 0\}$$

be the support of $\mathbf{m}$, ordered lexicographically (*i.e.* $(k,l) < (k',l')$ if $k < k'$, or $k = k'$ and $l < l'$). Put $(i,\ell) := \min S$. Then there is always an arrow $\mathbf{m}^-_{\ell,i} \xrightarrow{i} \mathbf{m}$ in the graph. Applying the same procedure to $\mathbf{n} = \mathbf{m}^-_{\ell,i}$, one obtains a minimal pair (j,k) such that $\mathbf{n}^-_{k,j} \xrightarrow{j} \mathbf{n}$ is an arrow in $B(\infty)$, and this can be repeated until one eventually reaches the origin $\emptyset$. As a result, one gets a certain path from $\emptyset$ to $\mathbf{m}$. The algorithm for τ obtained by using this particular path is precisely the one described by Moeglin and Waldspurger ([22], Lemme II.3). Note that here, we need a total order on the vertices i of the quiver, and there is no obvious analog of this procedure for the cyclic case.

2 In [3], Berenstein and Zelevinsky described for $\mathfrak{g} = \mathfrak{sl}_n$ the involutions of the canonical basis of $U^-(\mathfrak{g})$ corresponding to $\sharp$ and τ via Lemma 5.1 and Ariki's theorem. They have related them to the transition map between the two indexations of the basis associated with the two extreme reduced expressions of w_0 ([3], Prop. 3.3). They also noted the coincidence with the multisegment duality coming from the representation theory of $\mathfrak{p}$-adic $GL(m)$ (see [3], 3. Remark).

3 The Mullineux involution on n-regular partitions (see 2.4) is the restriction of the map $\mathbf{m} \mapsto \mathbf{m}^\sharp$ to the subgraph of $B(\infty)$ given by the crystal of the level 1 irreducible representation with highest weight Λ_0 (*cf.* 4.4 Remarks 2, and [16]).

5.3 Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. We briefly review the geometric construction of $B(\infty)$ conjectured by Lusztig ([18], 10.4) and proved by Kashiwara-Saito [13]. Let Γ be the graph associated to the root system of $\mathfrak{g}$. Let I be the set of vertices. Fix an orientation Ω of Γ and set $H = \Omega \cup \overline{\Omega}$, where $\overline{\Omega}$ denotes the orientation opposite to Ω. For any arrow $a \in H$ let $\mathrm{in}(a)$ and $\mathrm{out}(a)$ be the input and output vertices of a. Given a I-graded complex vector space $V = \bigoplus_{i \in I} V_i$, Lusztig has introduced [18, 19]

$$X_V = \bigoplus_{a \in H} \mathrm{Hom}(V_{\mathrm{in}(a)}, V_{\mathrm{out}(a)}),$$

and as above,

$$E_{V,\Omega} = \bigoplus_{a \in \Omega} \mathrm{Hom}(V_{\mathrm{in}(a)}, V_{\mathrm{out}(a)}).$$

The space $E^*_{V,\Omega}$ is naturally identified with $E_{V,\overline{\Omega}}$, and, thus, $X_V = E_{V,\Omega} \oplus E_{V,\overline{\Omega}}$ gets identified with the cotangent bundle of $E_{V,\Omega}$. It is therefore a symplectic manifold, and the natural action of G_V on X_V is Hamiltonian, with moment map $\mu : X_V \to \mathfrak{g}_V$, where $\mathfrak{g}_V$ is the Lie algebra of G_V. The i-th component of μ is the map

$$\mu_i : X_V \to \mathrm{End}(V_i), \qquad B \mapsto \sum_{\substack{a \in H \\ \mathrm{in}(a) = i}} \epsilon(a) B_{\overline{a}} B_a,$$

where $\epsilon(a) = 1$ if $a \in \Omega$ and -1 if $a \in \overline{\Omega}$. The set $\{B \in X_V \mid \mu(B) = 0\}$ is easily seen to be the union of the conormal bundles C_O of the G_V-orbits $O \in E_{V,\Omega}$.

An element $B \in X_V$ is nilpotent if there is an integer l such that $B_{a_1} \circ B_{a_2} \circ \cdots \circ B_{a_l} = 0$ for all sequences $(a_1, a_2, ..., a_l)$ such that the above expression is meaningful. Define

$$\Lambda_V = \{B \in X_V \mid \mu(B) = 0 \text{ and } B \text{ is nilpotent}\}.$$

It is known [18, 19] that Λ_V is a closed equidimensional G_V-stable subvariety of X_V. For Γ of type A_{n-1}, A_∞ or $A^{(1)}_{n-1}$, an element B of Λ_V can be identified with a pair (x, y) of commuting nilpotent endomorphisms of V of degree -1 and $+1$ respectively. Moreover, the irreducible components of Λ_V are the closures $\overline{C_O}$ of the conormal bundles to the nilpotent G_V-orbits O in $E_{V,\Omega}$ for type A_{n-1} or A_∞ ([19], 14), and to the aperiodic nilpotent G_V-orbits for type $A^{(1)}_{n-1}$ ([19], 15).

To the I-graded space V we associate the negative root $\nu = -\sum_i (\mathrm{dim} V_i)\alpha_i$. Let $B(\infty;\nu)$ be the set of irreducible components of Λ_V. Lusztig has introduced a crystal structure on the set $\sqcup_\nu B(\infty;\nu)$, and conjectured that this graph is isomorphic to the crystal $B(\infty)$ of $U^-_v(\mathfrak{g})$ ([18], 8.9). This was proved by Kashiwara-Saito [13]. More precisely, recall that the vertices of $B(\infty)$ are identified with orbits $O \in \mathcal{O}$ for type A_{n-1} or A_∞, and aperiodic orbits $O \in \mathcal{O}$ for type $A^{(1)}_{n-1}$. Then the map $\overline{C_O} \mapsto O$ is an isomorphism of crystals $\sqcup_\nu B(\infty;\nu) \to B(\infty)$.

The algebra $U_v(\mathfrak{g})$ admits an anti-automorphism $*$ such that $e_i^* = e_i$, $f_i^* = f_i$, and $(v^h)^* = v^{-h}$, which induces an involution (still denoted by $*$) of $B(\infty)$ [11]. It is shown in [13], Section 5.3 that the involution of $B(\infty;\nu)$ induced by the transpose map $B \mapsto {}^tB$ on Λ_V coincides with $*$. (This was also conjectured by Lusztig [18], 10.4.) More precisely, if $x \in O \subseteq N_{V,\Omega}$ then $*(\overline{C_O})$ is the irreducible component containing $({}^ty, {}^tx)$, where $y \in N_{V,\overline{\Omega}}$ is generic such that $[x, y] = 0$. Summarizing the discussion, and taking into account Lemma 5.1 (ii) and Ariki's theorem, we can state:

Proposition 5.2 *Let Γ be of type $A^{(1)}_{n-1}$ or A_∞. For x in $E_{V,\Omega}$, set*

$$Z(x) = \{y \in E_{V,\overline{\Omega}} \mid x \circ y = y \circ x\}.$$

Assume that x is nilpotent (and aperiodic in type $A^{(1)}_{n-1}$). Then,
(i) there is a unique G_V-orbit $O' \subset N_{V,\overline{\Omega}}$ such that $O' \cap Z(x)$ is Zariski dense and open in $Z(x)$, and this orbit only depends on the orbit O of x;
(ii) if $O = O_{\mathbf{m}}$ and $y \in O'$, then ${}^ty \in O_{\mathbf{m}^\tau}$, where the multisegment $\mathbf{m}^\tau$ is obtained from $\mathbf{m}$ as explained in 5.2.

For Γ of type A_∞, we recover the description of Zelevinsky's involution conjectured in [33] and established in [22].

6 Appendix

In [1] Ariki uses the non-vanishing theorem for affine Hecke algebras at roots of unity announced in [7]. The purpose of this appendix is to indicate a simple proof of this non-vanishing result for type A. S. Ariki has informed one of us (E.V.) that this simple argument was known to him and to G. Lusztig. For the sake of completeness, we include it here.

Fix $m \in \mathbf{N}^*$ and put $t = e^{2i\pi/n}$. We use the same notations as in Sections 3.1 and 5.3. In particular (Γ, Ω) is the cyclic quiver with n vertices and V is a I-graded vector space of dimension $\mathbf{d} \in \mathbf{N}^{(I)}$. Let $S_\mathbf{d}$ be the set of sequences $\mathbf{i} = (i_1, i_2, ..., i_m) \in I^m$ such that $\sharp\{k \,|\, i_k = i\} = d_i$. Given such a sequence let $F_\mathbf{i}$ be the variety of flags of type $\mathbf{i}$ in V (see [19], Section 1). Let $\tilde{F}_\mathbf{i}$ be the variety of pairs $(\phi, x) \in F_\mathbf{i} \times E_{V,\Omega}$ such that ϕ is x-stable. As usual, we denote by $\mathcal{D}(X)$ the bounded derived category of complexes of $\mathbf{C}$-sheaves on a complex algebraic variety X. Let $\pi_\mathbf{i} : \tilde{F}_\mathbf{i} \to E_{V,\Omega}$ be the second projection and put $\mathcal{L}_\mathbf{i} = (\pi_\mathbf{i})_!(\mathbf{C}) \in \mathcal{D}(E_{V,\Omega})$. By definition the degree-$\mathbf{d}$ component of Lusztig's canonical basis $\{b_O\}$ is labelled by the G_V-orbits $O \subset N_{V,\Omega}$ such that a shift of the intersection cohomology complex IC_O is a direct factor in $\bigoplus_{\mathbf{i} \in S_\mathbf{d}} \mathcal{L}_\mathbf{i}$ (use [19], Section 2.4).

Let F be the variety of complete flags in $\mathbf{C}^m$. The cotangent variety T^*F is identified with the set of pairs $(\phi, x) \in F \times \mathrm{End}(\mathbf{C}^m)$ such that ϕ is x-stable. If $s \in GL(m)$ is semi-simple let $G_s \subseteq GL(m)$ be its centralizer, let $N_s \subset \mathrm{End}(\mathbf{C}^m)$ be the set of nilpotent matrices x such that $sxs^{-1} = t^{-1}x$ and let $\tilde{F}_s$ be the set of pairs $(\phi, x) \in T^*F$ such that $x \in N_s$ and the flag ϕ is fixed by s. Let $\pi_s : \tilde{F}_s \to N_s$ be the second projection and set $\mathcal{L}_s = (\pi_s)_!(\mathbf{C}) \in \mathcal{D}(N_s)$. Let $\mathbf{d} \in \mathbf{N}^{(I)}$ be such that $\sum_i d_i = m$. If s is a diagonal matrix such that t^i has the multiplicity d_i in the spectrum of s then G_s, N_s are identified with G_V, $N_{V,\Omega}$ above. In this case set $\mathcal{L}_\mathbf{d} = \mathcal{L}_s$. Then, Ginzburg's geometric construction of the simple modules of the affine Hecke algebra (see [5]) implies that the simple modules of $\widehat{H}_m(t)$ are labelled by the G_V-orbits $O \subset N_{V,\Omega}$ such that a shift of the complex IC_O is a direct factor in $\bigoplus_\mathbf{d} \mathcal{L}_\mathbf{d}$.

In our case the non-vanishing result is precisely that the orbits labelling the degree-m component of the basis $\{b_O\}$ are the same as the orbits labelling the simple $\widehat{H}_m(t)$-modules. It is a consequence of the obvious equality of the complexes $\mathcal{L}_\mathbf{d}$ and $\bigoplus_{\mathbf{i} \in S_\mathbf{d}} \mathcal{L}_\mathbf{i}$ (simply observe that $F_\mathbf{d} = \sqcup_{\mathbf{i} \in S_\mathbf{d}} F_\mathbf{i}$).

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