# Canonical Bases of $q$-Deformed Fock Spaces

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**Abstract**

We define a canonical basis of the $q$-deformed Fock space representation of the affine Lie algebra $\widehat{\mathfrak{gl}}_n$. We conjecture that the entries of the transition matrix between this basis and the natural basis of the Fock space are $q$-analogues of decomposition numbers of the $q$-Schur algebras for $q$ specialized to a $n$th root of unity.

## 1 Introduction

The Fock space representation $\mathcal{F}$ of $\widehat{\mathfrak{sl}}_n$ is not irreducible. Its decomposition into simple $\widehat{\mathfrak{sl}}_n$-modules is given by [3]

$$\mathcal{F} \cong \bigoplus_{k\geq 0} M(\Lambda_0 - k\delta)^{\oplus p(k)}\,, \tag{1}$$

where $p(k)$ denotes the number of partitions $\lambda$ of $k$. Hence it is not obvious to apply Kashiwara's or Lusztig's method to define a canonical basis of $\mathcal{F}$.

In this note, we shall rather regard $\mathcal{F}$ as a representation of the enlarged algebra $\widehat{\mathfrak{gl}}_n$. Indeed, $\mathcal{F}$ is a simple $\widehat{\mathfrak{gl}}_n$-module, and a $q$-deformation $\mathcal{F}_q$ of this representation has been described in [11]. One can then define a natural semi-linear involution $v \to \overline{v}$ commuting with the action of the lowering operators of $\mathcal{F}_q$ and leaving invariant its highest weight vector. Using this involution, one obtains in an elementary way a canonical basis $\{G(\lambda)\}$ of $\mathcal{F}_q$.

By restriction to $U_q(\widehat{\mathfrak{sl}}_n)$, the space $\mathcal{F}_q$ decomposes similarly as

$$\mathcal{F}_q \cong \bigoplus_{k\geq 0} M_q(\Lambda_0 - k\delta)^{\oplus p(k)}\,, \tag{2}$$

where $M_q(\Lambda)$ denotes the simple $U_q(\widehat{\mathfrak{sl}}_n)$-module with highest weight $\Lambda$. The $G(\mu)$ indexed by $n$-regular partitions $\mu$ coincide with the elements of Kashiwara's global crystal basis of the basic representation $M(\Lambda_0)$. But we insist that the rest of our basis is not compatible with the decomposition (2).

We conjecture that for $q = 1$, the coefficients of the transition matrix of our canonical basis on the natural basis of the Fock space are equal to the decomposition numbers of $q$-Schur algebras at a $n$-th root of unity. A previous conjecture [14] on decomposition matrices of Hecke algebras having been recently established by Ariki and by Grojnowski, we can prove, by means of a $q$-analogue of Steinberg's tensor product theorem, that the present conjecture is verified for a large proportion of the decomposition numbers. Also, our basis can be explicitely computed by solving a system of linear equations.

Our notations for symmetric functions are as in [17]. Other notations are as in [19], except for the replacement of $q$ by $1/q$.

## 2 A $q$-analogue of the Fock space representation of $\widehat{\mathfrak{gl}}_n$

The Lie algebra $\widehat{\mathfrak{gl}}_n$ can be regarded as the sum $\widehat{\mathfrak{sl}}_n + \mathcal{H}_n$ where $\mathcal{H}_n$ is a Heisenberg algebra commuting with $\widehat{\mathfrak{sl}}_n'$ and such that $\widehat{\mathfrak{sl}}_n' \cap \mathcal{H}_n = \mathbf{C}c$, where $c$ is the central generator [4]. The bosonic Fock space

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$\mathcal{F} = \mathbf{C}[x_1, x_2, \ldots]$ can be interpreted as the algebra of symmetric functions in some infinite set of variables $t_i$ by means of the correspondence $x_k = \frac{1}{k} p_k$, where $p_k = \sum_i t_i^k$ are the power sums.

With this interpretation, the action of $\widehat{\mathfrak{gl}}_n$ on $\mathcal{F}$ can be described as follows. The generator $B_k$ of $\mathcal{H}_n$ acts by $nk \frac{\partial}{\partial p_{nk}}$ for $k > 0$ and as the multiplication by $p_{-nk}$ for $k < 0$. The action of the generators of $\widehat{\mathfrak{sl}}_n$ is particularly simple in the basis of Schur functions $s_\lambda$. For a node $\gamma$ of the Young diagram of a partition $\lambda$, located at the intersection of the $i$th row and the $j$th column of $\lambda$, define its residue (or colour) $r(\gamma) \in \{0, 1, \ldots, n-1\}$ as $r(\gamma) = j - i \bmod n$. Then, $e_i s_\lambda = \sum s_\nu$, $f_i s_\lambda = \sum s_\mu$, where $\nu$ (resp. $\mu$) runs through all partitions obtained from $\lambda$ by removing (resp. adding) a node of residue $i$.

The $q$-analogue of the Fock space representation of $\widehat{\mathfrak{sl}}_n$, in the form of a Fock space representation of the quantized enveloping algebra $U_q(\widehat{\mathfrak{sl}}_n)$, has been constructed by Hayashi [7] and further investigated by Misra and Miwa [18] who constructed the crystal basis. A quantum analogue $\mathcal{H}_n^q$ of the Heisenberg algebra $\mathcal{H}_n$ commuting with $U_q(\widehat{\mathfrak{sl}}_n')$ has been constructed only recently by Kashiwara, Miwa and Stern [11]. The sum $\mathbf{U} = U_q(\widehat{\mathfrak{sl}}_n) + \mathcal{H}_n^q$ is thus a $q$-analogue of the enveloping algebra $U(\widehat{\mathfrak{gl}}_n)$ and it would be interesting to compare it with the standard definition of $U_q(\widehat{\mathfrak{gl}}_n)$. The Fock space is an irreducible $\mathbf{U}$-module, and the canonical basis constructed in this paper will be adapted to this representation.

The Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ can be described as follows (note that our conventions are sligthly different from those of [18] and [11]). Let

$$\mathcal{F}_q = \bigoplus_\lambda \mathbb{Q}(q)|\lambda\rangle$$

be the $\mathbb{Q}(q)$ vector space with basis $|\lambda\rangle$ indexed by the set of all partitions. Let $\lambda$ and $\mu$ be two partitions such that $\mu$ is obtained from $\lambda$ by adding a node $\gamma$ of residue $i$. Let $I_i(\lambda)$ be the number of indent $i$-nodes of $\lambda$, $R_i(\lambda)$ the number of its removable $i$-nodes, $I_i^l(\lambda,\mu)$ (resp. $R_i^l(\lambda,\mu)$) the number of indent $i$-nodes (resp. of removable $i$-nodes) situated to the left of $\gamma$ ($\gamma$ not included), and similarly, let $I_i^r(\lambda,\mu)$ and $R_i^r(\lambda,\mu)$ be the corresponding numbers for nodes located on the right of $\gamma$. Set $N_i(\lambda) = I_i(\lambda) - R_i(\lambda)$, $N_i^l(\lambda,\mu) = I_i^l(\lambda,\mu) - R_i^l(\lambda,\mu)$ and $N_i^r(\lambda,\mu) = I_i^r(\lambda,\mu) - R_i^r(\lambda,\mu)$. Then,

$$f_i|\lambda\rangle = \sum_\mu q^{N_i^r(\lambda,\mu)}|\mu\rangle\,, \qquad e_i|\mu\rangle = \sum_\lambda q^{N_i^l(\lambda,\mu)}|\lambda\rangle \tag{3}$$

where in each case the sum is over all partitions such that $\mu/\lambda$ is a $i$-node,

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle \quad \text{and} \quad q^D|\lambda\rangle = q^{-N^0(\lambda)}|\lambda\rangle \tag{4}$$

where $D$ is the degree generator and $N^0(\lambda)$ the total number of 0-nodes of $\lambda$.

The $q$-Heisenberg algebra $\mathcal{H}_n^q$ of [11] is defined by means of Stern's construction of semi-infinite $q$-wedges [19]. The basis vector $|\lambda\rangle$ is interpreted as the semi-infinite wedge $u_I = u_{i_1} \wedge u_{i_2} \wedge \cdots$ where the sequence $I$ is defined by $i_k = \lambda_k - k + 1$. With our conventions, the commutation relations for $q$-wedges are the following. Suppose that $\ell < m$ and that $\ell - m \bmod n = i$. If $i = 0$ then $u_\ell \wedge u_m = -u_m \wedge u_\ell$ and otherwise,

$$u_\ell \wedge u_m = -\frac{1}{q} u_m \wedge u_\ell + \left(\frac{1}{q^2} - 1\right)\left(u_{m-i} \wedge u_{\ell+i} - \frac{1}{q} u_{m-n} \wedge u_{\ell+n} + \frac{1}{q^2} u_{m-n-i} \wedge u_{m+n+i} + \cdots\right)$$

where in the last sum one retains only the normally ordered terms.

Then, the generator $B_k$ of $\mathcal{H}_n^q$ acts on $u_I$ by

$$B_k u_I = \sum_{i \geq 1} u_{I + kn\delta^i} \tag{5}$$

where $\delta^i$ is the sequence $(\delta_j^i)_{j \geq 1}$ (Kronecker symbols). The semi-infinite wedge $u_I$ will be called the fermionic realization of the basis vector $|\lambda\rangle$. These operators verify the relations [11]

$$[B_k, B_\ell] = k \frac{1 - q^{-2nk}}{1 - q^{-2k}} \delta_{k+\ell,0}\ . \tag{6}$$

The action of $\mathcal{H}_n^q$ in the bosonic picture is better described in terms of other operators $U_k, V_k \in U(\mathcal{H}_n^q)$ [15]. Recall that for $k > 0$, $B_{-k}$ is a $q$-analogue of multiplication by $p_{kn} = p_k(t_1^n, t_2^n, \ldots) = \psi^n(p_k)$, where $\psi_n$ is the ring homomorphism raising the variables to the $n$th power. On the graphical representation by Young diagrams, multiplication by the complete homogeneous functions $\psi_n(h_k) = h_k(t_1^n, t_2^n, \ldots)$ of $n$th powers has a simple combinatorial description. Let

$$V_k = \sum_{m_1+2m_2+\cdots+km_k=k} \frac{1}{m_1!m_2!\cdots m_k!} \left(\frac{B_{-1}}{1}\right)^{m_1} \left(\frac{B_{-2}}{2}\right)^{m_2} \cdots \left(\frac{B_{-k}}{k}\right)^{m_k} \tag{7}$$

be the $q$-analogue of the multplication operator by $\psi^n(h_k)$. Then,

$$V_k|\lambda\rangle = \sum q^{-\mathbf{h}(\mu/\lambda)}|\mu\rangle \tag{8}$$

where the sum is over all partitions $\mu$ such that $\mu/\lambda$ is a horizontal $n$-ribbon strip of weight $k$, and

$$\mathbf{h}(\mu/\lambda) = \sum_R (\mathrm{ht}\,(R) - 1)$$

where the sum is over all the $n$ribbons $R$ tiling $\mu/\lambda$, $\mathrm{ht}\,(R)$ being the height of the ribbon $R$ (see [15]).

The scalar product on $\mathcal{F}_q$ is defined by $\langle\lambda|\mu\rangle = \delta_{\lambda\mu}$. The adjoint operator $U_k$ of $V_k$ acts by

$$U_k|\mu\rangle = \sum q^{-\mathbf{h}(\mu/\lambda)}|\lambda\rangle \tag{9}$$

where the sum is over all partitions $\lambda$ such that $\mu/\lambda$ is a horizontal $n$-ribbon strip of weight $k$.

Identifying $\mathcal{F}_q$ with $\mathbb{Q}(q) \otimes Sym$ by setting $|\lambda\rangle = s_\lambda$, one can define a linear operator

$$\psi_q^n : \mathcal{F}_q \longrightarrow \mathcal{F}_q$$

by specifying the image of the basis $(h_\lambda)$ as

$$\psi_q^n(h_\lambda) = V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_r}|\emptyset\rangle \ . \tag{10}$$

Then, the image $(\psi_q^n(g_\lambda))$ of any basis $(g_\lambda)$ of symmetric functions will be a basis of the space of $U_q(\widehat{\mathfrak{sl}}_n)$-highest weight vectors in $\mathcal{F}_q$.

# 3 An involution of the Fock space

Let $I \in \mathcal{J}_m$, and let $u_I$ be the associated basis vector of $\mathcal{F}_q$. We denote by $\alpha_{n,k}(I)$ the number of pairs $(r, s)$ with $1 \le r < s \le k$ and $r - s \not\equiv 0 \bmod n$.

**Proposition 3.1** *For $k \ge m$, the $q$-wedge*

$$\overline{u_I} = (-1)^{\binom{k}{2}} q^{\alpha_{n,k}(I)} u_{i_k} \wedge u_{i_{k-1}} \wedge \cdots \wedge u_{i_1} \wedge u_{i_{k+1}} \wedge u_{i_{k+2}} \wedge \cdots$$

*is independent of $k$.*

Define a semi-linear map $v \mapsto \overline{v}$ in $\mathcal{F}_q$ by

$$\overline{\sum_{I\in\mathcal{J}} \varphi_I(q) u_I} = \sum_{I\in\mathcal{J}} \varphi_I(q^{-1})\overline{u_I} \ . \tag{11}$$

**Theorem 3.2** (i) *$v \mapsto \overline{v}$ is an involution of $\mathcal{F}_q$.*

(ii) *$\overline{f_i v} = f_i \overline{v}$ and $\overline{B_{-k} v} = B_{-k}\overline{v}$ ($v \in \mathcal{F}_q$, $i \in \{0, \ldots, n-1\}$, $k > 0$).*

We note that there is a unique semi-linear map satisfying (ii) and $\overline{|\emptyset\rangle} = |\emptyset\rangle$. This implies that the restriction of the involution $v \mapsto \overline{v}$ to the subspace $M(\Lambda_0)$ of $\mathcal{F}_q$ coincides with the usual involution in terms of which the global crystal basis of $M(\Lambda_0)$ is defined.

Let $\lambda \vdash m$. Set

$$\overline{|\lambda\rangle} = \sum_{\mu \vdash m} a_{\lambda\mu}(q)|\mu\rangle \ .$$

**Theorem 3.3** (i) $a_{\lambda\mu}(q) \in \mathbb{Z}[q, q^{-}1]$.
(ii) $a_{\lambda\mu}(q) = 0$ *unless* $\lambda \trianglelefteq \mu$ *and* $\lambda$, $\mu$ *have the same* $n$*-core.*
(iii) $a_{\lambda\lambda}(q) = 1$.
(iv) $a_{\lambda\mu}(q) = a_{\mu'\lambda'}(q)$.

For $n = 2$, the matrices $A_m(q)$ of the involution in the subspaces spanned by partitions of weight $m$ for $m = 2, 3, 4$ are

$$\begin{bmatrix} 1 & 0 \\ \frac{q^2-1}{q} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{q^2-1}{q} & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{q^2-1}{q} & 1 & 0 & 0 & 0 \\ -\frac{q^2-1}{q^2} & \frac{q^2-1}{q} & 1 & 0 & 0 \\ 0 & q^2-1 & \frac{q^2-1}{q} & 1 & 0 \\ q^2-1 & 0 & -\frac{q^2-1}{q^2} & \frac{q^2-1}{q} & 1 \end{bmatrix}$$

(partitions are ordered in reverse lexicographic order, e.g., for $m = 4$, $(4), (31), (22), (211), (1111)$).

# 4 Canonical bases

Let $L$ (resp. $L^-$) be the $\mathbb{Z}[q]$ (resp. $\mathbb{Z}[q^{-1}]$)-lattice in $\mathcal{F}_q$ with basis $\{|\lambda\rangle\}$. Using theorems 3.2 and 3.3 one can construct "IC-bases of $\mathcal{F}_q$", in the terminology of Du [5].

**Theorem 4.1** *There exist bases* $\{G(\lambda)\}$, $\{G^-(\lambda)\}$ *of* $\mathcal{F}_q$ *characterized by:*
(i) $\overline{G(\lambda)} = G(\lambda)$, $\overline{G^-(\lambda)} = G^-(\lambda)$,
(ii) $G(\lambda) \equiv |\lambda\rangle \bmod qL$, $G^-(\lambda) \equiv |\lambda\rangle \bmod q^{-1}L^-$.

Set $G(\lambda) = \sum_\mu d_{\lambda\mu}(q)|\mu\rangle$ and $G^-(\lambda) = \sum_\mu e_{\lambda\mu}(q)|\mu\rangle$. Then $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$, $e_{\lambda\mu}(q) \in \mathbb{Z}[q^{-1}]$, and these polynomials are nonzero only if $\lambda \trianglelefteq \mu$ and $\lambda, \mu$ have the same $n$-core. Moreover, $d_{\lambda\lambda}(q) = e_{\lambda\lambda}(q) = 1$. Also, $\{G(\lambda) \mid \lambda \ n\text{-regular}\}$ coincides with the lower crystal basis of the basic representation $M(\Lambda_0)$ of $U_q(\widehat{\mathfrak{sl}}_n)$.

Let $\{G^\dagger(\lambda)\}$ be the adjoint basis of $\{G(\lambda)\}$ with respect to the scalar product of $\mathcal{F}_q$. It follows from theorem 3.3 (iv) that

$$G^\dagger(\lambda)' = G^-(\lambda') \tag{12}$$

where $v \mapsto v'$ denotes the semi-linear involution of $\mathcal{F}_q$ defined by $|\lambda\rangle' = |\lambda'\rangle$. Thus, for $\lambda$ $n$-regular,

$$G^-(\lambda') \equiv \left(q^{n|\lambda|}G^{\mathrm{up}}(\lambda)\right)' \bmod M(\Lambda_0)^\perp \ , \tag{13}$$

where $\{G^{\mathrm{up}}(\lambda)\}$ is the upper global crystal basis of $M(\Lambda_0)$ (cf. [14]).

We set $G^\dagger(\lambda) = \sum_\mu c_{\lambda\mu}(q)|\mu\rangle$, so that $c_{\lambda\mu}(q) = e_{\lambda'\mu'}(q)$ and $[c_{\lambda\mu}(q)] = [d_{\mu\lambda}(q)]^{-1}$.

For $n = 2$, the canonical basis is given up to partitions of 6 by the following matrices.

| | | |
|---|---|---|
| 2 | 1 | 0 |
| 11 | $q$ | 1 |

| | | | |
|---|---|---|---|
| 3 | 1 | 0 | 0 |
| 21 | 0 | 1 | 0 |
| 111 | $q$ | 0 | 1 |

| | | | | | |
|---|---|---|---|---|---|
| 4 | 1 | 0 | 0 | 0 | 0 |
| 31 | $q$ | 1 | 0 | 0 | 0 |
| 22 | 0 | $q$ | 1 | 0 | 0 |
| 211 | $q$ | $q^2$ | $q$ | 1 | 0 |
| 1111 | $q^2$ | 0 | 0 | $q$ | 1 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 5 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 41 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 32 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 311 | $q$ | 0 | $q$ | 1 | 0 | 0 | 0 |
| 221 | 0 | 0 | $q^2$ | $q$ | 1 | 0 | 0 |
| 2111 | 0 | $q$ | 0 | 0 | 0 | 1 | 0 |
| 11111 | $q^2$ | 0 | 0 | $q$ | 0 | 0 | 1 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 51 | $q$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 42 | 0 | $q$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 411 | $q$ | $q^2$ | $q$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 33 | 0 | 0 | $q$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 321 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 3111 | $q^2$ | $q$ | $q^2$ | $q$ | $q$ | 0 | 1 | 0 | 0 | 0 | 0 |
| 222 | 0 | 0 | $q^2$ | $q$ | $q$ | 0 | 0 | 1 | 0 | 0 | 0 |
| 2211 | 0 | $q^2$ | $q^3$ | $q^2$ | $q^2$ | 0 | $q$ | $q$ | 1 | 0 | 0 |
| 21111 | $q^2$ | $q^3$ | 0 | $q$ | 0 | 0 | $q^2$ | 0 | $q$ | 1 | 0 |
| 111111 | $q^3$ | 0 | 0 | $q^2$ | 0 | 0 | 0 | 0 | 0 | $q$ | 1 |

# 5 A $q$-analogue of Steinberg's tensor product theorem

Let $\alpha$ be a partition of $r$. Define an element $S_\alpha$ of $\mathbf{U}$ by

$$S_\alpha = \sum_{\beta \vdash r} \frac{\chi^\alpha_\beta}{z_\beta} B_{-\beta_1} B_{-\beta_2} \cdots B_{-\beta_k} \tag{14}$$

where $\beta = (\beta_1, \ldots, \beta_k) = (1^{m_1} 2^{m_2} \ldots r^{m_r})$, $z_\beta = 1^{m_1} m_1! \cdots r^{m_r} m_r!$ and $\chi^\alpha_\beta$ is the value of the irreducible character $\chi^\alpha$ of the symmetric group evaluated on a permutation of cycle type $\beta$.

The operator $S_\alpha$ is a $q$-analogue of the operator of multiplication by the plethysms $\psi^n(s_\alpha)$ in the ring of symmetric functions.

Let $\lambda$ be a partition such that $\lambda'$ is $n$-singular. We can write $\lambda = \mu + n\alpha$ where $n\alpha = (n\alpha_1, n\alpha_2, \ldots)$ and $\mu'$ is $n$-regular.

**Theorem 5.1** $G^-(\lambda) = S_\alpha\,(G^-(\mu))$.

This reduces the computation of $\{G^-(\lambda)\}$ to that of the subfamily $\{G^-(\mu) \,|\, \mu \ n\text{-regular}\}$.

**Conjecture 5.2** *The matrix $[d_{\lambda\mu}(1)]_{\lambda,\mu\vdash m}$ is equal to the decomposition matrix $\mathbf{D}_m$ of the $v$-Schur algebra $\mathcal{S}_m$ over a field of characteristic 0, for $v$ a primitive $n$-th root of unity.*

This conjecture, which generalizes conjecture 6.9 of [14], is already verified to a large extent. Indeed, on the one hand, Ariki has proved our previous conjecture [1], which means that the $d_{\lambda\mu}(1)$ for $\mu$ $n$-regular are equal to the corresponding decomposition numbers of $\mathcal{S}_m$. On the other hand, it follows from results of James [8] that the entries of the inverse matrices $\mathbf{D}_m^{-1}$ satify the same properties as those deduced from theorem 5.1 for the coefficients $c_{\lambda\mu}(1)$. Since we have verified that $\mathbf{D}_m^{-1} = {}^t[c_{\lambda\mu}(1)]$ for $m \le 10$ (any $n$), we deduce that an infinite number of $c_{\lambda\mu}(1)$ coincide with the corresponding entries of $\mathbf{D}_m^{-1}$.

The following refined conjecture has also been checked for small $m$. It generalizes the conjecture of section 9 of [14], due to Rouquier.

**Conjecture 5.3**

$$d_{\lambda\mu}(q) = \sum_{i\ge 0} [W(\lambda)^i / W(\lambda)^{i+1} : L(\mu)]\, q^i$$

*where* $(W(\lambda)^i)$ *is the Jantzen filtration of the Weyl module* $W(\lambda)$*, and* $L(\mu)$ *is the irreducible module corresponding to* $\mu$*.*

Finally, we note the following combinatorial description of certain polynomials $e_{\lambda\mu}(q)$ in the case $n = 2$, which proves that they are equal to the $q$-analogues of Littlewood-Richardson coefficients introduced in [2].

**Theorem 5.4** *Let* $n = 2$*. One has*

$$e_{2\lambda,\mu}(q) = \varepsilon_2(\mu) \sum_{T \in \mathrm{Yam}_2(\mu,\lambda)} q^{-2\mathrm{spin}}\,(T)$$

*where* $\mathrm{Yam}_2(\mu, \lambda)$ *denotes the set of Yamanouchi domino tableaux of shape* $\mu$ *and weight* $\lambda$*.*

## References


[1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, Preprint 1996.

[2] C. Carré and B. Leclerc, *Splitting the square of a Schur function into its symmetric and antisymmetric parts*, J. Alg. Comb. **??** (19??), ??-??.

[3] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in "Non-linear Integrable Systems – Classical Theory and Quantum Theory", Proceedings of RIMS Symposium, World Scientificm 1983, 39-119.

[4] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Paths, Maya diagrams, and representations of* $\widehat{\mathfrak{sl}}(r,C)$, Adv. Stud. Pure Math. **19** (1989), 149-191.

[5] J. Du, *??*

[6] V. Ginzburg, N. Reshetikhin and E. Vasserot, *quantum groups and flag varieties*, Contemp. Math. **175** (1994), 101-130.

[7] T. Hayashi, *q-analogues of Clifford and Weyl algebras - spinor and oscillator representations of quantum enveloping algebras*, Commun. Math. Phys., **127** (1990), 129-144.

[8] G. James, *The decomposition matrices of $GL_n(q)$ for $n \leq 10$*, Proc. London Math. Soc., **60** (1990), 225-265.

[9] G. James and A. Mathas, *A q-analogue of the Jantzen-Schaper theorem*, Preprint 1995.

[10] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.

[11] M. Kashiwara, T. Miwa and E. Stern, *Decomposition of q-deformed Fock spaces*, preprint RIMS-1035 (1995).

[12] D. Kazhdan and G. Lusztig, *Affine Lie algebras and quantum groups*, Int. Math. Res. Not. **2** (1991), 21-29.

[13] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Une conjecture pour le calcul des matrices de décomposition des algèbres de Hecke de type A aux racines de l'unité*, C. R. Acad. Sci. Paris **321** (1995), 511-516.

[14] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Commun. Math. Phys. (to appear).

[15] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties*, preprint q-alg/9512031.

[16] G. Lusztig, *Modular representations and quantum groups*, Contemporary Math. **82** (1989), 59-77.

[17] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford 1995.

[18] K.C. Misra and T. Miwa, *Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}_n)$*, Commun. Math. Phys. **134** (1990), 79-88.

[19] E. Stern, *Semi-infinite wedges and vertex operators*, Internat. Math. Research Notices **4** (1995), 201-220.