# Combinatorics of solvable lattice models, and modular representations of Hecke algebras

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*Dedicated to the memory of Abdus Salam*

**Abstract**

We review and motivate recently-observed relationships between exactly solvable lattice models and modular representations of Hecke algebras. Firstly, we describe how the set of $n$-regular partitions label both of the following classes of objects:
1. The spectrum of unrestricted solid-on-solid lattice models based on level-1 representations of the affine algebras $\widehat{\mathfrak{sl}}_n$,
2. The irreducible representations of type-A Hecke algebras at roots of unity: $H_m(\sqrt[n]{1})$.

Secondly, we show that a certain subset of the $n$-regular partitions label both of the following classes of objects:
1. The spectrum of restricted solid-on-solid lattice models based on cosets of affine algebras $C[n,1,1] = (\widehat{\mathfrak{sl}}_n)_1 \times (\widehat{\mathfrak{sl}}_n)_1 \; / \; (\widehat{\mathfrak{sl}}_n)_2$,
2. Jantzen-Seitz (JS) representations of $H_m(\sqrt[n]{1})$: irreducible representations that remain irreducible under restriction to $H_{m-1}(\sqrt[n]{1})$.

Using the above relationships, we characterise the JS representations of $H_m(\sqrt[n]{1})$ and show that the generating series that count them are branching functions of $\widehat{\mathfrak{sl}}_n$.

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# 0 Introduction

Exactly-solvable models in statistical mechanics and quantum field theory[1] offer concrete physical realisations of abstract mathematical structures. One can study them either from a physical or from a mathematical point of view.

From a physical point of view, one can say that though these models are highly non-trivial (they contain an infinite number of interacting degrees of freedom) they are solvable[2] because they are based on rich and consistent algebraic structures[3]. The most notable of these structures are the affine and Virasoro algebras[4]. Physicists are typically not very surprised by that. They are used to the fact that underlying all meaningful physical models there exist consistent mathematics, and they take it for granted that mathematics is useful to physics.

From a mathematical point of view, one can say that these models offer concrete realisations of highly abstract algebraic structures. Contrary to the physicists, mathematicians tend to be surprised by the fact that their abstractions turn out to be relevant to physical problems.

These relationships are important because they imply that one can establish a dictionary between the two pictures: the physical and the mathematical. For example, one can establish a dictionary between the representation theory of Lie algebras, their affine extensions, and their deformations, on the one hand, and the theory of exactly solvable lattice models on the other hand.

The above relationships have been repeatedly used as a guiding principle that enriched both subjects. For example, computations of short distance correlations in lattice models are based entirely on techniques developed in the context of the representation theory of highest weight modules [14, 27], while intuition gained from the physics of solvable lattice models has initiated the theory of quantum groups [9].

This motivates us to look for other examples of correspondences between the two subjects: exactly solvable models, and representation theory. In [18] we have described such a correspondence between the $(\widehat{\mathfrak{sl}}_n)_1 \times (\widehat{\mathfrak{sl}}_n)_1/(\widehat{\mathfrak{sl}}_n)_2$ coset models and representations of the symmetric group over a field of characteristic $n$ and of Hecke algebras at an $n$th root of 1 [5]. In this work, we present an

[1] For an overview of exactly solvable models, see [23].

[2] Solvable here means that one can—at least in principle—compute any physical quantity *exactly*. Exactly here means *completely*, in the sense of not being an approximation of anything in some perturbation expansion. However, *exactly* does not mean *rigorously*, since these computations typically make heavy use of certain assumptions. In other words, in principle, our computations are either exactly correct, or exactly wrong! The issue is usually settled by heavy numerical computations, or even comparisons with actual experimental results. However, none of this need to worry us. In this work, we are only interested in mathematical aspects of these models.

[3] For an introduction to the algebraic approach to exactly solvable lattice models, see [30].

[4] Standard references on affine and Virasoro algebras are [32, 33]. A particularly readable introduction to the subject is [37].

[5] Hecke algebras play an important role in the theory of exactly solvable models. For a

introduction to [18], and a review of its context.

Let us now describe briefly the contents of this paper. In Section 1, we introduce a class of exactly solvable 2-dimensional statistical mechanical models: the ABF models, outline how Baxter's corner transfer matrix (CTM) method can be used to reduce the computation of certain physical quantities, the 1-point functions, to a problem in $q$-counting: computing the generating function of a class of combinatorial objects called *paths*[6].

Solving the $q$-counting problem, we observe that the result is a character of the Virasoro algebra. This observation is by now classical, and is due to [12]. By the Goddard-Kent-Olive construction [20], the same expressions can also be obtained as branching functions of tensor products of level $\ell$ and level 1 representations of $\widehat{\mathfrak{sl}}_2$. It is in this sense that these models can be regarded as coset models of type $C[2,\ell,1] := (\widehat{\mathfrak{sl}}_2)_\ell \times (\widehat{\mathfrak{sl}}_2)_1/(\widehat{\mathfrak{sl}}_2)_{\ell+1}$.

Once the connection with the affine algebra $\widehat{\mathfrak{sl}}_2$ is observed, it is natural to seek the corresponding models that are related to $\widehat{\mathfrak{sl}}_n$. In Section 2, we introduce the representation theory and the combinatorics underlying these models, the JMO models based on the cosets $C[n,\ell,1]$. In this case, the CTM method can also be applied to reduce the computation of 1-point functions to the $q$-enumeration of a class of weighted paths [12]. The relevant highest weight modules are now those of the $\mathcal{W}_n$-algebras whose characters arise as branching functions of the coset $C[n,\ell,1]$.

In the case $\ell = 1$, another combinatorial description has been obtained in [19] in terms of coloured partitions. These partitions are characterised by certain simple conditions. Let us refer to these conditions as the FOW-conditions, and denote the set of these partitions by $FOW(n,j,k)$, where $n$ stands for $\widehat{\mathfrak{sl}}_n$ and $\{j,k\}$ determine the branching function which is computed, namely $b^{\Lambda_k+\Lambda_{j-k}}_{\Lambda_j,\Lambda_0}$.

It turns out that the FOW-partitions have a definite meaning in the representation theory of the symmetric groups over a field of characteristic $n$. In particular, if one disregards the $\{j,k\}$ labels in the FOW-conditions, one obtains the conditions satisfied by the Jantzen-Seitz (JS) partitions [26]. These are the partitions that label irreducible representations of a symmetric group over $\mathbb{F}_n$ that remain irreducible under restriction to the symmetric subgroup of immediately lower rank. Let us denote these partitions by $JS(n)$.

The first goal of this work is to explain why

$$\coprod_{j,k} FOW(n,j,k) = JS(n). \tag{1}$$

review, see [44]. The connections introduced in [18] are specifically related to the theory of modular representations, such as the JS irreducibility conditions, the concepts of cores, etc.

[6] Combinatorialists are familiar with these objects. They are called *Dyck paths*. However, the paths that we are interested in counting will satisfy extra conditions. In particular, they are required to start at a certain height, end at a certain height, do not go below the $x$-axis, and do not go above a certain model-dependent height.

In Section 3, we review some combinatorial aspects of the representations of symmetric groups over fields of characteristic $n$ and Hecke algebras $H_m(\sqrt[n]{1})$. Here of course, $n$ is assumed to be prime in the case of symmetric groups. Because our statistical mechanical models are based on affine algebras $\widehat{\mathfrak{sl}}_n$ for all $n > 1$, the Hecke algebras form the natural context in which we can work. We describe in particular the Jantzen-Seitz problem for representations of symmetric groups, whose complete solution was recently given by Kleshchev [39], and formulate a similar problem for Hecke algebras.

One of the main ingredients in our solution of the Hecke JS problem is a relationship between the representation theory of Hecke algebras $H_m(\sqrt[n]{1})$ and the basic representation of the affine algebra $\widehat{\mathfrak{sl}}_n$. Namely, the sum over $m$ of the Grothendieck groups associated with $H_m(\sqrt[n]{1})$ can be identified with the irreducible representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$, with Robinson's $i$-induction and $i$-restriction operators acting as the Chevalley generators. This relationship was described in [41, 42]. Moreover, it was conjectured there that Kashiwara's upper global crystal basis of $V(\Lambda_0)$ coincides with the natural basis of the sum of Grothendieck groups given by the classes of simple modules. This conjecture was proven by Ariki [2], and independently by Grojnowski using results of [21].

In Section 4, we review the basic facts that we need from the representation theory of quantum affine algebras. In particular, we describe the Fock space representations of $U_q(\widehat{\mathfrak{sl}}_n)$. The theory of crystal bases is reviewed in Section 5, and its relevance to the representation theory of Hecke algebras is explained.

In Section 6, using the above-mentioned relationship, we show that the Jantzen-Seitz problem for the Hecke algebras $H_m(\sqrt[n]{1})$ is equivalent to the computation via crystal graphs of the decomposition of tensor products of level 1 $\widehat{\mathfrak{sl}}_n$-modules. These are precisely the tensor products arising in the coset models $C[n, 1, 1]$ studied in [19]. Therefore we obtain the desired explanation of (1). Moreover, building upon results of [19], we can express the generating functions of JS-partitions having a given $n$-core in terms of branching functions of $\widehat{\mathfrak{sl}}_n$. This is to be compared with a well-known result on blocks of Hecke algebras. Indeed, the blocks of $H_m(\sqrt[n]{1})$ are labelled by $n$-cores, and the dimension of a block is the number of $n$-regular partitions of $m$ with the corresponding $n$-core. Using a formula first proved by Robinson (in the symmetric group case) one can compute the generating series of the dimensions of all blocks labelled by a given $n$-core, and recognize the string function of the level 1 $\widehat{\mathfrak{sl}}_n$-modules.

Thus our result shows that some branching functions of $\widehat{\mathfrak{sl}}_n$ other than the level 1 string function arise in a natural way in the modular representations of $H_m(\sqrt[n]{1})$. We conclude by discussing possible generalizations (Section 7).

In Appendix A, we recall a description of the Specht modules of $H_m(v)$.

# 1 From statistical mechanics to Virasoro highest weight modules

## 1.1 Introduction

A central activity in statistical mechanics is to describe critical phenomena in terms of exactly solvable lattice models. The most well-known of these models is the 2-dimensional Ising model, whose partition function was exactly calculated by Onsager in 1944 [46].

In 1980, the hard hexagon model was solved by Baxter [4]. In calculating order parameters, he found remarkable connections with pure mathematics: $q$-series identities of the Rogers-Ramanujan type, modular functions, etc. In 1984, this model was extended by Andrews, Baxter and Forrester (ABF) to an infinite series of models [1].

What ABF did can be summarised as follows. They constructed a series of restricted solid-on-solid (RSOS) models labelled by a positive integer $L$. Their Boltzmann weights are parametrised using elliptic functions, and satisfy the star-triangle relation or Yang-Baxter equation. Using the corner transfer matrix (CTM) method, they calculated certain physical quantities called local height probabilities (LHP's). From the latter, they computed the critical exponents: physical quantities that can be measured in actual experiments.

Subsequently, the Kyoto group [11] realised that the generating functions of certain combinatorial objects: the so-called *paths*, or *1-dimensional configurations*, that appear in the course of calculating the LHP's are precisely the branching functions for cosets of the affine algebras: $(\widehat{\mathfrak{sl}_2})_l \times (\widehat{\mathfrak{sl}_2})_1/(\widehat{\mathfrak{sl}_2})_{l+1}$, where the subscripts stand for the level of the representation.

This connection with affine algebras brought us an exciting game to play [12]. The rules of this game[7] are as follows[8]:

(R1) Find a solution of the Yang-Baxter equation related to the affine coset $(\widehat{\mathfrak{sl}_n})_{l_1} \times (\widehat{\mathfrak{sl}_n})_{l_2}/(\widehat{\mathfrak{sl}_n})_{l_1+l_2}$.

(R2) Show that the 1-dimensional configuration sum is a branching function for this coset.

In this work, we will not delve into the details of the above two steps. We will rather consider the solutions of the Yang-Baxter equations as given, outline the way that the CTM produces 1-dimensional configurations in the simpler case of the ABF models, and go straight to the combinatorial objects that it produces.

[7] Also known as *Baxter's ball game*.

[8] Although we only consider $(\widehat{\mathfrak{sl}_n})$ case here, the situations for the other affine Lie algebras are similar. There is also a vertex model version of this game. In this case, the coset should be replaced by the level $l$ representation of the affine algebra $(\widehat{\mathfrak{sl}_n})$, and the branching function by a string function.

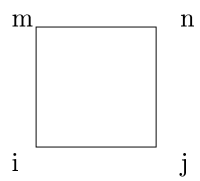
## 1.2 The ABF restricted solid-on-solid models

A restricted solid-on-solid (RSOS) model, as defined in [1], is a system of interacting discrete variables, called 'heights', $l$, defined on the sites of a square lattice. The heights, $l$, take values in the set of positive integers $l \in \{1, 2, \cdots, L-1\}$. $L$ characterises the model.

RSOS models are interaction-round-face (IRF) models: each configuration of 4 heights defined on the corners of a face on the lattice is assigned a Boltzmann weight. In the notation of [1], the non-vanishing weights are

$$\begin{aligned} W(l, l+1|l-1, l) &= W(l, l-1|l+1, l) = \alpha_l \\ W(l+1, l|l, l-1) &= W(l-1, l|l, l+1) = \beta_l \\ W(l+1, l|l, l+1) &= \gamma_l \\ W(l-1, l|l, l-1) &= \delta_l \end{aligned} \tag{2}$$

the labelling being given by



$$= W(m, n \mid i, j)$$

The Boltzmann weights are parametrised as

$$\begin{aligned} \alpha_l &= \rho h(v+\eta) \\ \beta_l &= \rho h(v-\eta) \frac{\left(h(w_{l-1}) h(w_{l+1})\right)^{1/2}}{h(w_l)} \\ \gamma_l &= \rho h(2\eta) \frac{h(w_l + \eta - v)}{h(w_l)} \\ \delta_l &= \rho h(2\eta) \frac{h(w_l - \eta + v)}{h(w_l)} \end{aligned} \tag{3}$$

$$h(u) = 2p^{1/4} \sin \pi u \prod_{n=1}^{\infty} \left(1 - 2p^n \cos \pi u + p^{2n}\right) \left(1 - p^{2n}\right)^2 \tag{4}$$

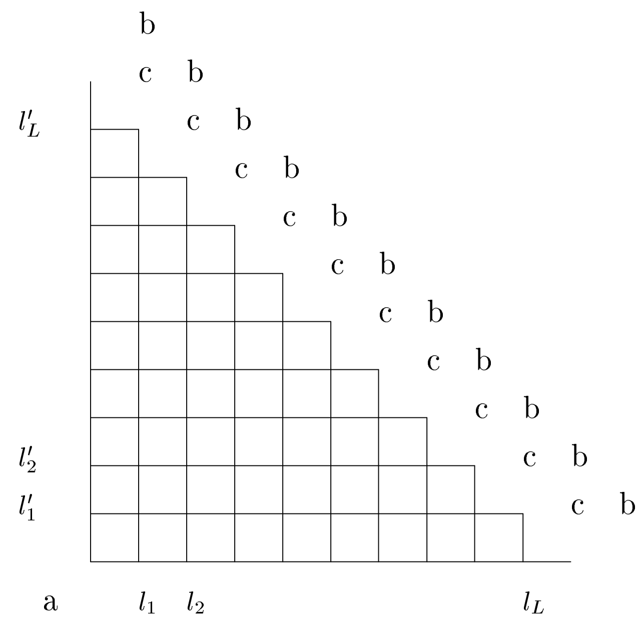
$\rho$ is a normalisation factor, $w_l = 2l\eta$. They satisfy the star-triangle, or Yang-Baxter equation

$$\begin{aligned} &\sum_g W(b, c\,|a, g) W'(a, g\,|f, e) W''(g, c\,|e, d) \\ = &\sum_g W''(a, b\,|f, g) W'(b, c\,|g, d) W(g, d\,|f, e) \end{aligned} \tag{5}$$

They are characterised by the property that the heights on neighbouring sites differ by $\pm 1$. Because of this, one obtains a non-trivial model only for $L-1 \geq 3$. For $L-1=3$, one obtains the IRF formulation of the Ising model. In the limit $L \to \infty$ one recovers the solid-on-solid version of Baxter's 8-vertex model [5]. For a complete discussion of the RSOS models, we refer to [1].

### 1.3 Corner transfer matrices

One can generate configurations in a model defined on a square lattice using corner transfer matrices CTM's[9]. One divides the square lattice into 4 quadrants. As we want to compute certain quantities in the infinite lattice limit, we may assume that each quadrant of the finite lattice is a right triangle bounded by two segments of the axes, with the external spins fixed to some ground state boundary condition, as in the following picture




Starting from a half-axis that extends from the centre of the lattice to a corner, *e.g.* the south one, a corner transfer matrix generates height configurations on the quadrant bounded by the initial half-axis and the half-axis that follows, as one turns the initial half-axis in a clockwise direction, keeping one end fixed at the centre of the lattice.

[9]There are two basic references to the CTM methods: 1. Chapter 13 of [5], and Appendix A of [1]. The first gets more into the physical background and assumptions that underlie the method. The second is a shorter, more technical outline of the method.

We denote the heights on the initial half-axis by the *height vector* $\mathbf{l} = \{l_1, l_2, \cdots, l_m\}$, and those on the final half-axis by $\mathbf{l}' = \{l'_1, l'_2, \cdots, l'_m\}$, with $l_1 = l'_1$, where $l_1$ is the height at the centre of the lattice.

The elements of the corner transfer matrix are sums over products of Boltzmann weights for configurations labelled by fixed initial and final height vectors. More precisely, the matrix elements of the corner transfer matrix $A$ of the lower right quadrant are defined by

$$A_{\mathbf{l},\mathbf{l}'} = A_{\mathbf{l},\mathbf{l}'}(b,c) = \begin{cases} \sum \prod W(\sigma_i \sigma_j | \sigma_m \sigma_n) & \text{if } \ l_1 = l'_1 \\ 0 \ \ \text{otherwise} \end{cases}$$

where the product is over all faces of the quadrant, and the sum runs over all admissible values of the heights on the circled vertices, the other boundary spins being fixed to $b, c$ as indicated on the picture.

One can think of the elements of a corner transfer matrix as the partition functions of a quadrant of a square lattice with fixed boundary conditions.

We denote the corner transfer matrices of the 4 quadrants by $A, B, C, D$ (in counterclockwise order).

We take $m \to \infty$, in the thermodynamic limit.

## 1.4 Local height probabilities

One can use the corner transfer matrices to compute 1-point functions, or equivalently order parameters. These functions are also known as the local height probabilities (LHP's) $P\,(a|b,c)$.

Following [5, 1], we consider a RSOS model on a square lattice and fix as above the height variables at the boundary of the lattice to the ground-state pair $\{b, c\}$, where $b - c = \pm 1$. The partition function $Z\,(b,c)$ of this configuration is

$$Z\,(b,c) = \text{Trace}\,(ABCD) \tag{6}$$

The product of the corner transfer matrices in (6) indicates identifying the heights on the common half-axes and summing over all possible heights. The trace indicates doing the same thing for the initial half-axis of $A$, and the final half-axis of $D$.

Computing $Z\,(b,c)$, using CTM's, allows us to fix the height at the centre of the lattice. We can do that by inserting the operator

$$(S_a)_{\mathbf{l},\mathbf{l}'} = \delta\,(l_1, a)\,\delta\,(\mathbf{l},\mathbf{l}')\,, \quad \delta\,(\mathbf{l},\mathbf{l}') \equiv \prod_{k=1}^{m} \delta\,(l_k, l'_k) \tag{7}$$

in (6) to obtain

$$Z\,(a|b,c) = \text{Trace}\,(S_a ABCD) \tag{8}$$

$Z(a|b,c)$ is the partition function of the configuration with fixed boundary conditions both at the centre and the boundary. 'Local height' refers to the height at the centre of the lattice. To obtain a probability, we normalise $Z(b,c)$ by $Z(a|b,c)$:

$$P(a|b,c) = \frac{\text{Trace}\,(S_a ABCD)}{\text{Trace}\,(ABCD)} \tag{9}$$

where $P(a|b,c)$ is the probability that the height variable at the centre of the lattice is $a$, while the heights on the boundary are $\{b,c\}$. It satisfies

$$\sum_a P(a|b,c) = 1 \tag{10}$$

## 1.5 From 2-dimensional to 1-dimensional configurations

We wish go through the derivation of LHP's, following [1]. Using the Yang-Baxter equations one can show that[10], in the thermodynamic limit (where the lattice becomes infinitely large), the CTM's can be written in the form

$$\begin{array}{llll} A(v) \sim \alpha(v) Q_1 M_1 & e^{v\mathcal{H}} & Q_2^{-1} \\ B(v) \sim \beta(v) Q_2 M_2 & e^{-v\mathcal{H}} & Q_3^{-1} \\ C(v) \sim \gamma(v) Q_3 M_3 & e^{v\mathcal{H}} & Q_4^{-1} \\ D(v) \sim \delta(v) Q_4 M_4 & e^{-v\mathcal{H}} & Q_1^{-1} \end{array} \tag{11}$$

where $\mathcal{H}, Q_1, \cdots, Q_4, M_1, \cdots, M_4$, are matrices that are independent of the spectral parameter $v$. Further, $\alpha, \beta, \gamma, \delta$ are scalar functions, and $\mathcal{H}, M_1, \cdots, M_4$ are diagonal. Substituting in (9), we obtain

$$P(a|b,c) = \frac{\text{Trace}(S_a M_1 M_2 M_3 M_4)}{\text{Trace}(M_1 M_2 M_3 M_4)} \tag{12}$$

Following a series of arguments given in detail in [5], one can show that, up to irrelevant scalar factors,

$$\begin{aligned} A(\eta) &= C(\eta) = I, \\ B(-\eta) &= D(-\eta) = R_1 \end{aligned} \tag{13}$$

where

$$(R_j)_{\mathbf{l},\mathbf{l'}} = \Big(h(w_{l_j})\Big)^{1/2} \delta(\mathbf{l},\mathbf{l'}),$$

[10] A better phrasing may be *"one can argue that"*.

and $\eta$ is the crossing parameter. Substituting the above in (11), and using the fact that both the $Q_i$'s and $\mathcal{H}$ are diagonal, we obtain[11]

$$A(\eta)B(-\eta)C(\eta)D(-\eta) = R_1^2 \tag{14}$$

$$M_1M_2M_3M_4 = R_1^2 e^{-4\eta\mathcal{H}}. \tag{15}$$

Evaluating the matrices in (15), one obtains

$$\left(e^{-4\eta\mathcal{H}}\right)_{\mathbf{ll}'} = q^{\left(-(2l_1-L)^2/16L+\Phi_m(\mathbf{l})\right)}\delta(\mathbf{l},\mathbf{l}') \tag{16}$$

where

$$\begin{aligned}\Phi_m(l_1,\cdots,l_{m+2}) &= \sum_{j=1}^{m} j\mathcal{H}(l_j,l_{j+1},l_{j+2})\\ \mathcal{H}(l_j,l_{j+1},l_{j+2}) &= \frac{1}{4}|l_j - l_{j+2}|\end{aligned} \tag{17}$$

and

$$\left(R_1^2\right)_{ll'} = \tau q^{(l_1-L)^2/8L}E(q^{l_1/2},q^{L/2})\delta(\mathbf{l},\mathbf{l}') \tag{18}$$

where $\tau = -\log q/(2\pi)$, and $E(z,q) = \Pi_{n=1}^{\infty}(1-q^{n-1}z)(1-q^nz^{-1})(1-q^n)$. Substituting the above results in (6), and (9), one obtains

$$P\left(a|b,c\right) = \frac{E(q^{a/2},q^{L/2})X_m(a,b,c|q)}{\sum_{1\le d<r}E(q^{d/2},q^{L/2})X_m(d,b,c|q)} \tag{19}$$

where $X_m(a,b,c|q)$ are 1-dimensional configuration sums, defined by

$$X_m(a,b,c|q) = \sum_{configuration} q^{\Phi_m(l_1,\cdots,l_{m+2})}. \tag{20}$$

The sum in $X_m(a,b,c|q)$ is taken over $l_2,\cdots,l_m$, with the boundary heights fixed at $(l_1 = a, l_{m+1} = b, l_{m+2} = c)$. There are $m+2$ sites on a half-diagonal, counted starting from the centre of the lattice, and proceeding towards an outer corner of the square lattice.

The 1-dimensional configuration sums $X_m(a,b,c,|q)$ are precisely the objects we are interested in. To compute the local height probabilities in the thermodynamic limit, one has to evaluate

$$X(a,b,c\,|\,q) = \lim_{m\ even\to\infty} X_m(a,b,c\,|\,q), \tag{21}$$

[11]This entire discussion is restricted to the anti-ferromagnetic, or so-called regime-III of the ABF models.

It is the configuration sum $X(a,b,c|q)$ that encodes the statistical mechanics of the model.

In [1], the 1-dimensional configuration sums were evaluated. For example, in the case under consideration (regime III), it is found that

$$X_m(a,b,c\,|\,q) = q^{a(a-1)/4}\left(F_m(a,b,c) - F_m(-a,b,c)\right)$$

where

$$F_m(a,b,c) = \sum_{n=-\infty}^{\infty} q^{n(L-1)(nL-a)+[bc-(2nL-a)(b+c-1)]/4} \begin{bmatrix} m \\ \frac{1}{2}(m+a-b)-nL \end{bmatrix}_q$$

In the thermodynamic limit, one obtains

$$X(a,b,c|q) = \frac{1}{\varphi(q)} q^{bc/4} \Delta(a, \frac{1}{2}(b+c-1)\,;\,q) \tag{22}$$

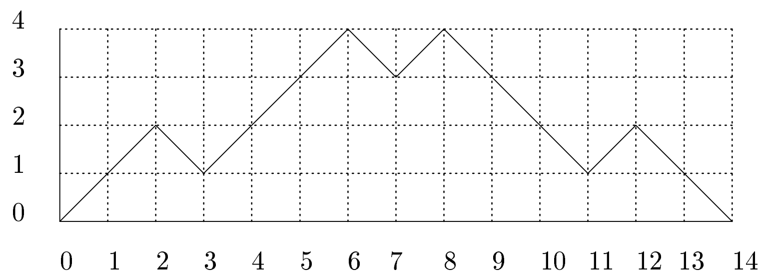
where

$$\varphi(q) = \prod_{n\geq 1}(1-q^n)$$

and

$$\Delta(a,d\,;\,q) = \sum_{n=-\infty}^{\infty} q^{L(L-1)n^2+Ldn+a(a-1)/4}\left(q^{-(L-1)an-ad/2} - q^{(L-1)an+ad/2}\right).$$

In the case of the ABF models, the 1-dimensional configurations $q$-counted by $X_m(a,b,c|q)$ can be represented graphically in terms of lattice paths (or Dyck graphs), defined by the points of coordinates $(k-1, l_k-1)$ $(k=1,\ldots,m)$ in the plane:




## 1.6 From 1-dimensional configurations to Virasoro characters

The 1-dimensional configurations of the ABF models can be regarded as combinatorial objects with shape-dependent weights. On that basis alone ABF evaluated the 1-dimensional sums, and obtained the result in terms of $q$-series.

The Kyoto group made the fundamental observation that the generating functions of the ABF 1-dimensional sums are the characters of highest weight modules of Virasoro algebras [12]. To be more precise, these are precisely the characters of the Virasoro modules arising in the discrete unitary conformal field theories discovered by Belavin, Polyakov, and Zamolodchikov [6].

### 1.7 From Virasoro characters to branching functions

The Kyoto group also observed that the ABF paths are intrinsically related to representation theory: they can be regarded as walks on a finite section of the weight lattice of the Lie algebra $\widehat{\mathfrak{sl}}_2$. This connection will be explained in more detail in the forthcoming section.

Furthermore, they showed that the Virasoro characters corresponding to configuration sums can be obtained as the branching coefficients of products of affine characters, as in the coset construction of Goddard-Kent-Olive [20].

This will also be explained in more details in latter sections. What we wish to point out here is that, once the connection with $\widehat{\mathfrak{sl}}_2$ is made, it is natural to look for models that are related to higher rank Lie algebras. These models do exist [31].

## 2 Combinatorics of solvable lattice models

The $\widehat{\mathfrak{sl}}_n$ analogues of the ABF models were obtained in [31]. For lack of space, we will not go into the definition of these models. Suffice it to say that they can be solved using the CTM, and that the result of the application of the CTM are configurations that can be interpreted as walks on restricted sections of the weight lattice of $\widehat{\mathfrak{sl}}_n$ as will be explained below.

To represent these generalised 1-dimensional configurations[12] the ABF-type paths are no longer adequate, and one has to work in terms of 'coloured' partitions.

### 2.1 Representation theory of $\widehat{\mathfrak{sl}}_n$

#### 2.1.1 The affine Lie algebra $\widehat{\mathfrak{sl}}_n$

Let $n \geq 2$ and let $\mathfrak{h}$ be a $(n+1)$-dimensional vector space over $\mathbb{Q}$ with basis $\{h_0, h_1, \ldots, h_{n-1}, D\}$. We let $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}, \delta\}$ be the corresponding dual basis of $\mathfrak{h}^*$, the dual space of $\mathfrak{h}$. That is,

$$\langle \Lambda_i, h_j \rangle = \delta_{ij}, \quad \langle \Lambda_i, D \rangle = 0, \quad \langle \delta, h_i \rangle = 0, \quad \langle \delta, D \rangle = 1.$$

[12]The can still be regarded as one dimensional configurations, in the sense that the can be represented as a linear sequence of sets of integers.

It will be convenient to extend the index set so that $\Lambda_i = \Lambda_{(i \bmod n)}$ for all $i \in \mathbb{Z}$. Then, for all $i \in \mathbb{Z}$, we set $\epsilon_i = \Lambda_{i+1} - \Lambda_i$ and $\alpha_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} + \delta_{i0}\,\delta$, where $\delta_{ij} = 1$ if $i - j \bmod n = 0$ and $\delta_{ij} = 0$ otherwise.

The $n \times n$ matrix $[\langle \alpha_i, h_j \rangle]$ is called the generalised Cartan matrix of type $A_{n-1}^{(1)}$. The associated (affine) Kac-Moody algebra is denoted by $\widehat{\mathfrak{sl}}_n$. It is defined as the algebra generated by $D$ and $e_i$, $f_i$, $h_i$ for $0 \le i < n$, subject to the relations:

$$\begin{gathered}
[h_i, h_j] = 0; \quad [h_i, D] = 0; \\
[h_i, e_j] = \langle \alpha_j, h_i \rangle e_j; \quad [D, e_j] = \delta_{j0} e_j; \\
[h_i, f_j] = -\langle \alpha_j, h_i \rangle f_j; \quad [D, f_j] = -\delta_{j0} f_j; \\
[e_i, f_j] = \delta_{ij} h_i; \\
(\operatorname{ad} e_i)^{1 - \langle \alpha_j, h_i \rangle} e_j = 0 \quad (i \neq j); \\
(\operatorname{ad} f_i)^{1 - \langle \alpha_j, h_i \rangle} f_j = 0 \quad (i \neq j),
\end{gathered}$$

where $(\operatorname{ad} a)\, b = [a, b]$.

The $\Lambda_i$ are known as the fundamental weights, the $\epsilon_i$ the fundamental vectors, and the $\alpha_i$ the simple roots of $\widehat{\mathfrak{sl}}_n$.

### 2.1.2 Weight lattices and root lattices

The root lattice of $\widehat{\mathfrak{sl}}_n$ is

$$Q = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i.$$

Its weight lattice is

$$P = \mathbb{Z}\delta \oplus \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i.$$

Elements of $P$ are known as weights. The dual weight lattice of $\widehat{\mathfrak{sl}}_n$ is defined to be

$$P^\vee = \mathbb{Z}\, D \oplus \bigoplus_{i=0}^{n-1} \mathbb{Z}\, h_i,$$

For $l \in \mathbb{Z}$, $P_l$ is defined by

$$P_l = \left\{ \sum_{i=0}^{n-1} a_i \Lambda_i \mid a_i \in \mathbb{Z} \text{ for } i = 0, 1, \ldots, n-1; \sum_{i=0}^{n-1} a_i = l \right\}.$$

For $l \in \mathbb{N}$, $P_l^+$ is defined by

$$P_l^+ = \left\{ \sum_{i=0}^{n-1} a_i \Lambda_i \in P_l \mid a_i \ge 0 \right\}.$$

Then define $P^+ = \bigcup_{l \geq 0} P_l^+$.

Strictly speaking, $P_l$ is the level $l$ integral weight lattice of $\widehat{\mathfrak{sl}}_n{}'$, which is the subalgebra of $\widehat{\mathfrak{sl}}_n$ generated by $e_i, h_i, f_i$ (without $D$).

### 2.1.3 $\widehat{\mathfrak{sl}}_n$-modules

Let $M$ be an $\widehat{\mathfrak{sl}}_n$-module. For each weight $\Lambda \in P$, the subspace of $M$ defined by

$$M_\Lambda = \{v \in M \mid h\,v = \langle \Lambda, h \rangle\, v,\ \ h \in \mathfrak{h}\}$$

is called the weight space of weight $\Lambda$ of $M$. If $v \in M_\Lambda$ then we write $\mathrm{wt}\,(v) = \Lambda$, and call such $v$ a weight vector of weight $\Lambda$.

The module $M$ is said to be integrable if:

1. $M = \bigoplus_{\Lambda \in P} M_\Lambda$;
2. $\dim M_\Lambda < \infty$ for each $\Lambda \in P$;
3. for each $i = 0, 1, \ldots, n-1$, $M$ decomposes into a direct sum of finite dimensional $\mathfrak{g}_i$-modules, where $\mathfrak{g}_i$ denotes the subalgebra of $\widehat{\mathfrak{sl}}_n$ generated by $e_i$, $f_i$, and $h_i$.

A weight vector $v \in M$ for which $e_i v = 0$ for all $i = 0, 1, \ldots, n-1$, is said to be a highest weight vector. If there exists a highest weight vector $v \in M$ such that $M = U(\widehat{\mathfrak{sl}}_n)\,v$, then $M$ is said to be a highest weight module. Then the weight of $v$ is called the highest weight of $M$.
If $l \geq 0$ then for all $\Lambda \in P_l^+$, there exists (up to equivalence) a unique irreducible integrable $\widehat{\mathfrak{sl}}_n$-module $V(\Lambda)$ with highest weight $\Lambda$ [32].

### 2.1.4 The Virasoro algebra

The Virasoro algebra is the complex Lie algebra $Vir$ generated by elements $L_n$ ($n \in \mathbf{Z}$) and $c$ subject to the relations

$$[L_m\,,\, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\, c\, \delta_{m+n,0}$$

and

$$[c\,,\, L_n] = 0\ .$$

It is the universal central extension of the Witt algebra, *i.e.* the Lie algebra generated by the differential operators $L_n = -z^{n+1}\frac{d}{dz}$. On an irreducible representation, $c$ is a scalar operator, whose eigenvalue $c$ is called *the central charge*. The eigenvalue of $L_0$ on the highest weight vector is called the conformal dimension $h$.

By using the Sugawara construction (see [32]), one can construct an action of $Vir$ on any representation of $\widehat{\mathfrak{sl}}_n$ of fixed level $l$. This can be done in particular for the tensor product of two irreducible representations [20].

### 2.1.5 Branching functions

Let $\Lambda'$, $\Lambda''$ be dominant integral weights of respective levels $l'$, $l''$. The tensor product of $V(\Lambda')$ by $V(\Lambda'')$ can be written as the finite direct sum

$$V(\Lambda') \otimes V(\Lambda'') = \bigoplus_{\Lambda \in P_l^+} \Omega^{\Lambda}_{\Lambda'\Lambda''} \otimes V(\Lambda) \tag{23}$$

where $l = l' + l''$ and $\Omega^{\Lambda}_{\Lambda'\Lambda''}$ is the space of highest weight vectors whose weights are congruent to $\Lambda$ modulo $\delta$. This is the decomposition into isotypic components for the action of $\widehat{\mathfrak{sl}}_n{}'$ (*i.e.* $\widehat{\mathfrak{sl}}_n$ without the generator $D$).

The action of $Vir$ on $V(\Lambda') \otimes V(\Lambda'')$ defined by the coset construction of [20] commutes with the action of $\widehat{\mathfrak{sl}}_n{}'$, so that each $\Omega^{\Lambda}_{\Lambda'\Lambda''}$ is a representation of $Vir$.

The formal character

$$b^{\Lambda}_{\Lambda'\Lambda''}(q) = \mathrm{ch}\,\Omega^{\Lambda}_{\Lambda'\Lambda''} = \sum_k q^k \dim\,(\Omega^{\Lambda}_{\Lambda'\Lambda''})_{\Lambda - k\delta}$$

is called a branching function (this definition differs from the usual Virasoro character by a factor $q^h$).

For $n = 2$, these Virasoro characters are precisely those of the discrete unitary series of irreducible representations. For $n > 2$, the spaces $\Omega^{\Lambda}_{\Lambda'\Lambda''}$ are generally not irreducible under the Virasoro algebra. They are irreducible representations of the $\mathcal{W}_n$-algebras, which are less well-understood.

The ABF models are related to the case $n = 2$. Let $V_{j,l}$ be the irreducible level $l$ representation $V((l-j)\Lambda_0 + j\Lambda_1)$ of $\widehat{\mathfrak{sl}}_2$, and let $b_{j_1 j_2 j_3}(q)$ be the character of $\Omega_{j_1 j_2 j_3}$, where

$$V_{j_1, L-3} \otimes V_{j_2, 1} = \bigoplus_{j_3} \Omega_{j_1 j_2 j_3} \otimes V_{j_3, L-2}$$

Then, the 1-dimensional configuration sums are given by

$$X(a, b, c|q) = q^{\gamma} b_{j_1 j_2 j_3}(q)$$

where

$$j_1 = \frac{1}{2}(b + c - 1) - 1, \quad j_2 = \frac{1}{2}(b - c + 1), \quad j_3 = a - 1$$

and $\gamma$ is a certain rational number [12, 13].

Let $L(c, h)$ be the irreducible representation of the Virasoro algebra with central charge $c$ and conformal weight $h$. In the $\widehat{\mathfrak{sl}}_2$ case, the decomposition of the tensor product reads

$$V_{j,l} \otimes V_{\varepsilon,1} = \bigoplus_s L(c, h_{rs}) \otimes V_{s-1, l+1}$$

where $r = j + 1$ and the sum runs over all $s$ such that $1 \le s \le l + 2$, $r - s$ even if $\varepsilon = 0$ or $r - s$ odd if $\varepsilon = 1$. The central charge $c$ is given by

$$c = 1 - \frac{6}{(l+2)(l+3)}$$

and

$$h_{rs} = \frac{[(l+3)r - (l+2)s]^2 - 1}{4(l+2)(l+3)}$$

The character $\chi_{r,s}(q)$ of $L(c, h_{rs})$ is given by the Rocha-Caridi formula [47]

$$\chi_{r,s}(q) = \frac{1}{\varphi(q)} \sum_{k=-\infty}^{\infty} \left( -q^{a(k)} + q^{b(k)} \right)$$

where, setting $m = l + 2$

$$\begin{aligned} a(k) &= \frac{[2m(m+1)k + (m+1)r + ms]^2 - 1}{4m(m+1)} \\ b(k) &= \frac{[2m(m+1)k + (m+1)r - ms]^2 - 1}{4m(m+1)} \,. \end{aligned} \tag{24}$$

## 2.2 Paths

As explained above, the notion of a *path* emerged from the corner transfer matrix method. Roughly speaking, a path corresponds to an eigenvector of the corner transfer matrix at the absolute temperature $q = 0$. It is defined as a certain sequence of points on the weight lattice of $\widehat{\mathfrak{sl}}_n$.

**Definition 2.1 (Paths)**

1. *A level $l$ path is a sequence*

$$p = (p_0, p_1, p_2, \ldots),$$

*for which $p_k \in P_l$ for all $k \ge 0$. The set of all level $l$ paths is denoted $\mathcal{P}_l$.*

2. *If $p \in \mathcal{P}_l$ and $\mu \in P_m$, define the level $l - m$ path $p - \mu$ by*

$$p - \mu = (p_0 - \mu, p_1 - \mu, p_2 - \mu, \ldots).$$

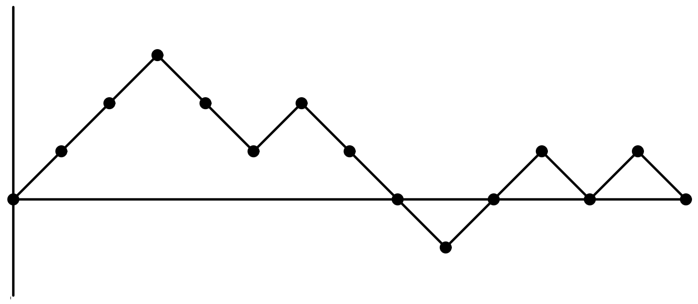
**Definition 2.2 (Unrestricted paths)**

1. *A $\Lambda_0$-path $p$ is a level 1 path for which, for all $k \ge 0$, there exists $\gamma(k)$ satisfying $0 \le \gamma(k) < n$, such that $p_{k+1} - p_k = \epsilon_{\gamma(k)}$, and for which for some $k_* \ge 0$, $p_k = \Lambda_k$ for all $k \ge k_*$.*

2. *The set of all $\Lambda_0$-paths is denoted $\mathcal{P}(\Lambda_0)$.*
3. *The length of $p \in \mathcal{P}(\Lambda_0)$ is the minimal $k_*$ for which $p_k = \Lambda_k$ for all $k \geq k_*$. We then define $l(p) = k_*$.*
4. *The ground state of $\mathcal{P}(\Lambda_0)$ is the unique path $\overline{p} \in \mathcal{P}(\Lambda_0)$ having length 0. Thus $\overline{p}_k = \Lambda_k$ for all $k \geq 0$.*

Note that a $\Lambda_0$-path $p$ is completely determined by the sequence $\gamma(0)$, $\gamma(1)$, $\gamma(2), \ldots$. The adjective *unrestricted* will be used when appropriate to distinguish the paths defined here from restricted paths which are defined later.

**Example 2.3** Let $n = 2$. The following diagram depicts the $\Lambda_0$-path ($\Lambda_0$, $\Lambda_1$, $2\Lambda_1 - \Lambda_0$, $3\Lambda_1 - 2\Lambda_0, 2\Lambda_1 - \Lambda_0, \Lambda_1, 2\Lambda_1 - \Lambda_0, \Lambda_1, \Lambda_0, 2\Lambda_0 - \Lambda_1, \Lambda_0, \Lambda_1, \Lambda_0$, $\Lambda_1$, $\Lambda_0, \ldots$).



This path has length 10.

With each $\Lambda_0$-path, we can associate an *energy*.

**Definition 2.4** *Let $p \in \mathcal{P}(\Lambda_0)$, and $\gamma(k)$ be defined by 2.2.1. Let $\overline{\gamma}(k)$ be the corresponding values for the ground state $\overline{p} \in \mathcal{P}(\Lambda_0)$: $\overline{\gamma}(k) = k \bmod n$ for all $k \geq 0$.*

1. *The energy of $p$ is defined to be*
$$E(p) = \sum_{k=1}^{\infty} k\Big(H\big(\gamma(k-1), \gamma(k)\big) - H\big(\overline{\gamma}(k-1), \overline{\gamma}(k)\big)\Big),$$
*where, for $0 \leq a, b < n$, we define*
$$H(a,b) = \begin{cases} 0 & \text{if } a < b; \\ 1 & \text{if } a \geq b. \end{cases}$$
2. *The weight of $p$ is then defined to be*
$$\mathrm{wt}\,(p) = p_0 - E(p)\delta,$$
*where $\delta$ is the null root of $\widehat{\mathfrak{sl}}_n$.*

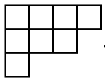
Note that the energy $E(p)$ of each path $p \in \mathcal{P}(\Lambda_0)$ is finite because $\gamma(k) = \overline{\gamma}(k)$ for all $k \geq l(p)$.

## 2.3 Partitions and paths

A partition $\lambda$ of $m$ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^{r} \lambda_i = m$. For $i > r$, we define $\lambda_i = 0$. Occasionally the notation $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})$ will be used to denote a partition, where for $i = 1, 2, \ldots, r$, the part $\lambda_i$ occurs in $\lambda$ with a multiplicity $a_i$.

The (finite) set of all partitions of $m$ is denoted $\Pi(m)$. Then define $\Pi = \cup_{m \geq 0} \Pi(m)$.

The diagram $F^\lambda$ associated with $\lambda \in \Pi(m)$ consists of $m$ nodes (or boxes) arranged in $r$ left adjusted rows. The number of nodes in the $i$th row is $\lambda_i$. For example:

$F^{(4,3,1)} =$  .

The partition $\lambda'$ conjugate to the partition $\lambda \in \Pi(m)$ is obtained by setting $\lambda'_i$ to be the length of the $i$th column of $F^\lambda$ for $1 \leq i \leq \lambda_1$ (reading from the left).

A partition $\lambda \in \Pi(m)$ is said to be $n$-regular if no part appears $n$ or more times. Equivalently: $\lambda'_i - \lambda'_{i+1} < n$ for all $i \geq 1$.

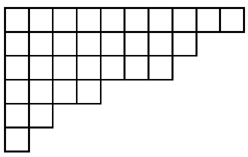
The set of $n$-regular partitions of $m$ is denoted $\Pi_n(m)$. Then define $\Pi_n = \cup_{m \geq 0} \Pi_n(m)$.
In the following definition, we associate an $n$-regular partition with each path $p \in \mathcal{P}(\Lambda_0)$.

**Definition 2.5 (Highest-lift of a path)** *Let $p \in \mathcal{P}(\Lambda_0)$ and $k_* = l(p)$. For $k \geq k_*$, let $t_k = 0$. Then for $k = k_* - 1, k_* - 2, \ldots, 0$, recursively calculate $t_k$ from $t_{k+1}$ by setting $t_k = k - \gamma(k) \pmod{n}$ with $0 \leq t_k - t_{k+1} < n$. The partition $\lambda = \lambda(p)$ is then defined such that its conjugate is*

$$\lambda' = (t_0, t_1, \ldots, t_{k_*-1}).$$

*The partition $\lambda(p)$ is known as the highest-lift of $p$.*

**Example 2.6** Consider the path described in Example 2.3. Here we calculate that for $k = 0, 1, .., 9$, the values of $\gamma(k)$ are $0, 0, 0, 1, 1, 0, 1, 1, 1, 0$, respectively. Hence, modulo 2, the respective values of $t_k$ are $0, 1, 0, 0, 1, 1, 1, 0, 1, 1$. Since the path has length 10, we set $t_{10} = 0$. Then, using $0 \leq t_k - t_{k+1} < n$, we obtain the values $t_9 = 1$, $t_8 = 1$, $t_7 = 2$, $t_6 = 3$, $t_5 = 3$, $t_4 = 3$, $t_3 = 4$, $t_2 = 4$, $t_1 = 5$, $t_0 = 6$. This yields the following partition:

 ,

and hence the highest-lift partition $(10, 8, 7, 4, 2, 1)$.

Note that $\lambda(\overline{p})$ is the empty partition and that each $\lambda(p)$ is $n$-regular. Moreover, each $n$-regular partition corresponds to a unique element of $\mathcal{P}(\Lambda_0)$. Thus the sets $\mathcal{P}(\Lambda_0)$ and $\Pi_n$ are in bijection.

It is often useful to fill each node of the partition corresponding to a path $p$ with an integer.

**Definition 2.7 (Colouring)** *A partition $\lambda$ is said to be coloured with $v$ if, for $i = 1, 2, \ldots$, and $j = 1, 2, \ldots, \lambda_i$, the $j$th node in the $i$th row is filled with the value $j - i + v \bmod n$. The value occupying a node is known as its colour charge. The colour charge $v$ of the leftmost node in the first row is referred to as the colour of the partition. The weight of the partition is $\Lambda_v$.*

For example, on colouring the partition $\lambda = (5, 5, 4, 1, 1)$ with 0 in the case $n = 3$, we obtain:

$$\begin{array}{|c|c|c|c|c|}\hline 0&1&2&0&1\\ \hline 2&0&1&2&0\\ \hline 1&2&0&1\\ \cline{1-4} 0\\ \cline{1-1} 2\\ \cline{1-1}\end{array}\,.$$

**Definition 2.8** *Colour the partition $\lambda$ with 0 and, for $0 \le i < n$, let $m_i$ be the multiplicity of the colour $i$ in $\lambda$. Then define:*

1. $E(\lambda) = m_0$;
2. $\mathrm{wt}\,(\lambda) = \Lambda_0 - \sum_{i=0}^{n-1} m_i \alpha_i$.

When $\lambda$ is interpreted as a basis vector $s_\lambda$ of the Fock space (see Section 4.2) $\mathrm{wt}\,(\lambda)$ is the $\mathfrak{sl}_n$-weight of $s_\lambda$, and $E(\lambda)$ its degree in the homogeneous gradation. It is straightforward to show that the energy and weight of a path coincide with those of the corresponding partition:

**Lemma 2.9** *Let $p \in \mathcal{P}(\Lambda_0)$ and $\lambda = \lambda(p)$, the corresponding partition.*

1. $E(p) = E(\lambda)$;
2. $\mathrm{wt}\,(p) = \mathrm{wt}\,(\lambda)$.

The next result, which was first given in [10], states that the character of the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$, amounts to enumerating the $n$-regular partitions, taking account of their colouring.

**Theorem 2.10** [10, 28] *The formal character of the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$ is*

$$\mathrm{ch}\, V(\Lambda_0) = \sum_{\lambda \in \Pi_n} e^{\mathrm{wt}\,(\lambda)}.$$

The principally specialised character of $V(\Lambda)$, defined by

$$\mathrm{Pr\,ch}\,V(\Lambda) = \sum_{m_0,\ldots,m_{n-1}} \dim V(\Lambda)_{\Lambda - m_0\alpha_0 - \cdots - m_{n-1}\alpha_{n-1}} q^{m_0+\cdots+m_{n-1}}, \qquad (25)$$

then follows in the $\Lambda = \Lambda_0$ case, as a direct corollary:

**Theorem 2.11** [32]

$$\mathrm{Pr\,ch}\,V(\Lambda_0) = \prod_{\substack{j\geq 1 \\ j\not\equiv 0 \bmod n}} \frac{1}{1-q^j}.$$

*Proof:* By Theorem 2.10 and Definition 2.8.2, the coefficient of $q^N$ on the right side of (25) is given by the number of $n$-regular partitions of $N$. The generating function for this number [25] then gives the result.

We will later describe a realisation of $V(\Lambda_0)$ that has a basis naturally indexed by the set of $n$-regular partitions $\Pi_n$.

## 2.4 Restricted paths

In Definition 2.2, we defined a set of unrestricted paths $\mathcal{P}(\Lambda_0)$. We now use these to define restricted paths.

**Definition 2.12 (Restricted paths)** *Let $\mu$ be a dominant integral weight of level $l$.*

1. *A path $p$ is said to be $(\mu, \Lambda_0)$-restricted if $p - \mu \in \mathcal{P}(\Lambda_0)$ and $p_k \in P^+_{l+1}$ for all $k \geq 0$.*
2. *The set of $(\mu, \Lambda_0)$-restricted paths is denoted $\mathcal{P}(\mu, \Lambda_0)$.*
3. *The length $l(p)$ of $p \in \mathcal{P}(\mu, \Lambda_0)$ is defined to be $l(p - \mu)$.*
4. *The ground state $\overline{p} \in \mathcal{P}(\mu, \Lambda_0)$ is defined to be $\overline{p} = \overline{p}' + \mu$ where $\overline{p}'$ is the ground state of $\mathcal{P}(\Lambda_0)$.*

   *Thus here, $\overline{p}_k = \mu + \Lambda_k$ for all $k \geq 0$.*

The elements of $\mathcal{P}(\mu, \Lambda_0)$ are called *restricted paths*, since they describe 'walks' on a restricted segment of a weight lattice. They also label the 1-dimensional configurations of certain restricted solid-on-solid models. Since $\{p - \mu | p \in \mathcal{P}(\mu, \Lambda_0)\} \subset \mathcal{P}(\Lambda_0)$, a partition may readily be associated with each restricted path.

**Definition 2.13 (Restricted partitions)**

1. *Let $p \in \mathcal{P}(\mu, \Lambda_0)$. Then the partition $\lambda(p)$ associated with $p$ is defined to be $\lambda(p - \mu)$.*

2. *Define $\mathcal{Y}(\mu, \Lambda_0) = \{\lambda(p) \mid p \in \mathcal{P}(\mu, \Lambda_0)\}$.*

Clearly $\mathcal{Y}(\mu, \Lambda_0) \subset \Pi_n$. In the case where $\mu$ is of level one, this set of partitions was first characterised in [19].

## 2.5 The FOW conditions

**Theorem 2.14 (FOW theorem)** [19] *Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r}) \in \Pi_n$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ and $0 < a_i < n$ for $i = 1, 2, \ldots, r$. Then $\lambda \in \mathcal{Y}(\Lambda_j, \Lambda_0)$ if and only if either $r = 0$, or*

$$a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod{n},$$

*for $i = 1, 2, \ldots, r-1$, and $j = \lambda_1 - a_1 \,(\mathrm{mod}\, n)$.*

Note that the value of $\lambda_i - \lambda_{i+1}$ is the length of a horizontal edge of the corresponding partition (it is the $i$th horizontal edge from the top, not counting the top border of the diagram). $a_i$ and $a_{i+1}$ are the lengths of its neighbouring vertical edges. Thus, we must check whether the sum of each horizontal edge and its neighbouring vertical edges is a multiple of $n$. This makes identifying these partitions particularly straightforward.

**Example 2.15** Consider the following partition, which is coloured with 0 in the case $n = 3$:

$$\lambda = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0&1&2&0&1&2&0&1&2&0&1&2&0\\ \hline
2&0&1&2&0&1&2&0&1&2&0&1&2\\ \hline
1&2&0&1&2&0&1&2&0&1\\ \cline{1-10}
0&1&2&0&1&2\\ \cline{1-6}
2&0&1&2&0\\ \cline{1-5}
1&2&0&1\\ \cline{1-4}
0\\ \cline{1-1}
2\\ \cline{1-1}
\end{array}\;.$$

Starting from the top right of this partition, the lengths of the vertical and horizontal edges taken in order along the border are 2,3,1,4,1,1,1,1,1,3,2,1. The above description requires that we sum the three successive values beginning at the 1st, 3rd, 5th, etc. We obtain the values 6,6,3,3,6. Since each is divisible by 3, we conclude that $\lambda \in \mathcal{Y}(\Lambda_j, \Lambda_0)$ for a certain $j$. Theorem 2.14 also gives $j = (\lambda_1 - a_1) \bmod n = (13 - 2) \bmod 3 = 2$.

The above vertical-horizontal-vertical edge lengths: 6,6,3,3,6; are not all zero modulo $n \neq 3$ and thus for $n \neq 3$, $\lambda \notin \mathcal{Y}(\Lambda_j, \Lambda_0)$ for all $j$.

As a further example, consider the partition $\mu = (10, 8, 7, 4, 2, 1)$ which was produced in Example 2.6. In this case, we find the sequence 1,1,1, of vertical-horizontal-vertical edge lengths. Since their sum is $3 \neq 0 \,(\mathrm{mod}\, 2)$, we conclude that in the case $n = 2$, $\mu \notin \mathcal{Y}(\Lambda_j, \Lambda_0)$ for all $j$. Hence, the path of Example 2.3 is not an element of $\mathcal{P}(\Lambda_j, \Lambda_0)$ for any $j$.

It is worth noting that the empty partition $\emptyset$ satisfies the conditions of Theorem 2.14 trivially and hence $\emptyset \in \mathcal{Y}(\Lambda_j, \Lambda_0)$ for all $j$ and all $n$. Correspondingly, the appropriate ground state path $\overline{p} \in \mathcal{P}(\Lambda_j, \Lambda_0)$.

The following result of [28] shows how the branching function of the tensor product of two level one $\widehat{\mathfrak{sl}}_n$ representations is obtained by the enumeration of restricted partitions.

**Theorem 2.16** [28] *Let $0 \leq j < n$. Then for each weight $\Lambda$, the branching function $b^{\Lambda}_{\Lambda_j,\Lambda_0}(q)$ is given by:*

$$b^{\Lambda}_{\Lambda_j,\Lambda_0} = \sum_{\substack{p \in \mathcal{P}(\Lambda_j,\Lambda_0)\\ p_0=\Lambda}} q^{E(p)} = \sum_{\substack{\lambda \in \mathcal{Y}(\Lambda_j,\Lambda_0)\\ \mathrm{wt}\,(\lambda)=\Lambda-\Lambda_j \ (\bmod \delta)}} q^{E(\lambda)}.$$

Here, the second equality results from the previously described mapping between paths and partitions.

If we now define $FOW(n,j,k)$ to be the subset of $\mathcal{Y}(\Lambda_j, \Lambda_0)$ comprising those partitions $\lambda$ for which $\mathrm{wt}\,(\lambda) = \Lambda_k + \Lambda_{j-k} - \Lambda_j \,(\bmod \delta)$, we immediately obtain the following:

**Theorem 2.17** [19] *Let $0 \leq j < n$ and $0 \leq k \leq (j-k) \bmod n$. Then the branching function $b^{\Lambda_k+\Lambda_{j-k}}_{\Lambda_j,\Lambda_0}(q)$ is given by:*

$$b^{\Lambda_k+\Lambda_{j-k}}_{\Lambda_j,\Lambda_0} = \sum_{\lambda \in FOW(n,j,k)} q^{E(\lambda)}.$$

## 2.6 Counting restricted paths

For $L \geq 0$, we set

$$\begin{aligned}
\mathcal{P}(\Lambda_j, \Lambda_0; L) &= \{p \in \mathcal{P}(\Lambda_j, \Lambda_0) \mid l(p) \leq L\}, && (26)\\
b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q; L) &= \sum_{\substack{p \in \mathcal{P}(\Lambda_j,\Lambda_0;L)\\ p_0=\Lambda}} q^{E(p)}, && (27)
\end{aligned}$$

whereupon $b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q; L)$ is a polynomial in $q$. In [19], a *constant sign q-series* for $b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q; L)$ was obtained. Since the expression in [19] is in the principal picture, we need to adjust the overall power of $q$. We set $\Lambda = \Lambda_s + \Lambda_t$ $(0 \leq s \leq t < n)$. Let $C$ be the Cartan matrix of $\mathfrak{sl}_n$, and $e_i$ be the $(n-1)$-dimensional unit vector $(0, \cdots, 0, \overset{i}{1}, 0 \cdots, 0)^t$. We set $e_n = 0$. The expression reads as follows.

$$\begin{aligned}
b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q; L) &= \sum_m q^{m^tC^{-1}m - m^tC^{-1}e_{s-t+n} + st/n} \prod_{i=1}^{n-1} \begin{bmatrix} l_i + m_i \\ m_i \end{bmatrix}, && (28)\\
l &= C^{-1}(Le_{n-1} + e_r + e_{s-t+n} - 2m), && (29)
\end{aligned}$$

where the sum is taken over all $m \in (\mathbb{Z}_{\geq 0})^{\times(n-1)}$ satisfying $t + \sum_{i=1}^{n-1} i m_i = 0 \ (\mathrm{mod}\, n)$, and $r$ is determined from $L - (s+t) \equiv r$, $0 < r \leq n$. Here, the notation means[13]

$$(q)_k = (1-q)(1-q^2)\cdots(1-q^k), \quad \begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q)_m}{(q)_{m-k}(q)_k}.$$

We refer to [19] for details of the proof. Taking the limit $L \to \infty$, we have

$$b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q) = \sum_{m} \frac{q^{m^t C^{-1} m - m^t C^{-1} e_{s-t+n} + st/n}}{\prod_{i=1}^{n-1} (q)_{m_i}},$$

with the same restriction on $m$.

# 3 Modular representations of symmetric groups and Hecke algebras

## 3.1 The symmetric groups

The symmetric group $\mathfrak{S}_m$ may be defined as the group generated by the elements $s_i$, for $i = 1, 2, \ldots, m-1$, subject to the relations:

$$\begin{aligned} & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \\ & s_i s_j = s_j s_i \qquad |i-j| > 1; \\ & s_i^2 = 1. \end{aligned}$$

In the realisation of $\mathfrak{S}_m$ as the permutation group on the set $\{1, 2, \ldots, m\}$, the generator $s_i$ is identified with the simple transposition $(i, i+1)$. The subgroup generated by $s_i$ $(i = 1, \ldots m-2)$ is isomorphic to $\mathfrak{S}_{m-1}$, and we shall be concerned with the problem of restricting representations of $\mathfrak{S}_m$ to this subgroup. Over fields of characteristic 0, the representation theory of $\mathfrak{S}_m$ has been understood since the beginning of the century.

**Theorem 3.1** *Over fields of characteristic 0, the inequivalent irreducible representations of $\mathfrak{S}_m$ are indexed by $\Pi(m)$, the set of partitions of $m$.*

There exist a number of ways to calculate the matrices of the irreducible representation of $\mathfrak{S}_m$ corresponding to a particular $\lambda \in \Pi(m)$. One way is by means of the Specht module $S^\lambda$ (see [25] for example), whereby the resulting matrices contain only integers. On restricting to $\mathfrak{S}_{m-1}$, the module $S^\lambda$ is no longer irreducible in general. One has the following well-known branching rule.

[13] The definition of $q$-binomial coefficient here is different from the one used in §4.

**Theorem 3.2** *Let $\lambda \in \Pi(m)$. Then*

$$S^\lambda \downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}} = \bigoplus_{\mu \in \mathcal{R}(\lambda)} S^\mu,$$

*where $\mathcal{R}(\lambda)$ is the set of all partitions of $m-1$ that can be obtained by removing a single node from $\lambda$.*

The following lattice of partitions, which is known as Young's lattice, results from joining a partition $\lambda \in \Pi(m)$ to a partition $\mu \in \Pi(m-1)$, if $S^\mu$ occurs in $S^\lambda \downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}}$:

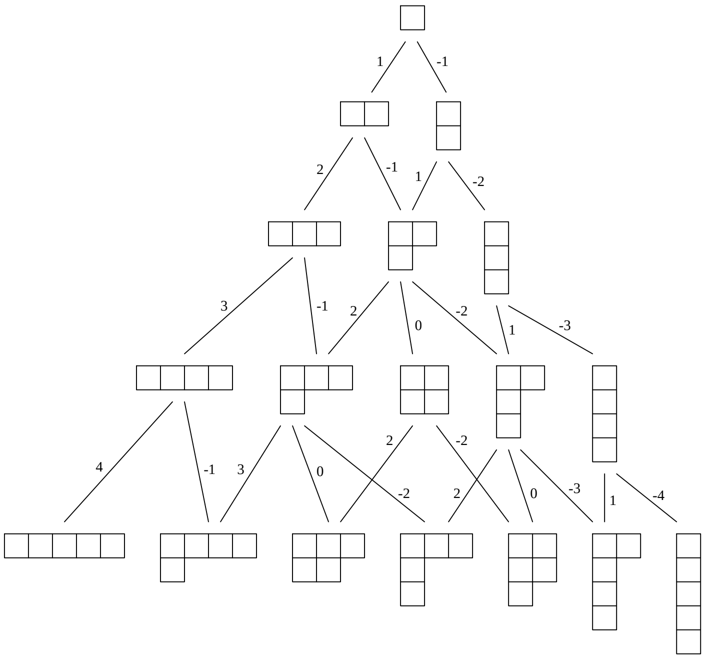



Figure 1.

As indicated in Fig. 1, each edge is naturally labelled by an integer. If the edge corresponds to the removal of a node sitting on row $i$ and column $j$, its label is $j-i$. The representation theoretical interpretation of these labels is as follows. The number associated with the edge linking $\lambda \in \Pi(m)$ and $\mu \in \Pi(m-1)$ is the eigenvalue of the eigenspace $S^\mu \subset S^\lambda$ of the Jucys-Murphy operator $L_m$, defined by

$$L_m = \sum_{i=1}^{m-1} (i,k).$$

## 3.2 Modular representations of $\mathfrak{S}_m$

On taking the entries of all of the matrices modulo a prime $n$, the Specht module representations remain well-defined and we obtain representations of $\mathfrak{S}_m$ over a field of characteristic $n$. However, it turns out that these representations are no longer irreducible in general. (In fact, they can be both reducible and indecomposable).

**Theorem 3.3** [24]

1. *Over a field of characteristic $n$, the inequivalent irreducible representations of $\mathfrak{S}_m$ are indexed by $\Pi_n(m)$, the set of $n$-regular partitions of $m$.*
2. *For each $\lambda \in \Pi_n(m)$, the corresponding irreducible module $D^\lambda$ is obtained by $D^\lambda = S^\lambda/\operatorname{rad} S^\lambda$, where $\operatorname{rad} S^\lambda$ is the maximal proper submodule of $S^\lambda$.*

Despite this result, much remains unknown about the $D^\lambda$. Of great interest is the search for a combinatorial algorithm giving the multiplicities of each $D^\mu$ amongst the composition factors of $S^\lambda$, not least because this would provide a hitherto unknown direct means to calculate the dimensions of the $D^\mu$. The multiplicities of the composition factors of $S^\lambda$ are conveniently written in a matrix with the rows labelled by $\lambda$ and the columns by $\mu$. This is the decomposition matrix of $\mathfrak{S}_m$ in characteristic n. In the case when $m = 5$ and $n = 2$, we have:

$$\begin{array}{cccc} & (5) & (41) & (32) \\ (5) & 1 & \cdot & \cdot \\ (41) & \cdot & 1 & \cdot \\ (32) & 1 & \cdot & 1 \\ (31^2) & 2 & \cdot & 1 \\ (2^21) & 1 & \cdot & 1 \\ (21^3) & \cdot & 1 & \cdot \\ (1^5) & 1 & \cdot & \cdot \end{array} \tag{30}$$

## 3.3 The Jantzen-Seitz problem for $\mathfrak{S}_m$

A modulo $n$ analogue of Theorem 3.2, giving a description of the restricted modules $D^\lambda\downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}}$, is also currently unknown.

In the $n = 2$ case, Benson [7] considered the somewhat simpler problem of determining which $\mathfrak{S}_m$-modules $D^\lambda$ remain irreducible on restriction to $\mathfrak{S}_{m-1}$. He conjectured that these modules are labelled by the partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ for which the $\lambda_i$ are pairwise distinct and all congruent modulo 2. Note that in the characteristic 0 case, Theorem 3.2 implies that this problem is solved by the rectangular partitions $\lambda = (k^l)$, since it is only in such a case that there is only one way to remove a node from $\lambda$ to leave a valid partition.

Later, Jantzen and Seitz considered the generalisation of Benson's conjecture to odd primes $n$. They proved that

**Theorem 3.4** [26] *Let* $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})$, *where* $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ *and* $0 < a_i < n$ *for* $i = 1, 2, \ldots, r$. *Then, over a field of characteristic* $n$, $D^\lambda \downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}}$ *is irreducible if*

$$a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod n,$$

*for* $i = 1, 2, \ldots, r-1$.

Jantzen and Seitz also conjectured that only for the $\lambda$ specified here, do the modules $D^\lambda$ remain irreducible on restriction. This conjecture was proved recently by Kleshchev [39].

In fact, Kleshchev managed to describe the *socle* (*i.e.* the sum of all simple submodules) of $D^\lambda \downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}}$.

**Theorem 3.5** [40] *Let* $\lambda$ *be a* $n$*-regular partition of* $m$. *Then*

$$\mathrm{Socle}(D^\lambda \downarrow^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}}) \cong \bigoplus_\mu D(\mu)$$

*where the sum is over all predecessors* $\mu$ *of* $\lambda$ *in Kleshchev's* $n$*-good lattice. In particular the sum is multiplicity-free and its description is combinatorial.*

Here Kleshchev's $n$-good lattice is a certain $n$-modular analogue of Young's lattice. It was observed in [41, 42] that this lattice is the same as the crystal graph of the basic representation of $\widehat{\mathfrak{sl}}_n$ computed in [45] (to be described below).

## 3.4 The Hecke algebras of type $A$

The (Iwahori-)Hecke algebra $H_m(v)$ of type $A_{m-1}$, is the $\mathbf{C}(v)$-algebra generated by the elements $T_1, \ldots, T_{m-1}$ subject to the relations:

$$\begin{aligned} &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \\ &T_i T_j = T_j T_i \qquad |i-j| > 1; \\ &T_i^2 = (v-1) T_i + v. \end{aligned}$$

If $w \in \mathfrak{S}_m$ and $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ is an expression for $w$ in terms of a minimal number of generators of $\mathfrak{S}_m$, define the element $T_w \in H_m(v)$ by $T_w = T_{i_1} T_{i_2} \cdots T_{i_l}$ (the defining relations of $H_m(v)$ ensure that $T_w$ is well-defined).

For generic values of $v$, $H_m(v)$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_m$. Thus, it is semisimple, and its irreducible representations are parametrised by partitions of $m$. Again a Specht module construction of the representation matrices may be undertaken (see Appendix A or [17]).

The entries in the representation matrices of the $T_w$ are now elements of $\mathbb{Z}[v]$. The branching graph of the Specht modules of $H_m(v)$ is the same as for symmetric groups in characteristic 0. In this case, the number $c$ associated with an edge linking $\lambda \in \Pi(m)$ and $\mu \in \Pi(m-1)$ indicates that the eigenvalue of the $v$-Jucys-Murphy operator $L_m(v)$ is $(v^c - 1)/(v-1)$ on the eigenspace $S^\mu \subset S^\lambda$. Here $L_m(v)$ is defined by

$$L_m(v) = \sum_{i=1}^{m-1} v^{i-m} T_{(i,m)}.$$

## 3.5 Modular representations of Hecke algebras

In the non-generic case when $v$ is a root of unity, although the Specht modules remain well defined, they are no longer irreducible in general.

**Theorem 3.6** [15] *Let $v$ be a primitive $n$th root of unity, that is, $v^n = 1$ and $v^k \neq 1$ for $1 \leq k < n$.*

1. *The inequivalent irreducible representations of $H_m(v)$ are indexed by $\Pi_n(m)$, the set of $n$-regular partitions of $m$.*
2. *For each $\lambda \in \Pi_n(m)$, the corresponding irreducible module $D^\lambda$ is obtained by $D^\lambda = S^\lambda / \mathrm{rad}\, S^\lambda$.*

Although the same labels $D^\lambda$ are traditionally used for the irreducible representations of $\mathfrak{S}_m$ over a field of characteristic $n$, and for the irreducible representations of $H_m(\sqrt[n]{1})$, the representations themselves should not be identified in any sense. Indeed, they may even have different dimensions. Moreover those of $\mathfrak{S}_m$ are defined only for $n$ prime.

As in the case of modular representations of $\mathfrak{S}_m$, one defines the decomposition matrices of $H_m(\sqrt[n]{1})$. Apart from their independent interest, they may be viewed as a stepping stone to the modular case of $\mathfrak{S}_m$. Compare the following table for $H_5(-1)$ (here $m = 5$ and $n = 2$) with that given in Section 3.1:

| | (5) | (41) | (32) |
|---|---|---|---|
| (5) | 1 | · | · |
| (41) | · | 1 | · |
| (32) | · | · | 1 |
| $(31^2)$ | 1 | · | 1 |
| $(2^2 1)$ | · | · | 1 |
| $(21^3)$ | · | 1 | · |
| $(1^5)$ | 1 | · | · |

(31)

Very recently, a combinatorial algorithm for calculating the decomposition matrices for $H_m(\sqrt[n]{1})$ was conjectured in [41, 42]. This algorithm computes in fact Kashiwara's global crystal basis of the basic representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ (see 5.3).

Lascoux, Leclerc and Thibon's interpretation of these bases in terms of the Grothendieck ring of representations of $H_m(\sqrt[n]{1})$ has now been proved by Ariki and Grojnowski.

This machinery enables us to determine for which $\lambda$ the restricted module $D^\lambda\downarrow^{H_m(\sqrt[n]{1})}_{H_{m-1}(\sqrt[n]{1})}$ is irreducible, and shows how to relate this Jantzen-Seitz type problem for Hecke algebras with the decomposition of tensor products of level 1 representations of $\widehat{\mathfrak{sl}}_n$.

# 4 The Fock representation of $U_q(\widehat{\mathfrak{sl}}_n)$

For our purposes, the relevant realisation of $\widehat{\mathfrak{sl}}_n$ and $U_q(\widehat{\mathfrak{sl}}_n)$ is the Fock representation. Moreover, the reduction from $H_m(v)$ to $H_m(\sqrt[n]{1})$ is reflected in the reduction from $\widehat{\mathfrak{gl}}_\infty$ to $\widehat{\mathfrak{sl}}_n$, so we shall first recall the Fock representation of $\widehat{\mathfrak{gl}}_\infty$.

## 4.1 The infinite rank affine algebra $\widehat{\mathfrak{gl}}_\infty$

Let $\overline{\mathfrak{gl}}_\infty$ denote the Lie algebra of complex $\mathbb{Z}\times\mathbb{Z}$-matrices $A=(a_{ij})$ such that $a_{ij}=0$ for $|i-j|\gg 0$. Therefore, $A$ has only a finite number of nonzero diagonals, and matrix multiplication makes sense, which allows to define the Lie bracket as the commutator $AB-BA$. Introduce the following block decomposition

$$A_{++}=(a_{ij})_{i,j>0},\ A_{+-}=(a_{ij})_{i>0,j\le 0},\ A_{-+}=(a_{ij})_{i\le 0,j>0},\ A_{--}=(a_{ij})_{i,j\le 0}.$$

Then (see [29]) $\widehat{\mathfrak{gl}}_\infty$ is defined as the one-dimensional central extension

$$\widehat{\mathfrak{gl}}_\infty=\overline{\mathfrak{gl}}_\infty\oplus\mathbf{C}\,c$$

with Lie bracket

$$\begin{aligned}[][A,B] &= AB-BA+\mathrm{tr}\,(A_{-+}B_{+-}-A_{+-}B_{-+})\,c, \quad (A,B\in\overline{\mathfrak{gl}}_\infty),\\ [c,\widehat{\mathfrak{gl}}_\infty] &= 0.\end{aligned}$$

The Chevalley generators of $\widehat{\mathfrak{gl}}_\infty$ are expressed in terms of the matrix units $E_{ij}$ by

$$\begin{aligned} e_i^\infty &= E_{i,i+1},\\ f_i^\infty &= E_{i+1,i},\\ h_i^\infty &= [e_i^\infty,f_i^\infty]=E_{i,i}-E_{i+1,i+1}+\delta_{i0}c, \quad (i\in\mathbb{Z}).\end{aligned}$$

By [29], $\widehat{\mathfrak{gl}}_\infty$ acts in a natural way on the polynomial ring

$$\mathcal{F}=\mathbf{C}[x_i, i\ge 1],$$

where the variable $x_i$ has degree $i$. This graded space is called the (bosonic) Fock space. The action is best described on a distinguished basis of homogeneous elements indexed by partitions, the Schur polynomials

$$s_\lambda = \sum_{\mu=(1^{m_1}\ldots r^{m_r})} \chi_\lambda(\mu)\,\frac{x_1^{m_1}\cdots x_r^{m_r}}{m_1!\cdots m_r!},$$

where for $\lambda$, $\mu$ partitions of $m$, $\chi_\lambda(\mu)$ denotes the irreducible character $\chi_\lambda$ of $\mathfrak{S}_m$ evaluated on a element of cycle-type $\mu$.

To write down the action, we need some combinatorial definitions. Let $\gamma$ be the node of the partition $\lambda$ situated on row $i$ and column $j$. The content $c = c(\gamma)$ of $\gamma$ is defined by $c(\gamma) = j - i$. The node $\gamma$ is then said to be a $c$-node. A removable $c$-node of $\lambda$ is a $c$-node $\gamma$ within $\lambda$, which if removed from $\lambda$ leaves a valid partition $\mu$. In this case, we also say that $\gamma$ is an addable $c$-node of $\mu$ and we use freely the notation $\lambda/\mu = \boxed{c}$ or $\lambda/\boxed{c} = \mu$ or $\mu \cup \boxed{c} = \lambda$.

**Example 4.1** To illustrate these definitions, consider the partition $(7,5,5,2,1,1)$, which has Young diagram:

$$\begin{array}{ccccccc}
\boxed{0}&\boxed{1}&\boxed{2}&\boxed{3}&\boxed{4}&\boxed{5}&\boxed{6}\\
\boxed{\bar{1}}&\boxed{0}&\boxed{1}&\boxed{2}&\boxed{3}&&\\
\boxed{\bar{2}}&\boxed{\bar{1}}&\boxed{0}&\boxed{1}&\boxed{2}&&\\
\boxed{\bar{3}}&\boxed{\bar{2}}&&&&&\\
\boxed{\bar{4}}&&&&&&\\
\boxed{\bar{5}}&&&&&&
\end{array}.$$

Here each node has been filled with its content, and for convenience the negative entries have the minus sign above the digit. We see that $\lambda$ has a removable $c$-node only if $c = -5,-2,2,6$ and an addable $c$-node only if $c = -6,-3,-1,4,7$. Taking an example from each of these cases, we have $\lambda/\mu = \boxed{\bar{2}}$ for $\mu = (7,5,5,1,1,1)$ and $\nu/\lambda = \boxed{7}$ for $\nu = (8,5,5,2,1,1)$.

We can now state

**Theorem 4.2** [29] *The Lie algebra $\widehat{\mathfrak{gl}}_\infty$ acts on $\mathcal{F}$ as follows:*

$$e_i^\infty s_\lambda = \begin{cases} s_\mu & \text{if } \lambda/\mu = \boxed{i}; \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

$$f_i^\infty s_\lambda = \begin{cases} s_\nu & \text{if } \nu/\lambda = \boxed{i}; \\ 0 & \text{otherwise,} \end{cases} \tag{33}$$

$$h_i^\infty s_\lambda = \begin{cases} -s_\lambda & \text{if } \lambda \text{ has a removable } i\text{-node;} \\ s_\lambda & \text{if } \lambda \text{ has an addable } i\text{-node;} \\ 0 & \text{otherwise,} \end{cases} \tag{34}$$

$$c\,s_\lambda = s_\lambda. \tag{35}$$

*This is an irreducible highest weight representation with highest weight $\Lambda_0$, that is, the highest weight vector $1 = s_0$ satisfies*

$$h_i^\infty s_0 = \delta_{i0}\, s_0.$$

Up to partitions with 5 nodes, the action of the generators is given by the following graph.

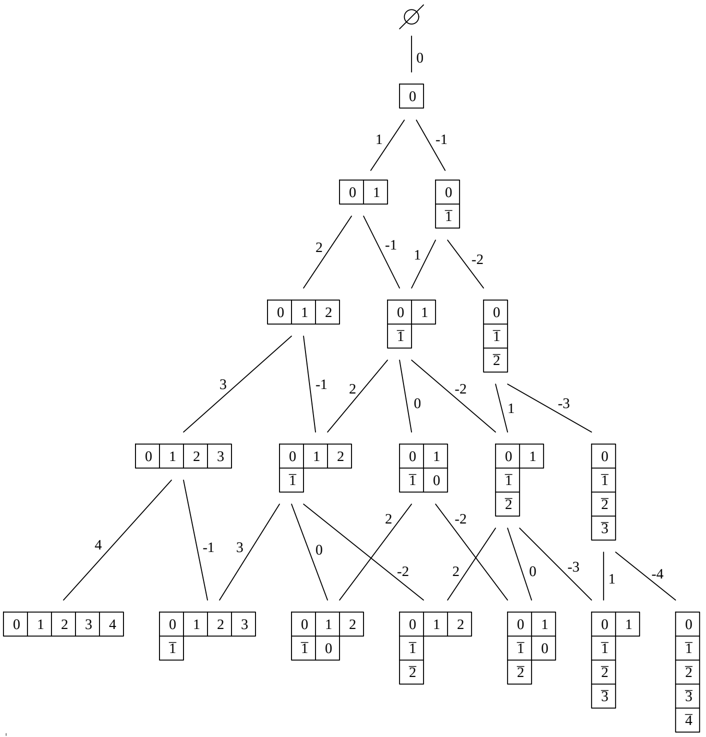


Figure 2.

We see that this graph is identical (apart from the inclusion of the empty partition $\emptyset$) to that in Fig. 1.

Moreover, if we define the complete restriction operator

$$e^{\infty} = \sum_{i=-\infty}^{\infty} e_i^{\infty},$$

the action of $e^{\infty}$ on $s_\lambda$ gives

$$e^{\infty} s_\lambda = \sum_{\mu \in \mathcal{R}(\lambda)} s_\mu,$$

where $\mathcal{R}(\lambda)$ is as given in Theorem 3.2. Thus, the Fock space representation of $\widehat{\mathfrak{gl}}_\infty$ yields the branching rule for $\mathfrak{S}_m$ in characteristic 0 (or $H_m(v)$ for generic $v$).

In what follows, a similar correspondence connecting the Hecke algebras $H_m(\sqrt[n]{1})$ and the Fock space representation of $\widehat{\mathfrak{sl}}_n$ is explained.

## 4.2 The embedding of $\widehat{\mathfrak{sl}}_n$ into $\widehat{\mathfrak{gl}}_\infty$

It is known that many affine Lie algebras can be realized as subalgebras of $\widehat{\mathfrak{gl}}_\infty$. This type of embedding has been used by the Kyoto group to exhibit relations between different hierarchies of soliton equations (see [29]). In particular, $\widehat{\mathfrak{sl}}_n$ can be realized as a subalgebra of $\widehat{\mathfrak{gl}}_\infty$ by taking as Chevalley generators of the former

$$\begin{aligned}
e_i &= \sum_{k=i\,(\bmod n)} e_k^\infty, \\
f_i &= \sum_{k=i\,(\bmod n)} f_k^\infty, \\
h_i &= \sum_{k=i\,(\bmod n)} h_k^\infty, \\
D &= -\sum_{k\in\mathbb{Z}} \left[\frac{k}{n}\right] E_{ii}.
\end{aligned}$$

Restricting to $\widehat{\mathfrak{sl}}_n$ the action of $\widehat{\mathfrak{gl}}_\infty$ on $\mathcal{F}$, one gets the Fock space representation of $\widehat{\mathfrak{sl}}_n$. Clearly, one has

$$\begin{aligned}
f_i s_\lambda &= \sum_{\mu:\mu/\lambda=\boxed{i}} s_\mu, \\
e_i s_\lambda &= \sum_{\mu:\lambda/\mu=\boxed{i}} s_\mu.
\end{aligned}$$

This action may be easily obtained from the graph in Fig. 2, after each of the labels (both in the partitions and on the edges) is replaced by its corresponding modulo $n$ value [11].

Under the action of $\widehat{\mathfrak{sl}}_n$, the Fock space $\mathcal{F}$ decomposes into

$$\mathcal{F} = \mathcal{T}_n \oplus \mathcal{S}_n \tag{36}$$

where

$$\mathcal{T}_n = U(\widehat{\mathfrak{sl}}_n)\, s_0 = \mathbf{C}[x_i, i \not\equiv 0 \bmod n],$$

and $\mathcal{S}_n$ is the ideal generated by the $x_{kn}$, $(k > 0)$. Moreover, as $\widehat{\mathfrak{sl}}_n$-modules,

$$\mathcal{T}_n \cong \mathcal{F}/\mathcal{S}_n \cong V(\Lambda_0),$$

the basic representation. Note that the natural map from $\mathcal{F}$ to $\mathcal{F}/\mathcal{S}_n$ obtained by setting $x_{kn} = 0$ is precisely the reduction of the KP-hierarchy to the KP$_n$-hierarchy (where KP$_2$ = KdV) (see [29]).

In our setting, the same map becomes the decomposition map from the Grothendieck group of $H_m(v)$-modules to that of $H_m(\sqrt[n]{1})$-modules (for all $m$).

## 4.3 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$

For $k, m \in \mathbb{Z}$, we define the following $q$-integers, factorials, and binomials

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]_q! = [k]_q\,[k-1]_q \cdots [1]_q, \quad \text{and} \quad \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]_q!}{[m-k]_q!\,[k]_q!}.$$

The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ is the unital associative algebra over $\mathbb{Q}(q)$ generated by the symbols $e_i$, $f_i$, $0 \le i \le n-1$, and $q^h$ for $h \in P^\vee$, subject to the relations:

$$q^h\,q^{h'} = q^{h+h'}; \quad q^0 = 1;$$
$$q^h e_j q^{-h} = q^{\langle \alpha_j, h\rangle} e_j;$$
$$q^h f_j q^{-h} = q^{-\langle \alpha_j, h\rangle} f_j;$$
$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}};$$
$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j\rangle \\ k \end{bmatrix}_q e_i^{1-\langle \alpha_i, h_j\rangle - k} e_j e_i^k = 0 \quad (i \ne j);$$
$$\sum_{k=0}^{1-\langle \alpha_i, h_j\rangle} (-1)^k \begin{bmatrix} 1 - \langle \alpha_i, h_j\rangle \\ k \end{bmatrix}_q f_i^{1-\langle \alpha_i, h_j\rangle - k} f_j f_i^k = 0 \quad (i \ne j).$$

The definitions concerning $U_q(\widehat{\mathfrak{sl}}_n)$-modules are direct analogues of those given for $\widehat{\mathfrak{sl}}_n$. Let $M$ be a $U_q(\widehat{\mathfrak{sl}}_n)$-module. For each weight $\Lambda \in P$, the subspace of $M$ defined by

$$M_\Lambda = \{v \in M \mid q^h\,v = q^{\langle \Lambda, h\rangle}\,v,\ h \in P^\vee\}$$

is called the weight space of weight $\Lambda$ of $M$. If $v \in M_\Lambda$ then we write $\mathrm{wt}\,(v) = \Lambda$, and call such $v$ a weight vector of weight $\Lambda$.

The module $M$ is said to be integrable if:

1. $M = \bigoplus_{\Lambda \in P} M_\Lambda$;
2. $\dim M_\Lambda < \infty$ for each $\Lambda \in P$;
3. for each $i = 0, 1, \ldots, n-1$, $M$ decomposes into a direct sum of finite dimensional $U_i$-modules, where $U_i$ denotes the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $e_i$, $f_i$, $q^{h_i}$ and $q^{-h_i}$.

A weight vector $v \in M$ for which $e_i v = 0$ for all $i = 0, 1, \ldots, n-1$, is said to be a highest weight vector. If there exists a highest weight vector $v \in M$ such that $M = U_q(\widehat{\mathfrak{sl}}_n)\,v$, then $M$ is said to be a highest weight module. The weight of $v$ is called the highest weight of $M$.

For $l \ge 0$ and all $\Lambda \in P_l^+$, there exists (up to equivalence) a unique integrable highest weight module $V(\Lambda)$ with highest weight $\Lambda$, and it is irreducible [9].

## 4.4 The Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$

We require further notation concerning removable and addable nodes. From now on, we say that $\gamma$ is a $i$-node if its content $c$ is such that $c = i \pmod{n}$.

**Definition 4.3** *Let $\lambda$ be a partition, $0 \le i < n$, $\gamma$ be an addable $i$-node of $\lambda$, and $\nu = \lambda \cup \gamma$. We set*

1. $N_i(\lambda) = \#\{$ *addable $i$-nodes of $\lambda$* $\} - \#\{$ *removable $i$-nodes of $\lambda$* $\}$,
2. $N_i^l(\lambda, \nu) = \#\{$ *addable $i$-nodes of $\lambda$ situated strictly to the left of $\gamma$* $\}$ $-\#\{$ *removable $i$-nodes of $\lambda$ situated to the left of $\gamma$* $\}$,
3. $N_i^r(\lambda, \nu) = \#\{$ *addable $i$-nodes of $\lambda$ situated strictly to the right of $\gamma$* $\}$ $-\#\{$ *removable $i$-nodes of $\lambda$ situated to the right of $\gamma$* $\}$,
4. $N^0(\lambda) = \#\{$ *0-nodes of $\lambda$* $\}$.

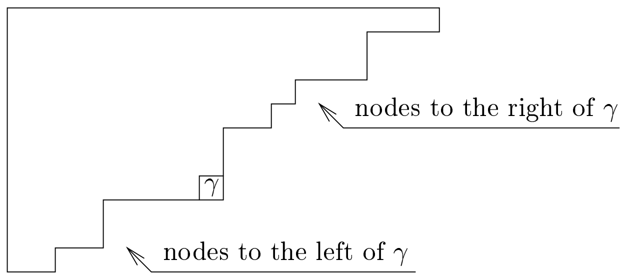



Figure 3.

The following construction of the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$ is due to Hayashi [22]. We give a formulation by Misra and Miwa [45].

**Theorem 4.4 ($q$-Fock space)** [22, 45] *Let the $q$-Fock space be defined by*

$$\mathcal{F} = \bigoplus_{\lambda \in \Pi} \mathbb{Q}(q)\, v_\lambda .$$

*Then $U_q(\widehat{\mathfrak{sl}}_n)$ acts on $\mathcal{F}$ by*

$$\begin{aligned} q^{h_i}\, v_\lambda &= q^{N_i(\lambda)}\, v_\lambda , \\ q^{D}\, v_\lambda &= q^{-N^0(\lambda)}\, v_\lambda , \\ f_i v_\lambda &= \sum_{\nu : \nu/\lambda = \boxed{i}} q^{N_i^r(\lambda,\nu)}\, v_\nu , \\ e_i v_\lambda &= \sum_{\nu : \lambda/\nu = \boxed{i}} q^{-N_i^l(\nu,\lambda)}\, v_\nu . \end{aligned}$$

*This representation is integrable.*

The "vacuum" vector $v_\emptyset$ labelled by the empty partition, is a highest weight vector of weight $\Lambda_0$. The decomposition of $\mathcal{F}$ under the action of $U_q(\widehat{\mathfrak{sl}}_n)$ is similar to (36). In particular one has

$$U_q(\widehat{\mathfrak{sl}}_n)\, v_\emptyset \cong V(\Lambda_0).$$

In the following section, we will describe a basis for this subrepresentation of $\mathcal{F}$. Of interest to us will be the transition matrices between that basis and the basis $(v_\lambda)$ of $\mathcal{F}$.

# 5 Canonical bases

Canonical bases of the irreducible highest weight modules of affine algebras have been introduced by Kashiwara as classical limits ($q \mapsto 1$) of the so-called global crystal bases of the corresponding quantum affine algebras. The first step towards obtaining the global crystal basis is to determine the crystal basis at $q = 0$ and the associated crystal graph, a combinatorial object describing the action of the quantum algebra in the crystal limit $q \mapsto 0$.

## 5.1 The crystal basis at $q = 0$

In this section, we will describe following [45] the crystal graph of $\mathcal{F}$ and isolate in it the crystal graph of $V(\Lambda_0)$.

To define a crystal basis at $q = 0$ (or simply a crystal basis), one needs to introduce Kashiwara's operators.

On restricting to $U_i$, the module $M$ decomposes into a direct sum of simple $U_i$-modules, with each $(l+1)$-dimensional $U_i$-module isomorphic to the $U_q(\mathfrak{sl}_2)$-module $V_l$ having highest weight $l\Lambda_1$:

$$M \cong \bigoplus_k V_{l_k}$$

Now, for arbitrary $l$, write

$$V_l = \bigoplus_{k=0}^{l} \mathbb{Q}(q)\, u_k^{(l)} ,$$

where

$$e_i\, u_0^{(l)} = f_i u_l^{(l)} = 0 \quad \text{and} \quad u_k^{(l)} = \frac{f_i^k}{[k]_q!}\, u_0^{(l)} .$$

Then the action of $\tilde{e_i}$ and $\tilde{f_i}$ on $V_l$ is defined by:

$$\tilde{f_i}\, u_k^{(l)} = u_{k+1}^{(l)} , \quad \tilde{e_i}\, u_k^{(l)} = u_{k-1}^{(l)} ,$$

where we understand $u^{(l)}_{-1} = u^{(l)}_{l+1} = 0$. The endomorphisms $\tilde{e}_i$, $\tilde{f}_i$ are then extended to $M$ using the above isomorphism of $U_i$-modules between $M$ and $\bigoplus_k V_{l_k}$.

Thus, any integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $M$ is equipped with a family of endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$.

Let $A \subset \mathbb{Q}(q)$ denote the ring of rational functions without a pole at $q = 0$. A *crystal lattice* of $M$ is a free $A$-module $L$ such that $M = \mathbb{Q}(q) \otimes_A L,\ L = \oplus_\Lambda L_\Lambda$ where $L_\Lambda = L \cap M_\Lambda$, and

$$\tilde{e}_i L \subset L, \quad \tilde{f}_i L \subset L, \quad (i = 0, 1, \ldots n-1).$$

In other words, $L$ spans $M$ over $\mathbb{Q}(q)$, $L$ is compatible with the weight space decomposition of $M$ and is stable under Kashiwara's operators. It follows that $\tilde{e}_i$, $\tilde{f}_i$ induce endomorphisms of the $\mathbb{Q}$-vector space $L/qL$ that we shall still denote by $\tilde{e}_i$, $\tilde{f}_i$. Now Kashiwara defines a *crystal basis* of $M$ (at $q = 0$) to be a pair $(L, B)$ where $L$ is a crystal lattice in $M$ and $B$ is a basis of $L/qL$ such that $B = \sqcup B_\Lambda$ where $B_\Lambda = B \cap (L_\Lambda / qL_\Lambda)$, and

$$\tilde{e}_i B \subset B \sqcup \{0\}, \quad \tilde{f}_i B \subset B \sqcup \{0\}, \quad i = 0, \ldots, n-1,$$

$$\tilde{e}_i v = u \quad \Longleftrightarrow \quad \tilde{f}_i u = v, \quad u, v \in B, \quad i = 0, \ldots, n-1.$$

Kashiwara has proven the following existence and uniqueness result for crystal bases [34, 35].

**Theorem 5.1** *Any integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $M$ has a crystal basis $(L, B)$. Moreover, if $(L', B')$ is another crystal basis of $M$, then there exists a $U_q(\widehat{\mathfrak{sl}}_n)$-automorphism of $M$ sending $L$ to $L'$ thence inducing an isomorphism of vector spaces from $L/qL$ to $L'/qL'$ which sends $B$ to $B'$. In particular, if $M = V(\Lambda)$ is irreducible, its crystal basis $(L(\Lambda), B(\Lambda))$ is unique up to an overall scalar multiple. It is given by*

$$L(\Lambda) = \sum_{0 \le i_1, i_2, \ldots, i_r \le n-1} A\, \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} u_\Lambda, \tag{37}$$

$$B(\Lambda) = \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r} u_\Lambda \mod qL(\Lambda) \,|\, 0 \le i_1, \ldots, i_r \le n-1\} \backslash \{0\}, \tag{38}$$

*where $u_\Lambda$ is a highest weight vector of $V(\Lambda)$.*

It follows that to each integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $M$, one can associate a well-defined coloured graph $\Gamma(M)$ whose vertices are labelled by the elements of $B$ and whose edges describe the action of the operators $\tilde{f}_i$ :

$$u \xrightarrow{i} v \iff \tilde{f}_i u = v.$$

$\Gamma(M)$ is called the *crystal graph* of $M$.

A crystal basis and the crystal graph of the $q$-Fock space $\mathcal{F}$ have been determined by Misra and Miwa.

**Theorem 5.2** [45] *The pair* $(L, B)$ *where*

$$L = \bigoplus_{\lambda \in \Pi} A\, v_\lambda\,,$$

*and* $B$ *is the basis of* $L/qL$ *given by*

$$B = \{v_\lambda \bmod qL \mid \lambda \in \Pi\}\,.$$

*is a crystal basis of* $\mathcal{F}$.

To describe the crystal graph we introduce the notion of *good node.* Let $\lambda$ be a partition and $0 \le i < n$. Construct a sequence of $A$s and $R$s, by from left to right, scanning the columns of $\lambda$ and noting the presence of an addable $i$-node by $A$ and the presence of a removable $i$-node by $R$. It will be useful to attach a subscript to each of these symbols to indicate the column from which it arose.

Now recursively remove $RA$ pairs (together with their subscripts) from this sequence until none remain. The sequence will then be of the form

$$AA\cdots AR\cdots RRR.$$

The node corresponding to the leftmost $R$ is termed a good (removable) $i$-node and that corresponding to the rightmost $A$ is termed a good addable $i$-node (note that there is at most one of each).

**Example 5.3** To illustrate this definition, consider $n = 3$ and the partition $\lambda = (16, 13, 11, 10, 9, 8, 7, 5, 2)$:

$$\begin{array}{cccccccccccccccc}
0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0\\
2&0&1&2&0&1&2&0&1&2&0&1&2&&&\\
1&2&0&1&2&0&1&2&0&1&2&&&&&\\
0&1&2&0&1&2&0&1&2&0&&&&&&\\
2&0&1&2&0&1&2&0&1&&&&&&&\\
1&2&0&1&2&0&1&2&&&&&&&&\\
0&1&2&0&1&2&0&&&&&&&&&\\
2&0&1&2&0&&&&&&&&&&&\\
1&2&&&&&&&&&&&&&&
\end{array}\,.$$

Here there are addable 0-nodes in columns 1,3,9,12 and 14, and removable 0-nodes in columns 5,7,10 and 16. Thus we form the following sequence:

$$A_1 A_3 R_5 R_7 A_9 R_{10} A_{12} A_{14} R_{16}.$$

The removal procedure first disposes of $R_7 A_9$ and $R_{10} A_{12}$, and then disposes of $R_5 A_{14}$ so that:

$$A_1 A_3 R_{16}$$

remains. Therefore $\lambda$ has a good addable node in the 3rd column and a good removable node in the 16th column. In the case of 1-nodes and 2-nodes, we first obtain the sequences:

$$A_6 A_8 R_9 A_{11} A_{17}$$

and

$$R_2R_8A_{10}R_{11}R_{13},$$

respectively which, after the removal procedure, produce:

$$A_6A_8A_{17}$$

and

$$R_2R_{11}R_{13}$$

respectively. Thus $\lambda$ has a good addable 1-node in column 17, but no good removable 1-node, and a good removable 2-node in column 2, but no good addable 2-node.

**Theorem 5.4** [45] *The crystal graph $\Gamma_n$ of $\mathcal{F}$ is the graph with vertices labelled by $\Pi$, the set of partitions, and whose arrows are given by:*

$$\lambda \xrightarrow{i} \mu \iff \mu \text{ is obtained from } \lambda \text{ by adding a good addable } i\text{-node.}$$

In such a case, we write $\tilde{f}_i(\lambda) = \mu$ and $\tilde{e}_i(\mu) = \lambda$. For each $\lambda \in \Pi$, the largest integer $k$ such that $\tilde{e}_i{}^k(\lambda) \neq 0$ is denoted $\varepsilon_i(\lambda)$. The largest integer $k$ such that $\tilde{f}_i{}^k(\lambda) \neq 0$ is denoted $\varphi_i(\lambda)$.

**Example 5.5** Up to partitions of weight 5, the crystal graph $\Gamma_2$ is as follows:

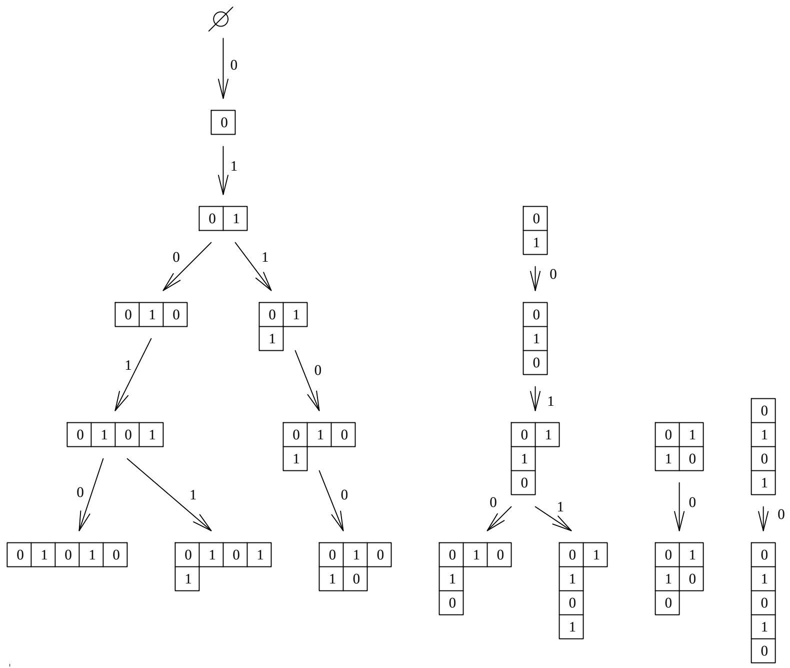


Figure 4.

It is clear from the definition that the crystal graph $\Gamma(M)$ of the direct sum $M = M_1 \oplus M_2$ of two $U_q(\widehat{\mathfrak{sl}}_n)$-modules is the disjoint union of $\Gamma(M_1)$ and $\Gamma(M_2)$. It follows from the complete reducibility of $M$ that the connected components of $\Gamma(M)$ are the crystal graphs of the irreducible components of $M$ [34, 35].

Thus, in Fig. 4, the splitting of $\Gamma_2$ into its connected components reflects the decomposition of the Fock representation of $U_q(\widehat{\mathfrak{sl}}_2)$.

In particular, the connected component of $\emptyset$ in $\Gamma_n$ is the crystal graph of the $U_q(\widehat{\mathfrak{sl}}_n)$-module $V(\Lambda_0)$. Its set of vertices is $\Pi_n$, the set of $n$-regular partitions. The crystal graph of $V(\Lambda_0)$ for $n = 3$ is shown in Fig. 5 (up to partitions of weight 8).

It is easily seen that each connected component of $\Gamma_n$ is headed by a unique partition $\mu$ of the form $\lambda^{*n}$, where if $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})$, we define $\lambda^{*n} = (\lambda_1^{na_1}, \lambda_2^{na_2}, \ldots, \lambda_r^{na_r})$. This is the combinatorial image of the known decomposition under $U_q(\widehat{\mathfrak{sl}}_n)$

$$\mathcal{F} \cong \bigoplus_{k\geq 0} V(\Lambda_0 - k\delta)^{\oplus p(k)} ,$$

where $p(k)$ denotes the number of partitions of $k$.

## 5.2 Tensor products

One of the main properties of crystal bases at $q = 0$ is that they behave well under tensor products [34, 35].

**Theorem 5.6** *Let $(L_1, B_1)$ and $(L_2, B_2)$ be crystal bases of integrable $U_q(\widehat{\mathfrak{sl}}_n)$-modules $M_1$ and $M_2$. Let $B_1 \otimes B_2$ denote the basis $\{u \otimes v,\ u \in B_1,\ v \in B_2\}$ of $(L_1/qL_1) \otimes (L_2/qL_2)$ (which is isomorphic to $(L_1 \otimes L_2)/q(L_1 \otimes L_2)$). Then, $(L_1 \otimes L_2,\ B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$, the action of $\tilde{e}_i$, $\tilde{f}_i$ on $B_1 \otimes B_2$ being given by*

$$\begin{aligned}
\tilde{f}_i(u \otimes v) &= \begin{cases} \tilde{f}_i u \otimes v & \text{if } \varphi_i(u) > \varepsilon_i(v); \\ u \otimes \tilde{f}_i v & \text{otherwise,} \end{cases} \\
\tilde{e}_i(u \otimes v) &= \begin{cases} \tilde{e}_i u \otimes v & \text{if } \varphi_i(u) \geq \varepsilon_i(v); \\ u \otimes \tilde{e}_i v & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking into account the fact that the decomposition of an integrable module $M$ into irreducible components is reflected by the splitting of its crystal graph into connected components, one sees that Theorem 5.6 gives a powerful way of computing tensor product multiplicities.

Let $\Lambda'$, $\Lambda''$, $\Lambda$ be dominant integral weights. The multiplicity $c^{\Lambda}_{\Lambda'\,\Lambda''}$ of $V(\Lambda)$ in the tensor product $V(\Lambda') \otimes V(\Lambda'')$ is equal to the number of vertices $b_1 \otimes b_2$ of $B(\Lambda') \otimes B(\Lambda'')$ that satisfy

$$\mathrm{wt}\,(b_1 \otimes b_2) = \Lambda \quad \text{and} \quad \tilde{e}_i(b_1 \otimes b_2) = 0 \quad (i = 0, \ldots, n-1).$$

By Theorem 5.6, this last condition is equivalent to the fact that $b_1 = b_{\Lambda'}$, the origin of the crystal graph of $V(\Lambda')$, and

$$\varepsilon_i(b_2) \le \langle \Lambda', h_i \rangle \qquad i = 0, 1, \dots, n-1.$$

Hence we get,

**Corollary 5.7** *The multiplicity $c^{\Lambda}_{\Lambda' \Lambda''}$ of $V(\Lambda)$ in $V(\Lambda') \otimes V(\Lambda'')$ is equal to the number of vertices $b_2$ of $B(\Lambda'')$ such that*

$$\mathrm{wt}\,(b_2) = \Lambda - \Lambda' \quad \textit{and} \quad \varepsilon_i(b_2) \le \langle \Lambda', h_i \rangle \qquad i = 0, 1, \dots, n-1.$$

## 5.3 The lower global crystal basis

The crystal basis of a $U_q(\widehat{\mathfrak{sl}}_n)$-module $M$ is not a basis of $M$, but only a combinatorial object reflecting the structure of $M$ in the crystal limit $q \mapsto 0$. To get a true basis of $M$, one needs to "globalize" this notion (Kashiwara's terminology) and define a global crystal basis. Kashiwara's theory enables such a basis to be obtained for all irreducible highest weight modules $V(\Lambda)$. However, if $M$ is not irreducible, there is no general way to obtain a global basis of $M$. For example, this is the situation for the Fock space $\mathcal{F}$ (note however, [43]).

In fact there are two global bases of $V(\Lambda)$, the lower one and the upper one, which are dual one to the other. In this section we consider the first and recall the definition of the lower global crystal basis of $V(\Lambda_0)$. We need to introduce an involution $v \longrightarrow \overline{v}$ of $V(\Lambda_0)$. We start from the involution $P \longrightarrow \overline{P}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ defined as the ring automorphism satisfying

$$\overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h}, \qquad (h \in P^\vee)\,,$$
$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad i = 0, 1, \dots, n-1\,.$$

Then, for $v = P\,v_\emptyset \in V(\Lambda_0)$, we set $\overline{v} = \overline{P}\,v_\emptyset$. Finally, we denote by $U^-_{\mathbb{Q}}$ the sub-$\mathbb{Q}[q, q^{-1}]$-algebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by $f_i^{(k)} := f_i^k/[k]!$, and we set $V_{\mathbb{Q}}(\Lambda_0) = U^-_{\mathbb{Q}}\,v_\emptyset$. We can now state

**Theorem 5.8** [35] *There exists a unique $\mathbb{Q}[q, q^{-1}]$-basis $\{G(\mu) \mid \mu \in \Pi_n\}$ of $V_{\mathbb{Q}}(\Lambda_0)$ such that:*

$$\begin{array}{ll} \text{(G1)} & G(\mu) \equiv v_\mu (\bmod qL), \\ \text{(G2)} & \overline{G(\mu)} = G(\mu). \end{array}$$

The basis $\{G(\mu)\}$ is called the lower global crystal basis of $V_{\mathbb{Q}}(\Lambda_0)$ (or simply of $V(\Lambda_0)$).

An efficient combinatorial means of computing the lower global crystal basis of $V(\Lambda_0)$ was given by [42]. In general, for each $\mu \in \Pi_n(m)$, this algorithm yields an expansion:

$$G(\mu) = \sum_{\lambda \in \Pi(m)} d_{\lambda\mu}(q) v_\lambda, \qquad (39)$$

where each $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$. In the case $n = 2$, for partitions of weight 5, the $d_{\lambda\mu}$ are given in the following table, where the columns are indexed by $\mu$ and the rows by $\lambda$.

$$\begin{array}{cccc} & (5) & (41) & (32) \\ (5) & 1 & \cdot & \cdot \\ (41) & \cdot & 1 & \cdot \\ (32) & \cdot & \cdot & 1 \\ (31^2) & q & \cdot & q \\ (2^21) & \cdot & \cdot & q^2 \\ (21^3) & \cdot & q & \cdot \\ (1^5) & q^2 & \cdot & \cdot \end{array} \tag{40}$$

The similarity with (31) led the authors of [42] to conjecture the following result, which was later proved by Ariki [2] and by Grojnowski.

**Theorem 5.9** *Let $d_{\lambda\mu}(q)$ be given by* (39). *Then the multiplicity of the irreducible module $D^\mu$ as a composition factor of the Specht module $S^\lambda$ of $H_m(v)$ when $v$ is a primitive $n$th root of unity is given by $d_{\lambda\mu}(1)$.*

It is interesting to reformulate this theorem in terms of Grothendieck groups [42]. Let $K_0(H_m(\sqrt[n]{1}))$ denote the Grothendieck group of the category of finitely generated projective $H_m(\sqrt[n]{1})$-modules. This is a free abelian group generated by the isomorphism classes of the indecomposable direct summands of $H_m(\sqrt[n]{1})$. If the $\widehat{\mathfrak{sl}}_n$-module $V(\Lambda_0)$ is regarded, via Robinson's $i$-induction and $i$-restriction operators, as the sum of Grothendieck groups

$$V(\Lambda_0) = \mathcal{K} = \bigoplus_m K_0(H_m(\sqrt[n]{1})),$$

then the theorem states that the global lower crystal basis (at $q = 1$) coincides with the canonical basis of $\mathcal{K}$ given by the classes of indecomposable summands of $H_m(\sqrt[n]{1})$.

Although only the values of $d_{\lambda\mu}(q)$ at $q = 1$ are required to determine the composition factors of $S^\lambda$, it is conjectured in [42], that the coefficients of the polynomials $d_{\lambda\mu}(q)$ are positive and provide information on the order of the composition factors in the composition series of $S^\lambda$.

## 5.4 The upper global crystal basis

The upper global crystal basis of $V(\Lambda_0)$ is defined as the basis adjoint to the lower global crystal basis with respect to the so-called Shapovalov form of $V(\Lambda_0)$. This bilinear form is characterized by the following properties

$$(v_\emptyset, v_\emptyset) = 1, \qquad (q^h u, v) = (u, q^h v), \qquad (f_i u, v) = (u, e_i v),$$

for all $u, v \in V(\Lambda_0)$, $h \in P^\vee$, and $0 \le i < n$. Thus the upper global crystal basis is the unique basis $\{G^{up}(\mu) \mid \mu \in \Pi_n\}$ of $V(\Lambda_0)$ such that

$$(G^{up}(\mu), G(\nu)) = \delta_{\mu\nu}, \quad (\mu, \nu \in \Pi_n).$$

The interpretation of the upper basis in terms of Grothendieck groups of Hecke algebras is the following. Let $G_0(H_m(\sqrt[n]{1}))$ be the Grothendieck group of the category of finitely generated $H_m(\sqrt[n]{1})$-modules. The elements of $G_0(H_m(\sqrt[n]{1}))$ are classes $[M]$ of modules, where $[M_1] = [M_2]$ if and only if the composition factors of $M_1$ occur in $M_2$ with identical multiplicity. (The order of the composition factors in the series is disregarded). The sum is defined by $[M]+[N] = [M \oplus N]$. It is known that this is a free abelian group with basis the set $\{[D^\mu]\}$ of classes of irreducible $H_m(\sqrt[n]{1})$-modules.

Then the direct sum

$$\mathcal{G} = \bigoplus_m G_0(H_m(\sqrt[n]{1}))$$

endowed with Robinson's $i$-induction and $i$-restriction operators becomes a $\widehat{\mathfrak{sl}}_n$-module isomorphic to $V(\Lambda_0)$ [42]. It has a canonical basis, namely $\{[D^\mu],\ \mu \in \Pi_n\}$. Another reformulation of Theorem 5.9 is that this basis coincides with the upper global crystal basis (at $q = 1$).

In particular, we have the following interpretation of the complete restriction operator in terms of the upper global basis of $V(\Lambda_0)$.

**Theorem 5.10** *Let $\lambda \in \Pi_n(m)$ and write*

$$\left(\sum_{i=0}^{n-1} e_i\right) G^{up}(\lambda) = \sum_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(q) G^{up}(\mu).$$

*Then*

$$[D^\lambda \downarrow^{H_m}_{H_{m-1}}] = \bigoplus_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(1)[D^\mu].$$

# 6 The Jantzen-Seitz problem for Hecke algebras at roots of unity

The results of [42] and [2] provide us with a correspondence between the theory of modular representations of Hecke algebras and solvable lattice models. Under this correspondence, the Hecke JS problem is reformulated as a problem that is already solved in the context of exactly solvable lattice models, namely, the restriction of a solid-on-solid model to the simplest possible non-trivial restricted one.

Let $\lambda \in \Pi_n$. One can write $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_r^{a_r})$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ and $0 < a_i < n$ for $i = 1, 2, \ldots, r$. We define the set $JS(n)$ as the set of all $\lambda \in \Pi_n$ such that

$$a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod{n},$$

for $i = 1, 2, \ldots, r-1$.

**Theorem 6.1** *Let $\lambda \in \Pi_n(m)$. Then $D^\lambda \downarrow_{H_{m-1}}^{H_m}$ is irreducible if and only if $\lambda \in JS(n)$.*

*Proof:* It is known (see [36] 5.3.8.–5.3.10.) that

$$e_i\, G^{up}(\lambda) = [\varepsilon_i(\lambda)]_q G^{up}(\tilde{e}_i\lambda) + \sum_{\mu \neq \lambda} E^i_{\lambda\mu}\, G^{up}(\mu) \tag{41}$$

where $E^i_{\lambda\mu}$ is a Laurent polynomial in $q$ invariant under $q \mapsto q^{-1}$, and such that:

$$\lim_{q \to 0} \frac{E^i_{\lambda\mu}}{[\varepsilon_i(\lambda)]_q} = 0\,.$$

If $\varepsilon_i(\lambda) = 1$ whereupon $[\varepsilon_i(\lambda)]_q = 1$, this implies that each $E^i_{\lambda\nu} = 0$ since for all $k$, the coefficients of $q^k$ and $q^{-k}$ in $E^i_{\lambda\mu}$ are equal. Thereupon $e_i\, G_q^{up}(\lambda) = G_q^{up}(\nu)$ where $\nu = \tilde{e}_i\lambda$. On the other hand, if $e_i\, G_q^{up}(\lambda) = G_q^{up}(\nu)$ for some $\nu$ then since $[x]_q = q^{x-1} + q^{x-3} + \cdots + q^{-x+1}$, (41) necessarily implies that $\varepsilon_i(\lambda) = 1$ whereupon $\nu = \tilde{e}_i\lambda$.

By Theorem 5.10, $D^\lambda\!\downarrow$ is irreducible if and only if $\sum e_i G^{up}(\lambda) = G^{up}(\nu)$ for some $\nu$. Therefore, by the above, $D^\lambda\!\downarrow$ is irreducible if and only if $\varepsilon_j(\lambda) = 1$ for some $j \in \{0, 1, \ldots, n-1\}$ and $\varepsilon_i(\lambda) = 0$ for $i \neq j$.

Now consider the branching function $b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q)$. By Corollary 5.7, the coefficient of $q^k$ in this function is equal to the number of vertices $b$ of the crystal graph of $B(\Lambda_0)$ such that:

$$\begin{aligned} &\mathrm{wt}(b) = \Lambda - \Lambda_j - k\delta\,; \\ &\varepsilon_j(b) \leq 1\,; \\ &\varepsilon_i(b) = 0 \quad (i \neq j). \end{aligned}$$

It follows from this discussion that the partitions $\lambda$ such that $D^\lambda \downarrow$ is irreducible can be identified with the set of vertices $b$ of the crystal graph $B(\Lambda_0)$ which contribute to a branching function $b^{\Lambda}_{\Lambda_j\,\Lambda_0}(q)$ for some $j$ and some $\Lambda \in P_2^+$. By 2.14 and 2.16, this set of partitions is precisely $JS(n)$, which completes the proof.

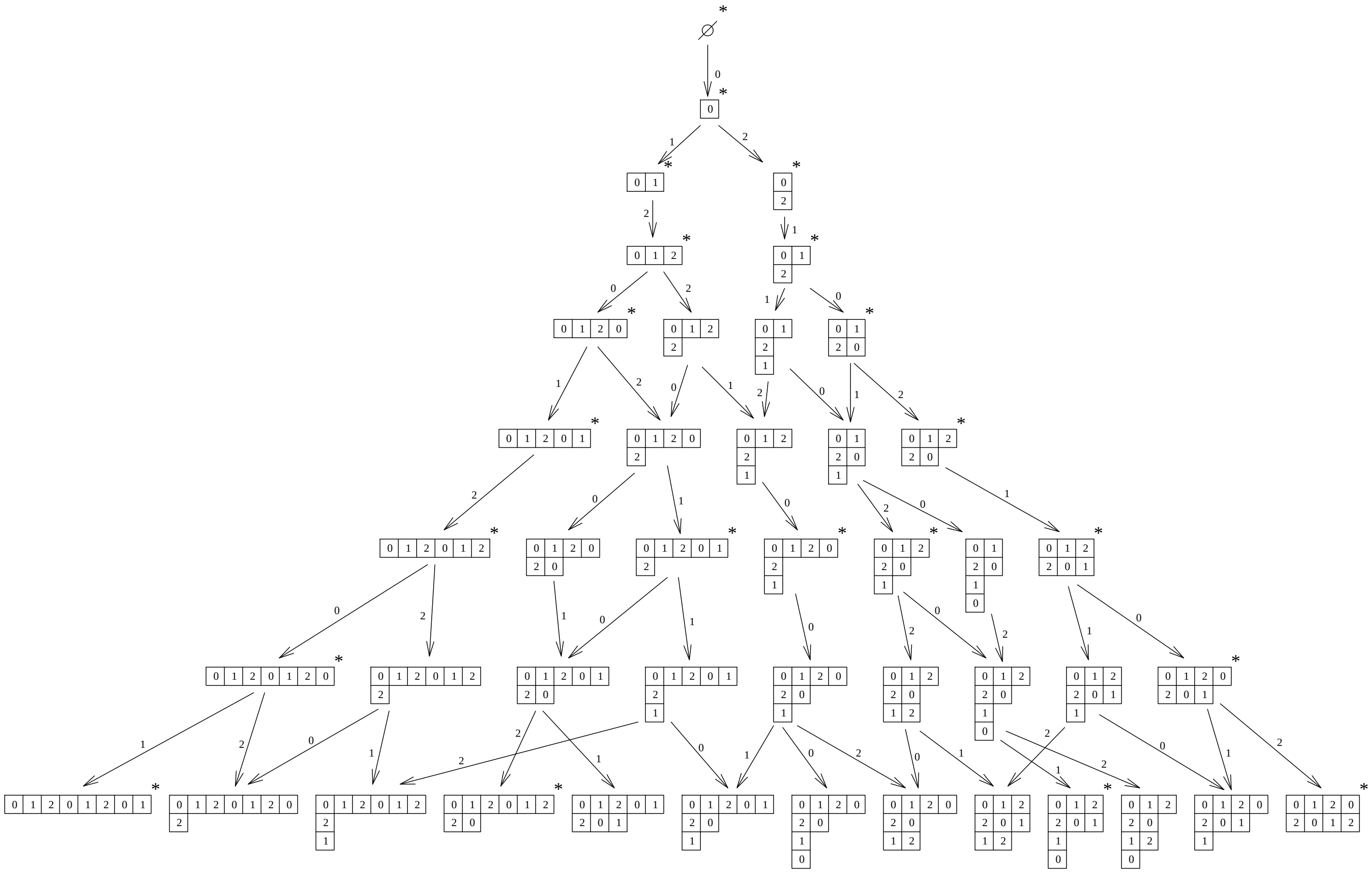


Figure 5.

**Example 6.2** The argument considered in the above proof may be illustrated by reference to the $\emptyset$-connected component of the crystal graph $\Gamma_3$ of $\widehat{\mathfrak{sl}}_3$ given (up to weight 8) in Fig. 5.
Here, those partitions $\lambda$ for which $\lambda \in JS(n)$ have been highlighted with an asterisk. As in the above proof, these partitions correspond to the nodes $b$ in this crystal graph for which, for some $j$, $\varepsilon_j(b) \leq 1$ and $\varepsilon_i(b) = 0$ for all $i \neq j$.

We will now give the interpretation in terms of Hecke algebras (or symmetric groups) of the splitting

$$JS(n) = \coprod_{j,k} FOW(n, j, k).$$

To do this we recall the notions of $n$-core and $n$-weight of a partition [25].

If $\lambda$ is a partition, then a sequence of $n$ adjoining nodes within $\lambda$ that begins at the rightmost end of a row, passes to the node directly below if one exists, else passes to the node directly on the left, and finishes at the bottom of a column, is known as an $n$-rim-hook of $\lambda$. For example, in the case $\lambda = (754^2)$, there are two 5-rim-hooks. In the following diagram, one comprises the nodes labelled 3,4,5,6 and 7, and the other comprises the nodes labelled 5,6,7,8 and 9:

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 2 | 1 | 0 |
| | | | 4 | 3 | | |
| | | | 5 | | | |
| 9 | 8 | 7 | 6 | | | |

.

(We also see that $\lambda$ has two 3-rim-hooks and four 4-rim-hooks.)

If $\lambda$ has no $n$-rim-hooks then $\lambda$ is termed an *$n$-core*.

Removing an $n$-rim-hook from $\lambda$ produces a valid partition. Thus a process of successive $n$-rim-hook removals may be carried out, the process eventually terminating with an $n$-core. It may be shown that the $n$-core $\mu$ so obtained is independent of the order of $n$-rim-hook removals (see [25]). This unique $\mu$ is termed the *$n$-core of* $\lambda$. For example, we find that $\lambda = (754^2)$ has 3-core $(421^2)$, 4-core $\emptyset$ and 5-core $(2^21)$.

The *$n$-weight* of $\lambda$ is the number of $n$-rim-hook removals that are required to produce its $n$-core. Thus $\lambda = (754^2)$ has 3-weight 4, 4-weight 5 and 5-weight 3.

The importance of the $n$-core derives from the Nakayama "Conjecture" for $\mathfrak{S}_m$ (see [25]) and was extended to $H_m$ in [16].

Let $JS(n, \mu, d)$ be the subset of $JS(n)$ comprising those partitions with $n$-core $\mu$ and $n$-weight $d$. Define the generating series

$$\chi_{n,\mu}(q) = \sum_{d \geq 0} \# JS(n, \mu, d) q^d.$$

**Theorem 6.3**

1. *If $\lambda \in JS(n)$ then the $n$-core $\mu$ of $\lambda$ is a rectangular partition $\mu = (k^l)$ such that $k + l \leq n$ (it is assumed here that if either $k = 0$ or $l = 0$ then $(k^l)$ means the empty partition).*

2. *If $k \neq 0 \neq l$ then:*
$$\chi_{n,(k^l)}(q) = q^{-s}\, b_{\Lambda_{k-l}\,\Lambda_0}^{\Lambda_k+\Lambda_{-l}}(q)\,,$$
*where $s = \min(k,l)$.*

3. $$\chi_{n,\emptyset}(q) = \left(\sum_{k=0}^{n-1} b_{\Lambda_k\,\Lambda_0}^{\Lambda_k+\Lambda_0}(q)\right) - (n-1)\,.$$

*Proof:* In the tensor product $V(\Lambda_j) \otimes V(\Lambda_0)$, all highest weights are of the form $\Lambda_k + \Lambda_{-l} - e\delta$ for $k - l = j \bmod n$ (for later convenience we use $-l$ and not $l$ here). We take $0 \leq k, l < n$ here and can also assume that $k \leq -l \bmod n$ whereupon $l + k \leq n$, and $l = 0$ only if $k = 0$. By definition, the multiplicity of $V(\Lambda_k + \Lambda_{-l} - e\delta)$ is given by the coefficient of $q^e$ in $b_{\Lambda_j,\Lambda_0}^{\Lambda_k+\Lambda_{-l}}(q)$ and hence, by Theorem 2.14 and Theorem 2.16, by the number of partitions $\lambda \in JS(n)$ for which $\mathrm{wt}\,(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_j - e\delta$. We claim that the $n$-core of such a $\lambda$ is the rectangular partition $\mu = (k^l)$, for which, by Lemma 2.9, we calculate $\mathrm{wt}\,(\mu) = \Lambda_k + \Lambda_{-l} - \Lambda_{k-l} - s\delta = \Lambda_k + \Lambda_{-l} - \Lambda_j - s\delta$, where $s = \min(k,l)$ is the multiplicity of the colour charge 0 in $\mu$. To establish this claim first note that for every string of weights $\Lambda, \Lambda - \delta, \Lambda - 2\delta, \ldots$, of the $\widehat{\mathfrak{sl}}_n$-module $V(\Lambda_0)$ (where $\Lambda + \delta$ is not a weight of $V(\Lambda_0)$), those partitions having these weights have the same $n$-core. This $n$-core has weight $\Lambda$. Then since $\mu = (k^l)$ with $k + l \leq n$, is manifestly an $n$-core, it follows that it is the $n$-core of $\lambda$, thence proving part 1.

Part 2 follows immediately since in the case $k \neq 0$ (so that $l \neq 0$), the partitions that enumerate the branching function $b_{\Lambda_{k-l}\Lambda_0}^{\Lambda_k+\Lambda_{-l}}$ are precisely those elements $\lambda \in JS(n)$ having weight $\mathrm{wt}\,(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_{k-l} - e\delta$, for some $e$, and hence $n$-core $(k^l)$.

For $k = 0$ and arbitrary $l$, each partition enumerating $b_{\Lambda_{-l}\Lambda_0}^{\Lambda_0+\Lambda_{-l}}$ has $n$-core $\emptyset$, and hence contributes to $\chi_{n,\emptyset}$. However, the empty partition occurs for each $l$, hence an adjustment of $n-1$ is needed after summing over all $l$. No other partition $\lambda$ is repeated since, as indicated by Theorem 2.14, the $b_{\Lambda_{-l}\Lambda_0}^{\Lambda_0+\Lambda_{-l}}$ to which it contributes is uniquely determined by $-l \bmod n = (\lambda_1 - a_1) \bmod n$. (The summation over $-l$ is replaced by one over $k$ to give the final result).

**Example 6.4** To illustrate this result, consider again the case $n = 3$, where we have the following branching functions (to three terms):

$$\begin{array}{lcl} b_{\Lambda_0,\Lambda_0}^{2\Lambda_0} & = & 1 + q^2 + \cdots; \\ b_{\Lambda_0,\Lambda_0}^{\Lambda_1+\Lambda_2} & = & q + 2q^2 + 2q^3 + \cdots; \\ b_{\Lambda_1,\Lambda_0}^{\Lambda_1+\Lambda_0} & = & 1 + q + 2q^2 + \cdots; \\ b_{\Lambda_1,\Lambda_0}^{2\Lambda_2} & = & q + q^2 + 2q^3 + \cdots; \\ b_{\Lambda_2,\Lambda_0}^{\Lambda_2+\Lambda_0} & = & 1 + q + 2q^2 + \cdots; \\ b_{\Lambda_2,\Lambda_0}^{2\Lambda_1} & = & q + q^2 + 2q^3 + \cdots. \end{array} \tag{42}$$

These are calculated from Theorem 2.14 which leads to the enumeration of the nodes of Fig. 5. labelled by asterisks. The only rectangular 3-cores are $\emptyset$, $(1)$, $(2)$ and $(1^2)$. Using Theorem 6.3, we thus obtain:

$$\begin{array}{lcl} \chi_{n,\emptyset} & = & 1 + 2q + 5q^2 + \cdots; \\ \chi_{n,(1)} & = & 1 + 2q + 2q^2 + \cdots; \\ \chi_{n,(2)} & = & 1 + q + 2q^2 + \cdots; \\ \chi_{n,(1^2)} & = & 1 + q + 2q^2 + \cdots. \end{array} \tag{43}$$

These correspond to the following sets:

$$\begin{array}{lcl} JS(3,\emptyset,0) & = & \{\emptyset\}; \\ JS(3,\emptyset,1) & = & \{(3),(21)\}; \\ JS(3,\emptyset,2) & = & \{(6),(51),(3^2),(41^2),(321)\}, \\ JS(3,(1),0) & = & \{(1)\}; \\ JS(3,(1),1) & = & \{(4),(2^2)\}; \\ JS(3,(1),2) & = & \{(7),(43)\}, \\ JS(3,(2),0) & = & \{(2)\}; \\ JS(3,(2),1) & = & \{(5)\}; \\ JS(3,(2),2) & = & \{(8),(3^21^2)\}, \\ JS(3,(1^2),0) & = & \{(1^2)\}; \\ JS(3,(1^2),1) & = & \{(32)\}; \\ JS(3,(1^2),2) & = & \{(62),(44)\}. \end{array} \tag{44}$$

# 7 Discussion

We have explained why the same set of combinatorial objects arises in two different contexts, namely the modular representations of symmetric groups and Hecke algebras, and the RSOS solvable lattice models based on the coset algebras $C[n,1,1]$. In fact the Jantzen-Seitz problem and the problem of evaluating one-dimensional configurations associated with these models are both equivalent to the computation of branching functions of tensor products of level 1 representations of $\widehat{\mathfrak{sl}}_n$.

It is very natural to ask what happens when one considers tensor products of higher level representations. By Ariki's theorem [2], the level $\ell$ representations of $\widehat{\mathfrak{sl}}_n$ can be interpreted as Grothendieck groups of some cyclotomic Hecke algebras [3, 8]. In particular for level 2 one gets Hecke algebras of type $B$ at roots of unity. On the other hand, removing the level 1 restriction, one obtains the more general RSOS models associated with cosets $C[n,\ell,1]$. This generalized correspondence is currently under investigation and we hope to report on this subject soon.

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*Note added* — A forthcoming preprint, [48], contains (among other things) an elementary purely combinatorial proof of part $(i)$ of 6.3.

# A The Specht module $S^\lambda$.

In this Appendix we give an explanation of the construction of the Specht module representations of $H_m(v)$ based on the account given in [38] which makes use of Young tableaux. An alternative construction is given in [17].

Let $\lambda \in \Pi(m)$. Filling the $m$ nodes of $\lambda$ with elements of $\{1,2,\ldots,m\}$ so that no element appears more than once, yields what is known as a Young tableau. A Young tableau of shape $\lambda$ is typically denoted $t^\lambda$.

A Young tableau $t^\lambda$ for which the entries increase from left to right across each row is said to be row-standard. A Young tableau $t^\lambda$ for which the entries increase from top to bottom down each column is said to be column-standard.

A Young tableau $t^\lambda$ which is both row-standard and column-standard is said to be standard.

The standard Young tableau of shape $\lambda$ for which the entries $1,2,\ldots,m$ occur successively down first the leftmost column then down subsequent columns taken in turn from left to right is denoted $t^{\lambda}_{-}$.

Let $1 \leq a, b \leq m$. The entry $a$ is said to precede $b$ in the tableau $t^\lambda$ if $a$ precedes $b$ on reading the entries of $t^\lambda$ down first the leftmost column of $t^\lambda$ and then down the remaining columns taken in turn from left to right. ($t^\lambda_-$ is such that $a$ precedes $b$ if and only if $a < b$.)

**Example A.1** In the case when $\lambda = (42)$, there are of course $6! = 720$ Young tableaux. Of these, 15 are row-standard, 180 are column standard and just the following nine are standard:

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6&\multicolumn{2}{c}{}\\\cline{1-2}\end{array},\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}.$$

The first tableau listed here is $t^{(42)}_-$.

**Definition A.2 (The Specht module)** [38] *The Specht module $S^\lambda$ of $H_m(v)$ is spanned by vectors $u_{t^\lambda}$ indexed by Young tableaux of shape $\lambda$ subject to the column and Garnir relations (defined below), and on which the action of $H_m(v)$ is generated by:*

$$T_i u_{t^\lambda} = \begin{cases} u_{x^\lambda} & \text{if } i \text{ precedes } i+1 \text{ in } t^\lambda; \\ v u_{x^\lambda} + (v-1) u_{t^\lambda} & \text{if } i+1 \text{ precedes } i \text{ in } t^\lambda, \end{cases}$$

*where the tableau $x^\lambda$ is obtained from $t^\lambda$ by interchanging $i$ and $i+1$.*

Note that the Specht modules for the symmetric group $\mathfrak{S}_m$ are obtained simply by setting $v = 1$ throughout this Appendix. In particular, we see that the action of $\mathfrak{S}_m$ on $u_{t^\lambda}$ does not depend on whether $i$ precedes $i+1$. Then the action of the generator $s_i$ of $\mathfrak{S}_m$ simply interchanges $i$ and $i+1$ in all cases.

**Definition A.3 (Column relations)** *If $x^\lambda$ differs from $z^\lambda$ only in that a single pair of entries within a column are transposed then:*

$$u_{z^\lambda} = -v_{x^\lambda}.$$

**Example A.4** For example (denoting $u_{t^\lambda}$ by $t^\lambda$ for typographical reasons),

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&8&5&10&4&12\\\hline 6&11&3&\multicolumn{3}{c}{}\\\cline{1-3} 9&2&7&\multicolumn{3}{c}{}\\\cline{1-3} 13&\multicolumn{5}{c}{}\\\cline{1-1}\end{array} = - \begin{array}{|c|c|c|c|c|c|}\hline 1&8&5&10&4&12\\\hline 6&2&3&\multicolumn{3}{c}{}\\\cline{1-3} 9&11&7&\multicolumn{3}{c}{}\\\cline{1-3} 13&\multicolumn{5}{c}{}\\\cline{1-1}\end{array} = - \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&10&4&12\\\hline 6&8&5&\multicolumn{3}{c}{}\\\cline{1-3} 9&11&7&\multicolumn{3}{c}{}\\\cline{1-3} 13&\multicolumn{5}{c}{}\\\cline{1-1}\end{array}.$$

Let $z^\lambda$ be column-standard but not row-standard. Then an adjacent pair of entries exists with that on the left greater than that on the right. Consider these two entries together with all those below the left one and all those above the right one (see the example below). Now form all possible tableaux $t^\lambda$ by permuting these entries in all ways such that the permuted entries are increasing down the portions of each of the two columns being considered.

**Definition A.5 (Garnir relations)** *The Garnir relation is the following expression in which the sum is over all such tableaux:*

$$(-v)^{l(w_{z^\lambda})} \sum_{t^\lambda} (-v)^{-l(w_{t^\lambda})} u_{t^\lambda} = 0,$$

*where $w_{t^\lambda} \in \mathfrak{S}_m$ maps $t^\lambda_-$ to $t^\lambda$ and $l(w)$ is the length of the permutation $w \in \mathfrak{S}_m$. (The factor at the front is included to make all coefficients postive powers of $v$ and the coefficient of $u_{z^\lambda}$ one).*

This enables $u_{z^\lambda}$ to be written in terms of other tableaux.

**Example A.6** Consider the following tableau:

$$z^\lambda = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{3}&10&4&12\\\hline 6&\mathbf{8}&\mathbf{5}\\\cline{1-3} 9&\mathbf{11}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} .$$

If we consider the pair of entries 8 and 5 which violate the row-standard condition, we are led to permuting the entries which are highlighted here. The following Garnir relation results:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{3}&10&4&12\\\hline 6&\mathbf{8}&\mathbf{5}\\\cline{1-3} 9&\mathbf{11}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} - v \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{3}&10&4&12\\\hline 6&\mathbf{5}&\mathbf{8}\\\cline{1-3} 9&\mathbf{11}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} + v^2 \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{5}&10&4&12\\\hline 6&\mathbf{3}&\mathbf{8}\\\cline{1-3} 9&\mathbf{11}&7\\\cline{1-3} 13\\\cline{1-1}\end{array}$$

$$+v^2 \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{3}&10&4&12\\\hline 6&\mathbf{5}&\mathbf{11}\\\cline{1-3} 9&\mathbf{8}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} - v^3 \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{5}&10&4&12\\\hline 6&\mathbf{3}&\mathbf{11}\\\cline{1-3} 9&\mathbf{8}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} + v^4 \begin{array}{|c|c|c|c|c|c|}\hline 1&2&\mathbf{8}&10&4&12\\\hline 6&\mathbf{3}&\mathbf{11}\\\cline{1-3} 9&\mathbf{5}&7\\\cline{1-3} 13\\\cline{1-1}\end{array} = 0.$$

By employing a lexicographical ordering on the set of tableaux, it may be seen that each element of $S^\lambda$ may be written as a linear combination of elements $u_{t^\lambda}$, by repeated use of the column and Garnir relations. In fact, we have the following:

**Theorem A.7** [15]

1. *The set*

$$\{u_{t^\lambda} \mid t^\lambda \text{ is a standard tableau}\}$$

*is a basis for $S^\lambda$.*

2. *If $v$ is not a root of unity, then*

$$\{S^\lambda \mid \lambda \in \Pi(m)\}$$

*is a complete set of mutually inequivalent irreducible $H_m(v)$-modules.*

Note that the construction implies that each matrix element of the resulting representation matrices is an element of $\mathbb{Z}[v]$. This implies that these representation matrices remain well defined on specialising $v$ to any value or on taking the entries modulo some field characteristic, and indeed provide representation matrices in these circumstances.

**Example A.8** To illustrate the construction of explicit representation matrices using the Specht module approach, consider representing $T_1 \in H_5(v)$ in the Specht module $S^{(32)}$, by acting on each $u_{t^{(32)}}$ for which $t^{(32)}$ is standard (once more $u_{t^\lambda}$ is written as $t^\lambda$):

$$
\begin{aligned}
T_1 \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} &= \begin{array}{|c|c|c|}\hline 2&3&5\\\hline 1&4\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}, \\
T_1 \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} &= \begin{array}{|c|c|c|}\hline 2&1&5\\\hline 3&4\\\cline{1-2}\end{array} = v\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} - v^2\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}, \\
T_1 \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} &= \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&5\\\cline{1-2}\end{array} = -\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}, \\
T_1 \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} &= \begin{array}{|c|c|c|}\hline 2&1&4\\\hline 3&5\\\cline{1-2}\end{array} = v\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} - v^2\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}, \\
T_1 \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} &= \begin{array}{|c|c|c|}\hline 2&1&3\\\hline 4&5\\\cline{1-2}\end{array} = v\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} - v^2\begin{array}{|c|c|c|}\hline 1&4&3\\\hline 2&5\\\cline{1-2}\end{array} = v\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} - v^3\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} + v^4\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}.
\end{aligned}
$$

Here, column relations have been used in the first and third calculations, and Garnir relations have been used in the second, fourth and last (twice), to express each result in terms of the standard tableaux. Consequently, in the representation labelled by the partition (32), $T_1$ is mapped to the matrix (where zeros are denoted by dots):

$$
\begin{pmatrix}
-1 & -v^2 & . & . & v^4 \\
. & v & . & . & . \\
. & . & -1 & -v^2 & -v^3 \\
. & . & . & v & . \\
. & . & . & . & v
\end{pmatrix}.
$$

The matrices representing the generators $T_i$ of $H_m(v)$ in each irreducible representation for $m \le 5$ given in [38] have been produced in a similar way.

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