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An Introduction to Noncommutative Symmetric Functions

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Abstract. We give a short introduction to the theory of Noncommutative Symmetric Functions, including noncommutative Vieta formulae, relations with Solomon's descent algebras, quasi-symmetric functions and Hecke algebras at $q = 0$.

Keywords. Symmetric functions, noncommutative polynomials, quasi-symmetric functions, descent algebras, Hecke algebras

1. Symmetric functions, physics, and computers

The aim of these notes is to present a short introduction to the recent theory of *noncommutative symmetric functions*. It is an extension of the usual theory of symmetric functions, and since this is a school on Physics and Computer Sciences, I have to say a few words about the role of symmetric functions in these disciplines.

As everybody knows, the *elementary* theory of symmetric functions has to do with algebraic equations, and algebraic equations occur in every field involving a bit of mathematics. So it is not this point that I have in mind. I rather want to stress the *advanced* part of the theory, which is related to Group representations, integrable systems, invariant theory and algebraic geometry, among others.

From the very beginning, computers have been used for physical calculations. Many of these were related to group representations. Although the standard references for the theory were for a long time Hermann Weyl's books [41,42], it was rather in the book by Francis D. Murnaghan [36] and then mostly in the one by Dudley E. Littlewood [24] that practical methods of calculation were presented. All of these methods were based on non-trivial identities on symmetric functions. The first computer programs devoted to group theorical calculations in Atomic Spectroscopy and Nuclear Physics were based on Littlewood's methods. The algorithms are described in [44,45,46]. Up to 1979, Littlewood's book remained the only reference dealing with these advanced aspects of symmetric functions. Then appeared the first edition of Ian G. Macdonald's book [28], giving a modernized presentation of the previous topics, and including some new ones, in particular the theory of Hall-Littlewood functions. To cut the story short, Hall-Littlewood functions, discovered by Littlewood in 1959 [25], are certain symmetric functions depending on a parameter q, which realize what is called the Hall algebra. The interest of this discovery is that James A. Green had expressed the character table of the general

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linear group over a finite field $GL(n, \mathbb{F}_q)$ in terms of the structure constants of this Hall algebra [17]. Litlewood's discovery allowed then for an efficient algorithm to compute these tables. The same Hall-Littewood functions were rediscovered by Macdonald, this time in the guise of spherical functions on p-adic groups [29]. Since then, they have arised in various mathematical problems, and also in the analysis of certain exactly solvable models of statistical mechanics [19,37].

In the meantime, knowing that real and p-adic Lie groups had many properties in common, and that a one-parameter family of symmetric functions (the Jack polynomials) interpolated between spherical functions of the different series, Macdonald looked for, and eventually found [30], a two parameter family of symmetric functions (the famous Macdonald polynomials) containing as specializations the Jack and Hall-Littlewood families. These new polynomials are know to be related to conformal field theory (they express singular vectors of Virasoro and W-algebras in various representations) and to exactly solvable one-dimensional n-body problems in the quantum and relativistic cases (cf. [8]).

Research in this area is still very active, and relies a lot upon Combinatorics and Computer Algebra. Now, after the advent of Quantum Groups and Noncommutative Geometry, it became a natural reflex to look for noncommutative analogues of all interesting mathematical theories. At the time, Israel M. Gelfand and Vladimir S. Retakh were developing a general theory of noncommutative determinants [13,14]. The classical theory of symmetric functions involves a lot of determinantal identities and the story began with an attempt to lift these to the Gelfand-Retakh theory [12].

2. The quest of the Noncommutative Symmetric Functions

2.1. Back to basics

The relations between the coefficients of a polynomial and its roots

$$P(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{k=0}^{n}(-1)^k e_k(X) x^{n-k}, \tag{1}$$

attributed to François Viète, were known in the sixteenth century, and certainly to the ancient civilisations in the case of quadratic polynomials.

Our first requirement for a theory of noncommutative symmetric functions will then be that it should have something to say about roots of polynomials with coefficients in noncommutative rings (or, at least, in skew fields), and about the expansion of products of linear factors $(x - x_i)$.

The advent of linear algebra and matrices allowed for a fresh interpretation of the Viète formulas. If $P(x)$ is the characteristic polynomial of a matrix M, the x_i are its eigenvalues, and $e_k(X) = \operatorname{tr} \Lambda^k(M)$, where $\Lambda^k(M)$ is the kth exterior power of M, i.e., the matrix whose entries are the minors of order k of M.

It is often more convenient to assume that the x_i are the reciprocals of the eigenvalues, so that

$$|I - xM| = \prod_{i=1}^{n}(1 - x_i x) = \sum_{k=0}^{n} e_k(X)(-x)^k \tag{2}$$

is invertible as a formal power series, and its inverse

$$|I - xM|^{-1} = \prod_{i=1}^{n}(1 - x_i x)^{-1} = \sum_{k\geq 0} h_k(X)x^k \tag{3}$$

has as coefficients the complete homogeneous symmetric functions $h_k(X)$, which can be interpreted as the traces of the symmetric powers $S^k(M)$. This last statement is essentially McMahon's "Master Theorem".

The power sums $p_k(X) = \sum x_i^k$ are obviously the traces of the powers M^k, and at a more advanced level, one knows that the traces of the images of M under the irreducible polynomial representation of GL_n, labelled by partitions λ, are the so-called Schur functions $s_\lambda(X)$ [24,28].

Another natural requirement for a theory of noncommutative symmetric functions is therefore that some of these properties should have an analogue for matrices with entries in a noncommutative field.

2.2. *Roots of noncommutative polynomials*

We first have to choose a reasonable definition of noncommutative polynomials, i.e., with coefficients in a noncommutative algebra R. If our variable x does not commute with R, the resulting algebra is in general intractable. So we will rather assume that x commutes with R, and introduce the notion of left and right roots. For

$$P(x) = \sum_{k=0}^{n} a_k x^k\,, \quad a_k \in R\,, \tag{4}$$

we say that $c \in R$ is a right root of P if

$$\sum_{k=0}^{n} a_k c^k = 0 \tag{5}$$

Now, if $x_1, \dots, x_n \in R$, where we assume that R is a division ring, can we find

$$P(x) = x^n - \Lambda_1 x^{n-1} + \Lambda_2 x^{n-2} - \cdots + (-1)^n \Lambda_n \tag{6}$$

such that the x_k are right roots of $P(x)$? If the x_i are pairwise non conjugate, the unique solution is [4]

$$P(x) = \begin{vmatrix} 1 & \dots & 1 & 1 \\ x_1 & \dots & x_n & x \\ x_1^2 & \dots & x_n^2 & x^2 \\ \vdots & & \vdots & \vdots \\ x_1^n & \dots & x_n^n & \boxed{x^n} \end{vmatrix} \tag{7}$$

where for a matrix $A = (a_{ij})$ with coefficients in a noncommutative ring, the notation $|A|_{ij}$, which can be displayed like a determinant with the entry a_{ij} in a box, denotes ones of the *quasi-determinants* of A. The quasi-determinants of A can be defined for a generic matrix by the formulae [13,14]

$$|A|_{ij} = (A^{-1})_{ji}^{-1} \tag{8}$$

(these are analogues of the ratio of a determinant and one of its principal minors).

One finds for example

$$\Lambda_1(x_1, x_2) = (x_2^2 - x_1^2)(x_2 - x_1)^{-1},$$

$$\Lambda_2(x_1, x_2) = (x_2^2 - x_1 x_2)(x_1^{-1} x_2 - 1)^{-1}$$

These expressions are symmetric functions of the x_i, but are not polynomials. The correct setting for developing their rigorous theory is the *free field* generated by the x_i [15].

2.3. Products of linear factors

Since the Viète formulae are not anymore true for noncommutative polynomials, we should also have a look at the expansion of products of linear factors. Let $Y = \{y_1 < \ldots < y_n\}$ be an ordered alphabet of non-commuting variables. We could define elementary symmetric functions of Y by

$$\lambda_t(Y) = (1 + ty_n)(1 + ty_{n-1}) \cdots (1 + ty_1) = \sum_{k=0}^{n} \Lambda_k(Y) t^k \tag{9}$$

where t commutes with the y_i.

An immediate objection to this naive definition would be that the $\Lambda_k(Y)$ are not symmetric. This is an illusion. First, they are actually symmetric, but for a special action of the symmetric group, the plactic action of Lascoux and Schützenberger, now known to be a particular case Kashiwara's action of Weyl groups on crystal graphs [26]. The Λ_k do not generate the full algebra of invariants of this action, but a very interesting subalgebra denoted by $\mathbf{Sym}(Y)$, and called the algebra of noncommutative symmetric polynomials over Y, the full algebra of invariants being $\mathbf{FSym}(Y)$, the algebra of free symmetric polynomials [9].

Suppose now that $x = \{x_1, \ldots, x_n\}$ is a generic set of right roots of $P(x)$, in the sense that the Vandermonde quasi-determinants

$$v_k = \begin{vmatrix} x_1^{k-1} & x_2^{k-1} & \cdots & \boxed{x_k^{k-1}} \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots 1 & \end{vmatrix} \neq 0\,. \tag{10}$$

Then, if we define

$$y_k = v_k x_k v_k^{-1}\,, \tag{11}$$

we have

$$P(x) = (x - y_1)(x - y_2) \cdots (x - y_n) = \sum_{k=0}^{n} (-1)^k \Lambda_k(Y) x^{n-k}\,. \tag{12}$$

This is the relation between the two kinds of symmetric functions already considered. Moreover, R. Wilson has shown that any polynomial in the y_i which is symmetric in the x_i is a polynomial in the $\Lambda_i(Y)$ [43].

Therefore, the two questions lead to the same algebra, although in a nonobvious manner.

These considerations can lead to new results, even in elementary linear algebra. For example, Connes and Schwarz [6], obtained the following identity. Suppose that $R = M_k(\mathbb{C})$, that is, $a_1, \dots, a_n, x_1, \dots, x_n$ are $k \times k$ complex matrices such that the x_i form a nondegenerate set of right roots of

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n .$$

Then,

$$\det(1 - \lambda x_1) \cdots \det(1 - \lambda x_n) = \det(1 + a_1 \lambda + \cdots + a_n \lambda^n) . \tag{13}$$

In particular,

$$\operatorname{tr} a_1 = -(\operatorname{tr} x_1 + \cdots + \operatorname{tr} x_n) \tag{14}$$

$$\det a_n = (-1)^{nk} \det x_1 \cdots \det x_n . \tag{15}$$

a result previously obtained by D. Fuchs and A. Schwarz [11].

For Y infinite, following the usage of the commutative theory, we speak of the $\Lambda_i(Y)$ as noncommutative symmetric *functions* (instead of polynomials).

2.4. Matrices over noncommutative rings

Quasideterminants allow one to define analogues of the characteristic polynomial for a $n \times n$ matrix M with noncommutative entries. Instead of a single characteristic polynomials, we get (in general) n *quasi-characteristic polynomials*

$$|1 + tM|_{ii} = \lambda_t(\alpha^{(i)}) \tag{16}$$

where it is convenient to introduce n “virtual alphabets” $\alpha^{(i)}$ of quasi-eigenvalues.

For example, with $M = E_n - (n-1)I$, where

$$E_n = \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ e_{21} & e_{22} - 1 & \dots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \dots & e_{nn} - n + 1 \end{pmatrix} \tag{17}$$

(the Capelli matrices of classical invariant theory), in which the e_{ij} are the generators of the universal enveloping algebra $U(\mathfrak{gl}_n)$ corresponding to the matrix units,

$$e_{ij} = \sum_{k=1}^{n} x_{ik} \frac{\partial}{\partial x_{kj}}$$

the coefficients of the quasi-characteristic polynomials provide new generators of its center $Z(\mathfrak{gl}_n)$

This has been generalized by A. Molev to the case of Sklyanin determinants in twisted Yangians [34].

2.5. Formal noncommutative symmetric functions

We can see from the previous considerations that there are several reasonable but non equivalent notions of noncommutative elementary symmetric functions. However, in each case, we end up with sequence of elements Λ_k, which do not commute, and are to be considered as being of degree k in some reasonable sense.

We should therefore introduce an algebra of *Formal Noncommutative Symmetric Functions*

$$\mathbf{Sym} = \mathbb{K}\langle \Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots \rangle \tag{18}$$

as the free associative algebra over an infinite sequence of noncommuting indeterminates $\Lambda_k, k \geq 1$, with Λ_k of degree k, and try to generalize as much as possible the classical constructions on the algebra of commutative symmetric functions, which, after all, is nothing but

$$Sym = \mathbb{K}[e_1, e_2, \ldots, e_n, \ldots] \tag{19}$$

The previous examples should then be considered as specializations.

3. Hopf algebras

3.1. Commutative symmetric functions

The ring of symmetric polynomials in n variables $Sym(X) = \mathbb{K}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ ($\mathbb{K}$ is some field of characteristic 0) is freely generated by the *elementary symmetric polynomials*

$$e_k = \sum_{i_1 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad (k = 1, 2, \ldots, n)\,.$$

Thus, $Sym(X)$ is just a polynomial algebra $\mathbb{K}[e_1, \ldots, e_n]$, with a particular grading defined by $\deg(e_k) = k$.

The algebra Sym of *symmetric functions* is obtained by letting $n \to \infty$. The classical theory of symmetric functions can therefore be regarded as the study of the polynomial algebra $\mathbb{K}[e_k \,:\, k \geq 1]$ over an infinite sequence of indeterminates, graded by $\deg(e_k) = k$. *Note that the original variables x_i can be eliminated from this definition.*

As an algebra, Sym has several distinguished sets of generators. Let $\lambda(t) = \sum_{k \geq 0} e_k t^k$ be the generating series of elementary symmetric functions. Then, the *complete homogeneous functions* h_n can be defined by their generating series $\sigma(t) = \lambda(-t)^{-1} = \sum_{k \geq 0} h_k t^k$, and the power-sums p_n by $\psi(t) = \sum_{k \geq 1} p_k t^{k-1} = \frac{d}{dt} \log \sigma(t)$ (the Newton formulas). One has as well

$$Sym = \mathbb{K}[h_1, h_2, \ldots] = \mathbb{K}[p_1, p_2, \ldots]\,.$$

A Hopf algebra structure is defined by means of the comultiplication $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$, and the antipode $\tilde{\omega}(p_k) = -p_k$. It can be shown that Sym is isomorphic to its graded dual [28].

The dimension of the homogeneous component of degree n of Sym is $\dim Sym_n = |\mathrm{Part}\,(n)| = p(n)$, the number of partitions of n, i.e., non-decreasing sequences of positive integers summing to n. Linear bases of Sym are therefore naturally labelled by partitions. The simplest ones are m_λ (monomial symmetric functions), $e_\lambda, h_\lambda, p_\lambda$ (products $e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_r}$ etc.), and the Schur functions $s_\lambda = |h_{\lambda_i+j-i}|$, which are the fundamental ones in representation theory.

There is a canonical scalar product, defined by either of the equivalent formulas

$$\langle s_\lambda, s_\mu\rangle = \langle m_\lambda, h_\mu\rangle = \langle p_\lambda, p_\mu^*\rangle = \delta_{\lambda\mu}$$

(where $p_\mu^* = p_\mu / z_\mu$ and $z_\mu = \prod i^{m_i} m_i!$, m_i being the multiplicity of i in μ). This scalar product materializes the self duality of Sym, in that Δ is the adjoint of the multiplication map $f \otimes g \mapsto fg$. The equivalence of these definitions, as well as the self duality, is a consequence of the following property: for two bases (u_λ) and (v_λ) of Sym,

$$\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu} \Leftrightarrow \prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda u_\lambda(X) v_\lambda(Y)$$

and of the classical Cauchy identity for Schur functions

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j} (1 - x_i y_j)^{-1} .$$

There is also a second comultiplication defined by $\delta(p_k) = p_k \otimes p_k$. This comultiplication is dual to a multiplication known as the *internal product* $*$, i.e., $\langle f * g, h\rangle = \langle f \otimes g, \delta(h)\rangle$.

3.2. Noncommutative symmetric functions

Imitating the description of Sym as a polynomial algebra in independent indeterminates e_k, graded by $\deg(e_k) = k$, we have defined the algebra $\mathbf{Sym}$ of formal noncommutative symmetric functions as the free associative algebra on an infinite sequence Λ_k of noncommuting indeterminates (the noncommutative elementary functions), graded by $\deg(\Lambda_k) = k$. As in the commutative case, one introduces the generating series

$$\lambda(t) = \sum_{n \geq 0} \Lambda_n t^n , \quad \sigma(t) = \lambda(-t)^{-1} = \sum_{n \geq 0} S_n t^n , \tag{20}$$

of elementary and complete functions, where t is an indeterminate in the ground field $\mathbb{K}$.

A concrete realization $\mathbf{Sym}(A)$ of this algebra can be given by taking an infinite sequence $A = \{a_n | n \geq 1\}$ of noncommuting indeterminates of degree 1, by setting

$$\lambda(A; t) = \sum_{n \geq 0} \Lambda_n(A) t^n = \overleftarrow{\prod_{i \geq 1}} (1 + t a_i) , \tag{21}$$

so that $\Lambda_n(A)$ gets identified with the sum of all strictly decreasing words of length n, and $S_n(A)$ with the sum of all nondecreasing words of the same length, which are respectively represented as column-shaped and row-shaped Young tableaux.

The algebra homomorphism $\mathbf{Sym}(A) \to Sym(X)$ defined by $a_i \mapsto x_i$, called the *commutative image* $F \mapsto \underline{F}$, maps Λ_n to e_n, so that $\mathbf{Sym}$ is actually a noncommutative lifting of the algebra of symmetric functions.

The first really interesting question is "what are the noncommutative power sums ?". There are indeed several possibilities. If one starts from the classical expression

$$\sigma(t) = \exp\left\{\sum_{k\geq 1} p_k \frac{t^k}{k}\right\} , \tag{22}$$

one can choose to define noncommutative power sums Φ_k by the same formula

$$\sigma(t) = \exp\left\{\sum_{k\geq 1} \Phi_k \frac{t^k}{k}\right\} , \tag{23}$$

but a noncommutative version of the Newton formulas

$$nh_n = h_{n-1}p_1 + h_{n-2}p_2 + \cdots + p_n \tag{24}$$

which are derived by taking the logarithmic derivative of (22) leads to different noncommutative power-sums Ψ_k inductively defined by

$$nS_n = S_{n-1}\Psi_1 + S_{n-2}\Psi_2 + \cdots + \Psi_n \, . \tag{25}$$

Introducing the generating function $\psi(t) = \sum_{k\geq 1} \Psi_k t^{k-1}$, one may also regard $\sigma(t)$ as the unique solution of the differential equation

$$\frac{d}{dt}\sigma(t) = \sigma(t)\psi(t) \tag{26}$$

satisfying the initial condition $\sigma(0) = 1$. The generating function of the Φ_k, taken in the form

$$\Phi(t) = \sum_{k\geq 1} \Phi_k \frac{t^k}{k} \tag{27}$$

is then the logarithm of this solution. From this, one realizes that the relation between the two kinds of noncommutative power-sums is of a rather complicated nature. The expression of $\Phi(t)$ as a function of the Ψ_k is known as the *continuous Baker-Campbell-Hausdorff* (BCH) series [27,3,33]. It can be written as a Chen series (iterated integrals) in a quite explicit form, and it is usually interpreted as expressing the logarithm of the "evolution operator" $\sigma(t)$ in terms of the "Hamiltonian" $\psi(t)$ [3]. It is known that the continuous BCH series is a Lie series, so that the Φ_k are elements of the free Lie algebra $\mathcal{L}$ generated by the Ψ_k, of which they form another system of generators. In fact, any sequence (F_n) of generators of $\mathcal{L}$, with $\deg(F_n) = n$ can be shown to provide an admissible family of noncommutative power sums, in the sense that the commutative image of F_n is a nonzero multiple of p_n. Moreover, the Poincaré-Birkhoff-Witt theorem shows that $\mathbf{Sym}$ can be identified with the universal enveloping algebra $U(\mathcal{L})$ of $\mathcal{L}$. As such, $\mathbf{Sym}$ is endowed with a canonical comultiplication Δ, for which $\mathcal{L}$ is the space of primitive elements (Friedrich's theorem, see [38]). In particular,

$$\Delta\Psi_k = \Psi_k \otimes 1 + 1 \otimes \Psi_k \qquad \Delta\Phi_k = \Phi_k \otimes 1 + 1 \otimes \Phi_k \,, \tag{28}$$

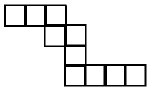
and also

$$\Delta\Lambda_n = \sum_{k=0}^{n} \Lambda_k \otimes \Lambda_{n-k} \,, \quad \Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \,. \tag{29}$$

That is, the comultiplication of $\mathbf{Sym}$ is mapped onto the usual one on Sym under the commutative image homomorphism. There is also an analogue of the canonical involution $\omega : e_n \leftrightarrow h_n$, defined in the same way by $\omega(\Lambda_n) = S_n$, and it is easy to check that the signed version $\tilde{\omega}(\Lambda_n) = (-1)^n S_n$ is an antipode for Δ.

As for ordinary symmetric functions, we first define linear bases by taking monomials in the various families of algebraic generators. Here, our generators do not commute, so that basis elements of the homogeneous component $\mathbf{Sym}_n$ of degree n will be labelled by *compositions* of n, i.e., ordered sequences $I = (i_1, \ldots, i_r)$ of positive integers with $i_1 + i_2 + \cdots + i_r = n$. For a sequence (G_n) of homogeneous generators with $\deg(G_n) = n$, we set $G^I = G_{i_1} G_{i_2} \cdots G_{i_r}$. Therefore, we already have the four bases Λ^I, S^I, Φ^I and Ψ^I.

A composition I of n is conveniently pictured as a *ribbon diagram*, which is a rimhook shaped skew Young diagram whose succesive rows have lengths $i_1, i_2, \ldots, i_r$ (read from top to bottom in the French convention). For example, the ribbon diagram of shape $I = (3, 2, 1, 4)$ is



The number of compositions of n is equal to 2^{n-1}. A useful way to realize this is to encode the ribbon diagram of a composition I of n by the subset $\mathrm{Des}\,(I) = \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_{r-1}\}$ of $\{1, 2, \ldots, n-1\}$. The elements of $\mathrm{Des}\,(I)$ are called the *descents* of the composition.

The next basis that should be defined, to pursue the parallel with the commutative theory, would be the analogue of monomial symmetric functions. However, we have no way to achieve this at this point. This is because in the classical case, the monomial basis (m_λ) is dual to the homogeneous one (h_λ), of which we already have the noncommutative analogue (S^I). But since the comultiplication Δ of $\mathbf{Sym}$ is obviously cocommutative, $\mathbf{Sym}$ cannot be self-dual, and the analogues of the monomial functions will have to live in the dual Hopf algebra $\mathbf{Sym}^*$, to be discussed in the next section.

On the other hand, we can define the analogues of Schur functions. These are the so-called *ribbon Schur functions*.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4). The ribbon Schur function (R_I) is defined by the alternating sum

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)}\, S^J \qquad \Longleftrightarrow S^I = \sum_{J \preceq I} R_J \,, \tag{30}$$

where $\ell(I)$ denotes the length of I. Clearly, (R_I) is also a homogenous basis of $\mathbf{Sym}$.

The commutative image of a ribbon Schur function R_I is the corresponding ordinary ribbon Schur function, which will be denoted by by r_I. The r_I were first introduced by MacMahon [31]. An important property of the ribbon Schur functions is their very simple multiplication formula

$$R_I\, R_J = R_{I\triangleright J} + R_{I\cdot J} \tag{31}$$

where $I \triangleright J$ denotes the composition $(i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$ and $I \cdot J$ the composition $(i_1, \ldots, i_r, j_1, \ldots, j_s)$.

4. Quasi-symmetric functions

Recall that the Hopf algebra of commutative symmetric functions is self-dual. This is not anymore true of $\mathbf{Sym}$, since it is obviously cocommutative. We must therefore compute the dual Hopf algebra $\mathbf{Sym}^*$ (we mean the graded dual). As pointed out in the preceding section, we need it to define the proper generalization of monomial symmetric functions. In the commutative case, the duality between monomial symmetric functions m_λ and homogeneous products h_λ can be traced back to the Cauchy-type identity

$$\sigma(XY;1) = \prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda m_\lambda(X) h_\lambda(Y)\,, \tag{32}$$

where the sum runs over all partitions. By analogy, let us consider the ordered product

$$\sigma(XA;1) := \overrightarrow{\prod_{i\geq 1}}\, \sigma(A;x_i) = \overrightarrow{\prod_{i\geq 1}}\,\overrightarrow{\prod_{j\geq 1}}(1 - x_i a_j)^{-1} \tag{33}$$

where A is a totally ordered set of noncommutative variables, and X a totally ordered set of *commutative* variables, also commuting with those of A. This is a natural choice, since we know that the dual of $\mathbf{Sym}$ will be commutative. Expanding the product, we find

$$\sigma(XA;1) = \sum_I M_I(X) S^I(A)\,, \tag{34}$$

where the polynomials M_I are

$$M_I(X) = \sum_{j_1<j_2<\ldots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r}\,. \tag{35}$$

Thus, the M_I are pieces of monomial symmetric functions. That is, m_λ is equal to the sum of the M_I labelled by the distinct rearrangements I of the partition λ. The M_I form the basis of a subalgebra of $\mathbb{K}[X]$, already introduced in 1984 by Gessel [16] under the name *quasi-symmetric functions*. It is usually denoted by $QSym$. It is naturally graded, and its homogeneous component $QSym_n$ has, like $\mathbf{Sym}_n$, dimension 2^{n-1}. One can then define a pairing $\langle\ ,\ \rangle$ between $QSym$ and $\mathbf{Sym}$ by

$$\langle M_I\,,\, S^J\rangle = \delta_{IJ}\,. \tag{36}$$

With this at hand, it is easy to see that $QSym$ is indeed the graded dual Hopf algebra of $\mathbf{Sym}$ ([32], see also [12,20]). That is,

$$\langle f \otimes g \, , \, \Delta(P) \rangle = \langle fg \, , \, P \rangle \, , \tag{37}$$

$$\langle \gamma(f) \, , \, P \otimes Q \rangle = \langle f \, , \, PQ \rangle \, , \tag{38}$$

where the comutiplication γ of $QSym$ maps a quasi-symmetric function $f = f(X)$ to $f(X \hat{+} Y)$, where $X \hat{+} Y$ denotes the ordered sum of two disjoint ordered sets, and $u(X)v(Y)$ is identified with $u \otimes v$. The dual basis of (R_I), denoted by (F_I), which is in some sense a quasi-symmetric analogue of the Schur basis, is given by

$$F_I = \sum_{J \succeq I} M_J \, . \tag{39}$$

Finally, since any symmetric function f is in particular quasi-symmetric, one can expand it on the various bases of $QSym$. A useful property is then

$$\langle f \, , \, G \rangle = \langle f \, , \, g \rangle \, , \tag{40}$$

where, on the right-hand side, $g = \underline{G}$ is the commutative image of G and the brackets denote the ordinary scalar product of symmetric functions [16].

5. Representation theoretical interpretations

5.1. Commutative case: symmetric groups and generic Hecke algebras

It is known that the irreducible characters χ^λ of the symmetric group $\mathfrak{S}_n$ are parametrized by the partitions λ of n. The *Frobenius characteristic map* is a linear correspondence between characters and symmetric functions, defined by $\chi^\lambda \mapsto \mathrm{ch}\,(\chi^\lambda) = s_\lambda$. This is an isometric isomorphism $R(\mathfrak{S}_n) \to Sym_n$, where $R(\mathfrak{S}_n)$ is the vector space spanned by the irreducible characters. One has then $\mathrm{ch}\,(\phi\psi) = \mathrm{ch}\,(\phi) * \mathrm{ch}\,(\psi)$, where $*$ is the internal product. The *Frobenius character formula* expresses the value $\chi(\sigma)$ of a character χ on a permutation σ of cycle type μ as $\chi^\lambda(\mu) = \langle s_\lambda, p_\mu \rangle$.

Another important point is the induction formula, interpreting the ordinary multiplication of symmetric functions in terms of induced representations. If $f = \mathrm{ch}\,(\xi)$ and $g = \mathrm{ch}\,(\eta)$ are the characteristics of two representations of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, then $fg = \mathrm{ch}\,(\chi)$ where χ is the character of $\mathfrak{S}_{m+n}$ induced from the character $\xi \times \eta$ of its subgroup $\mathfrak{S}_m \times \mathfrak{S}_n$.

Schur functions can also be interpreted as characters of $GL(n, \mathbf{C})$. The polynomial representations V_λ of $GL(n, \mathbf{C})$ are parametrized by partitions of length at most n, and $s_\lambda(x_1, \ldots, x_n) = \mathrm{trace}_{V_\lambda}(X)$ where X is the diagonal matrix $X = \mathrm{diag}\,(x_1, \ldots, x_n)$.

Schur functions correspond in a similar way to the irreducible representations of the q-deformed structures $H_n(q)$, $U_q(gl_n)$ or $F_q(GL_n)$, for generic q. In the case of the Hecke algebra $H_n(q)$, which is the algebra generated by elements $T_1, \ldots, T_{n-1}$ satisfying the braid relations together with $(T_i + 1)(T_i - q) = 0$, the character formula can be written as [5]

$$\chi^\lambda_\mu(q) = \langle s_\lambda(X) \, , \, (q-1)^{-\ell(\mu)} h_\mu((q-1)X) \rangle \, , \tag{41}$$

where the "λ-ring notation" $h_\mu((q-1)X)$ means the following. In general, if X and Y are two (multi-) sets of variables, identified with the formal sum of their elements, the symmetric functions of $X+Y$, $X-Y$ and XY are defined by

$$p_k(X \pm Y) = p_k(X) \pm p_k(Y)\,, \quad p_k(XY) = p_k(X)p_k(Y)\,, \tag{42}$$

and then by expressing any symmetric function as a polynomial in the power sums. Therefore, the symmetric functions of $(q-1)X = qX - X$ are the images of those of X under the ring homomorphism $p_k \mapsto (q^k - 1)p_k$.

The Hecke algebra $H_n(q)$ was introduced by Iwahori in [18]. The original definition was as follows. Let $G = GL(n, \mathbb{F}_q)$ and B be the subgroup of G formed by upper triangular matrices. Then, G acts on the vector space $M = \mathbb{C}\,G/B$ spanned by the left cosets of B. To decompose this representation, one can use Schur's lemma and look at the centralizer of $\mathbb{C}G$ in $\mathrm{End}\,(M)$. Iwahori showed that this centralizer is isomorphic to $H_n(q)$.

5.2. *Noncommutative case:* 0-*Hecke algebras*

When q is a generic complex number, i.e., neither zero nor a nontrivial root of unity, the $H_n(q)$ are semi-simple (in fact isomorphic to $\mathbb{C}\mathfrak{S}_n$) and the direct sum

$$\mathcal{R}(q) = \bigoplus_{n\geq 0} R(H_n(q)) \tag{43}$$

where $R(H_n(q))$ is the vector space spanned by isomorphism classes $[M]$ of finite dimensional $H_n(q)$-modules, with addition induced by direct sum, can be identified with Sym, the simple modules $S_\lambda(q)$ (q-Specht modules) being represented by Schur functions. This defines a characteristic map $\mathrm{ch}\ :\ R(H_n(q)) \to Sym_n$. The usual product of symmetric functions corresponds then to induction from $H_m \otimes H_n$ to H_{m+n}, i.e., for $\lambda \vdash m$ and $\mu \vdash n$,

$$\mathrm{ch}\,\left(S_\lambda(q) \otimes S_\mu(q) \uparrow_{H_m(q)\otimes H_n(q)}^{H_{m+n}(q)}\right) = s_\lambda s_\mu\,. \tag{44}$$

This statement summarizes a good deal about the representation theory of Hecke algebras in the generic case.

For non generic values of the parameter, when $H_n(q)$ is not semi-simple, it is a difficult task to describe $R(H_n(q))$ as defined above, since this would amount to understand *all* the indecomposable representations up to isomorphism. The usual strategy to investigate a non semi-simple algebra $\mathcal{A}$ is to introduce two kinds of *Grothendieck groups* (cf. [7]). The first one, usually denoted by $G_0(\mathcal{A})$ is the quotient of $R(\mathcal{A})$ by the relations $[M] = [M'] + [M'']$ whenever there is a short exact sequence $0 \to M' \to M \to M'' \to 0$. In $G_0(\mathcal{A})$, $[M] = [N]$ whenever M and N have the same simple composition factors, occuring with the same multiplicities. The second one, denoted by $K_0(\mathcal{A})$, is the free abelian group generated by isomorphism classes of finite dimensional projective $\mathcal{A}$-modules. Here, addition corresponds to direct sum. In the sequel, we will rather mean by G_0 and K_0 the complexified Grothendieck groups.

A particularly interesting example of this situation occurs for $\mathcal{A} = H_n(\zeta)$, where ζ is a primitive k-th root of unity. Then, the direct sums

$$\mathcal{G}(\zeta) = \bigoplus_{n\geq 0} G_0(H_n(\zeta)) \quad \text{and} \quad \mathcal{K}(\zeta) = \bigoplus_{n\geq 0} K_0(H_n(\zeta)) \tag{45}$$

can be respectively identified with a quotient and with a subalgebra of Sym. Precisely, if we denote by $\mathcal{J}_k$ the ideal of Sym generated by the power sums p_{nk}, $n \geq 1$, and by $\mathcal{S}_k$ the subalgebra generated by the p_m such that $m \not\equiv 0 \bmod k$, then

$$\mathcal{G}(\zeta) \simeq Sym/\mathcal{J}_k \quad \text{and} \quad \mathcal{K}(\zeta) \simeq \mathcal{S}_k\,. \tag{46}$$

The characteristic map ch : $\mathcal{G}(\zeta) \to Sym/\mathcal{J}_k$ realizing the first isomorphism is compatible with the previous one, in the sense that the specialized Specht module $S_\lambda(\zeta)$ is mapped to the class $\bar{s}_\lambda = s_\lambda \bmod \mathcal{J}_k$. The induction formula remains valid but does not tell us that much about the representation theory of $H_n(\zeta)$. This is because $S_\lambda(\zeta)$ is usually not irreducible, and essential information is contained in the multiplicities $d_{\lambda\mu}$ of its simple composition factors D_μ. These numbers are called the decomposition numbers, and they form the *decomposition matrix* of $H_n(\zeta)$. These matrices have been determined only recently [23,1].

On the other hand, the non semi-simple Hecke algebras $H_n(0)$ are quite well-understood. The representation theory of 0-Hecke algebras for general type has been worked out by Norton [35], and special combinatorial features of type A have been described by Carter [5].

There are 2^{n-1} simple $H_n(0)$-modules, which are all one-dimensional [5,35]. These irreducible representations are obtained by sending a set of generators to -1 and its complement to 0. We shall label these representations by compositions of n rather than by subsets of generators. Let I be a composition of n and let $\mathrm{Des}\,(I)$ the associated subset of $\{1,\ldots,n-1\}$. The irreducible representation φ_I of $H_n(0)$ is defined by

$$\varphi_I(T_i) = \begin{cases} -1 & \text{if } i \in \mathrm{Des}\,(I)\,, \\ 0 & \text{if } i \notin \mathrm{Des}\,(I)\,. \end{cases} \tag{47}$$

The corresponding $H_n(0)$-module is denoted by $\mathbf{C}_I$. These modules (when I runs over all compositions of n) form a complete system of simple $H_n(0)$-modules.

The direct sum of the Grothendieck groups

$$\mathcal{G}(0) = \bigoplus_{n\geq 0} G_0(H_n(0)) \tag{48}$$

has therefore a canonical basis, the classes $[\mathbf{C}_I]$ of the simple modules, which is naturally labelled by compositions. One may then look for a characteristic map with values in $\mathbf{Sym}$ or in $QSym$. The correct choice is to identify $\mathcal{G}(0)$ with the algebra of quasi-symmetric functions, the characteristic map being given by

$$\mathrm{ch}\,([\mathbf{C}_I]) = F_I\,. \tag{49}$$

This map is again compatible with the characteristic map of the generic case. One has a decomposition map $d\,:\,\mathcal{G}(q) \to \mathcal{G}(0)$ sending the class $[S_\lambda(q)]$ of a generic Specht module to the class $[S_\lambda(0)]$ of its 0-specialization, which is usually not irreducible, nor even semi-simple. The choice (49) has the property that

$$\mathrm{ch} \circ d = \mathrm{ch} \tag{50}$$

that is, $\mathrm{ch}\,(S_\lambda(0)) = s_\lambda$, regarded as a quasi-symmetric function.

From this, one can easily recover Carter's combinatorial description of the decomposition matrix of $H_n(0)$. The multiplicity $d_{\lambda I}$ of the simple module $\mathbf{C}_I$ as a composition factor of $S_\lambda(0)$ is equal to the coefficient of F_I in the quasi-symmetric expansion

$$s_\lambda = \sum_I d_{\lambda I} F_I \tag{51}$$

so that by duality, $d_{\lambda I} = \langle s_\lambda \, , \, R_I \rangle$, and by (40), this is equal to the ordinary scalar product of symmetric functions $\langle s_\lambda \, , \, r_I \rangle$.

The characteristic map at $q = 0$ is also compatible with the induction product, that is, once again,

$$\mathrm{ch}\left([M \otimes N] \uparrow_{H_m(0) \otimes H_n(0)}^{H_{m+n}(0)} \right) = \mathrm{ch}\,([M])\mathrm{ch}\,([N]) \tag{52}$$

In particular, the composition factors of the induction product of two simple modules $\mathbf{C}_I$ and $\mathbf{C}_J$ are described by the product $F_I F_J$, which is given by an interesting combinatorial formula. To state it, let us first define the shape composition $I = C(\sigma)$ of a permutation σ as the unique composition of n such that $\mathrm{Des}\,(\sigma) = \mathrm{Des}\,(I)$. Now, suppose that $|I| = m$ and $|J| = n$. Take any permutation $\alpha \in \mathfrak{S}_m$ such that $C(\alpha) = I$ and any permutation $\beta \in \mathfrak{S}_n$ such that $C(\beta) = J$. We consider permutations as words on the letters $1, 2, \ldots$, and we denote by $\beta[m]$ the shifted word

$$\beta[m] = (\beta_1 + m) \cdot (\beta_2 + m) \cdots (\beta_n + m)\,. \tag{53}$$

Recall that the *shuffle* product on words is defined inductively by

$$au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v)\,, \quad a, b \in A \;\; u, v \in A^* \tag{54}$$

(where A^* is the set of words on the alphabet A) the initial condition being that the empty word is the unit element. The product formula is then [32]

$$F_I F_J = \sum_\sigma (\alpha \sqcup\!\sqcup \beta[m] \, , \, \sigma) F_{C(\sigma)} \tag{55}$$

where $(\alpha \sqcup\!\sqcup \beta[m] \, , \, \sigma)$ denotes the coefficient of σ in $\alpha \sqcup\!\sqcup \beta[m]$.

It is tempting to interpret this formula as the shadow of something like

$$F_\alpha F_\beta = \sum_\sigma (\alpha \sqcup\!\sqcup \beta[m] \, , \, \sigma) F_\sigma \tag{56}$$

in a noncommutative algebra based on all permutations. Such an algebra does exist, it is the Hopf algebra of Free Quasi-Symmetric Functions (cf. [32,9] and F. Hivert's article in this volume).

Let us now look at the K_0-groups of the 0-Hecke algebras. The indecomposable projective $H_n(0)$-modules have also been classified by Norton (cf. [5,35]). From the general theory of finite dimensional algebras, one knows that for each simple module S, there is a unique indecomposable projective module M such that $S = M/\mathrm{rad}\,(M)$. Thus, one associates to each composition I of n the unique indecomposable projective $H_n(0)$-module $\mathbf{M}_I$ such that $\mathbf{M}_I/\mathrm{rad}\,(\mathbf{M}_I) \simeq \mathbf{C}_I$. The canonical duality between $G_0(\mathcal{A})$ and $K_0(\mathcal{A})$, defined by the pairing

$$\langle N\,,\,M\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(M,N) \tag{57}$$

tells us that the direct sum of Grothendieck groups

$$\mathcal{K}(0) = \bigoplus_{n\geq 0} K_0(H_n(0)) \tag{58}$$

has to be identified with the algebra of noncommutative symmetric functions. The functorial duality between induction and restriction implies that the characteristic map

$$\mathrm{Ch}\;:\;\mathcal{K}(0) \longrightarrow \mathbf{Sym} \tag{59}$$

$$[\mathbf{M}_I] \longmapsto R_I \tag{60}$$

is a ring homomorphism, i.e.,

$$\mathrm{Ch}\,(\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_m(0)\otimes H_n(0)}^{H_{m+n}(0)}) = R_I\,R_J = R_{I\cdot J} + R_{I\triangleright J}\,. \tag{61}$$

The family $(\mathbf{M}_I)_{|I|=n}$ forms a complete system of indecomposable projective $H_n(0)$-modules, and the decomposition of the left regular representation is multiplicity free:

$$H_n(0) = \bigoplus_{|I|=n} \mathbf{M}_I\,. \tag{62}$$

6. Descent algebras and internal product

It remains to introduce an important operation on $\mathbf{Sym}$, namely, the noncommutative analogue of the internal product $*$ of Sym. Recall that it is the dual of the comultiplication $\delta(f) = f(XY)$. It occurs in algebraic identities such as

$$\sigma(XYZ;1) := \prod_{i,j,k}(1 - x_iy_jz_k)^{-1} = \sum_{\lambda,\mu} s_\lambda(X)s_\mu(Y)(s_\lambda * s_\mu)(Z) \tag{63}$$

which generalizes the Cauchy identity to three sets of variables.

The comultiplication $f \mapsto f(XY)$ makes sense as well for quasi-symmetric functions, if ones takes care of the order on the product set XY. That is, X and Y need to be totally ordered, and one can set $x_iy_j < x_ky_l$ whenever $(i,j) < (k,l)$ for the lexicographic order. Let us denote by $X\hat{\times}Y$ the product XY endowed with this order. Then, we can extend δ from Sym to $QSym$ by setting [16]

$$\delta(f) = f(X\hat{\times}Y)\,. \tag{64}$$

The internal product $*$ of $\mathbf{Sym}$ can now be defined as the dual of δ, that is,

$$\langle f\,,\,P * Q\rangle = \langle \delta(f)\,,\,P\otimes Q\rangle\,. \tag{65}$$

Clearly, each homogeneous component $\mathbf{Sym}_n$ is a $*$-subalgebra. In fact, it follows from results of Gessel [16] that this algebra is *anti*-isomorphic to the so-called *descent algebra* Σ_n of $\mathfrak{S}_n$. The descent algebras have been introduced by Solomon [39] for general finite Coxeter groups in the following way. Let (W,S) be a Coxeter system. One says that

$w \in W$ has a descent at $s \in S$ if w has a reduced word ending by s. For $W = \mathfrak{S}_n$ and $s_i = (i, i+1)$, this means that $w(i) > w(i+1)$, whence the terminology. In this case, we rather say that i is a descent of w. Let $\mathrm{Des}\,(w)$ denote the descent set of w, and for a subset $E \subseteq S$, set

$$D_E = \sum_{\mathrm{Des}\,(w)=E} w \quad \in \mathbb{Z}W\,. \tag{66}$$

Then, Solomon showed that the D_E span a $\mathbb{Z}$-subalgebra of $\mathbb{Z}W$. Moreover

$$D_{E'} D_{E''} = \sum_E c^E_{E' E''} D_E \tag{67}$$

where the coefficients $c^E_{E'E''}$ are nonnegative integers.

The canonical anti-isomorphism $\alpha \,:\, \Sigma_n \rightarrow \mathbf{Sym}_n$ maps the descent class D_E to the ribbon Schur function R_I, with I such that $E = \mathrm{Des}\,(I)$. From one of Solomon's formulas, one obtains the following multiplication rule.

Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ be two compositions of n. Then,

$$S^I * S^J = \sum_{M \in \mathrm{Mat}\,(I,J)} S^M \tag{68}$$

where $\mathrm{Mat}\,(I, J)$ denotes the set of matrices of nonnegative integers $M = (m_{ij})$ of size $p \times q$ such that $\sum_s m_{rs} = i_r$ and $\sum_r m_{rs} = j_s$ for $r \in [1, p]$ and $s \in [1, q]$, and where

$$S^M = S_{m_{11}}\, S_{m_{12}} \, \cdots \, S_{m_{1q}} \, \cdots \, S_{m_{p1}} \cdots S_{m_{pq}} \,.$$

Note that by definition, if F and G are homogeneous of different degrees, $F * G = 0$, and that S_n is the unit element of the $*$-subalgebra $\mathbf{Sym}_n$.

Let $h_\lambda = h_I = \underline{S^I}$ and $h_\mu = h_J = \underline{S^J}$ be the commutative images of S^I and S^J. From the known expression of $h_\lambda * h_\mu$ in the commutative case, one can see that $\underline{(S^I * S^J)} = \underline{S^I} * \underline{S^J}$, so that in general

$$\underline{F * G} = \underline{F} * \underline{G}\,, \tag{69}$$

that is, the commutative image is a homomorphism for the internal products.

From equation (68), one derives a fundamental formula, whose commutative version is just a special case of the Mackey formula for the restriction of an induced character. Let $F_1, F_2, \ldots, F_r, G \in \mathbf{Sym}$. Then,

$$(F_1 F_2 \cdots F_r) * G = \mu_r \left[(F_1 \otimes \cdots \otimes F_r) * \Delta^r G \right] \tag{70}$$

where in the right-hand side, μ_r denotes the r-fold ordinary multiplication and $*$ stands for the operation induced on $\mathbf{Sym}^{\otimes n}$ by $*$ [12].

In the commutative case, the power-sums p_n and more generally the power-sum products are quasi-idempotents (i.e., idempotents up to a scalar factor) for the internal product. Precisely,

$$p_\mu * p_\mu = z_\mu\, p_\mu \tag{71}$$

where for $\mu = (1^{m_1} 2^{m_2} \cdots n^{m_n})$, $z_\mu = \prod_i i^{m_i} \cdot m_i!$. Therefore, the commutative images of noncommutative power sums and their products are quasi-idempotents, and one may wonder whether there are true quasi-idempotents among them. Thanks to the anti-isomorphism α with the descent algebra, we could then use them to construct idempotents in the group algebras of symmetric groups.

As an illustration of the formalism, let us try this program with the power sums Ψ_n. We want to know whether $\Psi_n * \Psi_n = n\Psi_n$. To this end, we can write a generating function for the $*$-squares in the form

$$\sum_{n\geq 1} (xy)^{n-1} (\Psi_n * \Psi_n) = \psi(x) * \psi(y) , \tag{72}$$

since $\Psi_i * \Psi_j = 0$ for $i \neq j$. Now, writing (26) in the form

$$\psi(t) = \sum_{n\geq 1} t^{n-1} \Psi_n = \lambda(-t)\sigma'(t)$$

and applying the splitting formula (70), we get

$$\begin{aligned}\sum_{n\geq 1} (xy)^{n-1} (\Psi_n * \Psi_n) &= \lambda(-x)\sigma'(x) * \psi(y) \\ &= \mu\left[(\lambda(-x) \otimes \sigma'(x)) * (\psi(y) \otimes 1 + 1 \otimes \psi(y))\right] \\ &= \mu\left[(\lambda(-x) * 1) \otimes (\sigma'(x) * \psi(y))\right] ,\end{aligned}$$

(since $\sigma'(x)$ has no degree zero terms)

$$= \left(\sum_{n\geq 1} n x^{n-1}\, S_n \right) * \left(\sum_{n\geq 1} y^{n-1}\, \Psi_n \right) = \sum_{n\geq 1} (xy)^{n-1}\, n\, \Psi_n ,$$

the last equality following from the fact that $S_n * F = F$ for $F \in \mathbf{Sym}_n$.

Hence, $\Psi_n * \Psi_n = n\Psi_n$, so that $\theta_n = \alpha^{-1}(\Psi_n)$ is a quasi-idempotent of Σ_n. To see what it looks like, we have to expand Ψ_n on the ribbon basis. The linear recurrence (25) together with the multiplication formula for ribbons (31) (recall that $S_k = R_k$) yields

$$\Psi_n = R_n - R_{1,n-1} + R_{1,1,n-2} - \cdots = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k} , \tag{73}$$

which is analogous to the classical expression of p_n as the alternating sum of hook Schur functions. Therefore, in the descent algebra,

$$\theta_n = \sum_{k=0}^{n-1} (-1)^k\, D_{\{1,2,\ldots,k\}} . \tag{74}$$

On this expression, we can recognize a famous element of the group algebra of the symmetric group, namely, Dynkin's left-bracketing operator (see [38]). The *standard left bracketing* of a word $w = x_1 x_2 \cdots x_n$ is the Lie polynomial

$$L_n(w) = [\cdots[[[x_1,x_2],x_3],x_4],\ldots,x_n]\ . \tag{75}$$

This formula defines a linear operator L_n on the homogeneous component $\mathbb{K}_n\langle A\rangle$ of the free associative algebra $\mathbb{K}\langle A\rangle$. In terms of the right action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{K}_n\langle A\rangle$, defined on words by

$$x_1\,x_2\,\cdots\,x_n\cdot\sigma = x_{\sigma(1)}\,x_{\sigma(2)}\,\cdots\,x_{\sigma(n)}\ , \tag{76}$$

one can write

$$L_n(w) = \sum_{\sigma\in\mathfrak{S}_n} a_\sigma\,(w\cdot\sigma) = w\cdot\theta_n$$

the coefficient a_σ being ±1 or 0, according to whether σ is a "hook permutation" or not. To see this, one has only to write the permutations appearing in the first θ_i as ribbon tableaux and then to argue by induction. For example,

$$\theta_3 = [[1,2],3] = 123\ -\ \begin{matrix}2&\\1&3\end{matrix}\ -\ \begin{matrix}3&\\1&2\end{matrix}\ +\ \begin{matrix}3\\2\\1\end{matrix}$$

and it is clear that when expanding $\theta_4 = [\theta_3,4]$ one will only get those (signed) tableaux obtained from the previous ones by adding 4 at the end of the last row, minus those obtained by adding 4 on the top of the first column. Thus, we have proved that $\frac{1}{n}L_n$ is a projector, whose image is obviously a subspace of the free Lie algebra. By iteration of Jacobi's identity, it is easy to see that any Lie element can be written as a linear combination of standard left bracketings, so that what we have actually obtained is a proof of Dynkin's characterization of Lie elements: a noncommutative homogeneous polynomial $P\in\mathbb{K}_n\langle A\rangle$ is a Lie polynomial if and only if $L_n(P)=nP$ (cf. [38]).

Idempotents such as $\frac{1}{n}\theta_n$, acting as projectors onto the free Lie algebra are usually called *Lie idempotents* [38].

So far, our formalism has just led us to an exotic proof of a classical result. Let us now see whether the method contains the germ of some generalization. One ingredient in our proof was the analogue (73) of the expansion of a power sum as an alternating sum of hook Schur functions. This expression has a well known q-analogue, namely, the one involved in the character formula for Hecke algebras (41). It can be written in the form

$$\frac{h_n((1-q)X)}{1-q} = \sum_{k=0}^{n-1}(-q)^k s_{n-k,1^k}\ . \tag{77}$$

Let us look for a noncommutative analogue of this expression. To this aim, it will be convenient to extend as much as possible the λ-ring notation of the classical theory. Given two totally ordered sets A and B of noncommuting variables, we can define the virtual alphabet $A-B$ by specifying its complete symmetric functions $S_n(A-B)$. Their generating series is defined by

$$\sigma(A-B;t) := \lambda(B;-t)\sigma(A;t)\,. \tag{78}$$

One also defines the symmetric functions of $A+B$ by

$$\sigma(A+B;t) := \sigma(A;t)\sigma(B;t) \tag{79}$$

where now, A and B can be either real or virtual, and for a real ordered commutative alphabet X, the virtual alphabet XA is defined by

$$\sigma(XA;t) = \overrightarrow{\prod_{i\geq 1}} \sigma(A, tx_i)\,. \tag{80}$$

These definitions imply a definition of quasi-symmetric functions of a difference $f(X-Y)$, which is the same as the one obtained by composing the comultiplication and the antipode, and we can now give a meaning to the noncommutative symmetric functions of all virtual alphabets of the type $(X\pm Y)(A\pm B)$.

The case we have in mind corresponds to $X=\{1\}$ and $Y=\{q\}$. Here,

$$\begin{aligned}\sigma((1-q)A;t) &= \sigma(A-qA;t) = \lambda(qA;-t)\sigma(A;t)\\ &= \textstyle\sum_{k\geq 0} t^k(-q)^k\Lambda_k(A)\sum_{l\geq 0} t^l S_l(A)\,.\end{aligned}$$

Taking into account the fact that $\Lambda_k = R_{1^k}$ and applying the multiplication rule for ribbons, we find that

$$\Theta_n(q) := \frac{S_n((1-q)A)}{1-q} = \sum_{k=0}^{n-1}(-q)^k R_{1^k,n-k}\,, \tag{81}$$

is the q-analogue we were looking for. Moreover, it is easy to see that the image of $\Theta_n(q)$ under the isomorphism α^{-1} is the left q-bracketing of the standard word $1\,2\,\ldots\,n$, that is,

$$\alpha^{-1}(\Theta_n(q)) = \theta_n(q) = [[\ldots[[\,1,\,2\,]_q,\,3\,]_q,\,\ldots]_q,\,n\,]_q$$

where $[R,S]_q = RS - q\,SR$. Now, we can prove the following q-analog of Dynkin's theorem, which, according to what we have already seen, can be understood as describing the left q-bracketing of a homogeneous Lie polynomial:

$$\Theta_n(q) * \Psi_n = [n]_q\,\Psi_n\,. \tag{82}$$

The proof works exactly as the previous one. We have

$$\vartheta(t) = \sum_{n\geq 1}\Theta_n(q)t^{n-1} = \frac{\sigma((1-q)A;t)-1}{(1-q)t} = \lambda(A;-qt)\sigma'_q(A;t)\,. \tag{83}$$

where σ'_q denotes the the q-derivative with respect to t. Then,

$$\vartheta(t) * \Psi_n = \mu((\lambda(A;-qt)\otimes\sigma'_q(A;t)) * \Delta(\Psi_n))$$

$$= \mu((\lambda(A;-qt)\otimes\sigma'_q(A;t)) * (1\otimes\Psi_n + \Psi_n\otimes 1))\,,$$

which implies that

$$\vartheta(t) * \Psi_n = (\lambda(A; -qt) * 1)\,(\sigma'_q(A;t) * \Psi_n) = [n]_q\, \Psi_n\, t^{n-1}\ .$$

Equation (82) means that homogeneous Lie polynomials of degree n are again eigenvectors of the left q-bracketing operator, now with the q-integer $[n]_q$ as eigenvalue. They actually constitute the $[n]_q$ eigenspace. However, the q-Dynkin element $\theta_n(q)$ is invertible in the group algebra for generic values of q, and its other eigenvalues are nonzero. Its spectral decomposition has been obtained in [20]. The case $q = -1$ is of special importance: it governs the combinatorics of the *peak algebras* [2].

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