Fast and Stable Computation of Binary Error Rates in Rapid Rayleigh Fading Channels

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We present a fast and numerically stable procedure for evaluating the binary error rate of optimal and suboptimal quadratic receivers in a rapid Rayleigh fading channel.

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1 Introduction

The analysis of many practical digital transmission systems involves the computation of the probability that a given Hermitian quadratic form in complex normal variates is negative. Numerous such examples can be found in Proakis's standard textbook (cf [1]). This kind of problem appears in particular in the context of performance analysis of classical demodulation protocols of modulated signals transfered on noisy Gaussian channels.

A first expression for the probability that a given Hermitian quadratic form in complex normal variates is negative was derived by Turin (cf [2]) and used later by Barrett ([3]) to unveil a closed form expression for this probability as a rational function of the eigenvalues of the corresponding covariance matrix, which can always be put in diagonal form. Barett shows in particular that the general problem discussed above can be reduced to the study of the following probability

$$P\left(\sum_{j=0}^{N-1} u_j^* u_j < \sum_{j=0}^{N-1} v_j^* v_j\right) = P(U < V) \tag{1}$$

where the u and v's are independent complex centered Gaussian random variables with arbitrary variances respectively denoted (for every j in $[0,N]$)

$$E[u_j^* u_j] = \chi_j \ , \tag{2}$$

$$E[v_j^* v_j] = \delta_j \, . \tag{3}$$

Barett's formula for (1) above is as follows:

$$P(U<V) = \sum_{k=0}^{N-1} \prod_{j\neq k} \frac{1}{1-\delta_k^{-1}\delta_j} \prod_{j=0}^{N-1} \frac{1}{1+\delta_k^{-1}\chi_j} \, . \tag{4}$$

Barrett's formula is presently perhaps the best approach from a practical point of view. Alternate methods involving direct contour integration of the known characteristic function along a carefully selected path so as to maximise numerical stability, algebraic manipulations like in [1] (Annex B) or [5] provide another approach involving numerical quadrature of trigonometric functions.

A brute force approach involving the determination of the respective probability distribution functions of U and V above results in a formula in the same vein as (2), but exhibiting additional singularities when some of the χ_i's are identical, while (2) proves undefined only when some of the δ_i's collapse to a common value.

None of those alternate methods offer a nice and compact formulation comparable with that of [3] which we consider to be the state of the art regarding the problem of numerical evaluation of (1), despite the fact that it proves undefined in many cases of real interest, and numerically delicate to use in the vicinity of the singular points.

The first thing to realise is that the underlying rational function of $2N$ variables is not intrinsically singular at all, it is a perfectly smooth function of its argument. It can be put explicitly, as shown in Section 2 under the form of a sum of positive terms with no cancellation between numerator and denominator, the associated penalty being a much longer formula, which eventually proves useless as N becomes large.

The remaining of this paper discusses many interesting properties of this function and culminates in a numerical engine that keeps all positive attributes of Barrett's formulation, especially its $O(N^2)$ operation count, while offering the additional benefits of being defined and fully stable across the whole domain of definition.

2 Exact expression in terms of Schur functions

Formula (4) can be recast into the form

$$P(U<V) = \frac{F(\delta,\chi)}{\prod_{j,k}(\delta_j+\chi_k)} \tag{5}$$

where $F(\delta,\chi)$ is a polynomial which is symmetric with respect to the δ_j and to the χ_k. Using some rather involved techniques of symmetric function theory (*cf.* [6]) it is possible to express $F(\delta,\chi)$ in terms of the Schur symmetric functions (see the Appendix for the explanation of notation):

$$F(\delta,\chi) = \sum_{\lambda\subseteq(N^{N-1})} s_{(N,\lambda)}(\delta)s_{\lambda^\vee}(\chi)\,. \tag{6}$$

This means that $F(\delta,\chi)$ are the sums of all the monomials obtained by taking the product of all entries of a Young tableau of shape (N,λ) by all entries of a Young tableau of shape $\lambda^\vee$ for all partitions λ contained in the rectangle N^{N-1}. The reader may easily reformulate this property by noticing that these monomials are exactly obtained by taking a square of size $N\times N$ in which one considers the monomials obtained from the combinatorial structure consisting of

- a Young tableau containing the first row filled with elements of the alphabet δ,
- the complement of the previous Young tableau in the square filled with elements of the alphabet χ.

Since (6) will not be needed in the sequel, the details of its derivation will not be reproduced here. However, some comments are in order. We first note that the denominator of (5) is just the resultant of the polynomials $X(z)$ and $D(z)$ having as roots the χ_j and the $-\delta_j$. It can therefore be computed from their coefficients only, and the same is true of the numerator $F(\delta,\chi)$, thanks to its symmetry properties.

In fact, application of various determinantal identities for Schur functions leads to the conclusion that $P(U<V)$ can be obtained directly from the solution of a linear system constructed form the coefficients of $X(z)$ and $D(z)$. The solution of this system can finally be identified with the vector of coefficients of the polynomial $\pi(z)$, introduced in the forthcoming section. Once this has been realized, it becomes possible to give a simple and direct derivation of the resulting algorithm, which will be done in Section 4.

3 A stable algorithm for evaluating $P(U < V)$

Our procedure consists in the following three steps:

1. Compute the polynomials $\chi(z) = \prod_{i=1}^{N}(1-\chi_i z)$ and $\Delta(z) = \prod_{i=1}^{N}(1+\delta_i z)$;
2. Compute $\pi(z)$, defined as the multiplicative inverse of $\chi(z)$ modulo $\Delta(z)$, i.e. the unique polynomial $\pi(z)$ of degree less or equal to $N-1$ such that one can write
$$\pi(z)\,\chi(z) + \mu(z)\,\Delta(z) = 1$$
with some other polynomial $\mu(z)$ (of degree less or equal to $N-1$);
3. The constant term $\pi(0)$ is equal to $P(U < V)$.

The roots of the monic polynomials $\chi(z)$ and $\Delta(z)$ are the inverses of the variances χ_i and $-\delta_i$, respectively. When the roots are simple, we find that the polynomial $\pi(z)$ passes through the points $(z_i, \pi(z_i)) = (-\delta_i^{-1}, \chi(-\delta_i^{-1})^{-1})$, and the procedure outlined above is tantamount to a kind of polynomial extrapolation from the z_i towards 0. Standard Lagrange interpolation methods applied directly to the multiplicative inversion in step 2 above would result in the same kind of problems we face with the straight evaluation of (4). The crux of the stabilized method is that we do not *need* to mechanize step 2 using one of the usual interpolation engines and that the canonical method for multiplicative inversion, the Extended Euclidean Algorithm, performs a much better job, especially when the roots are clustered or of multiplicity greater than 1.

4 Proof of the algorithm

Let us now proceed with the formal proof of the previously defined method. We shall show that the claim in step 3 above is true whenever all δ_i are distinct, which, together with the rational and continuous nature of the desired probability as a function of its numerous arguments, and the fact that all quantities above are still well-defined even when Δ has multiple roots, is enough to validate the claimed attractive attributes of the resulting procedure.

The unique polynomial of degree less than N which stisfies the N conditions $\pi(z_j) = t_j$ may be found using the classical Lagrange interpolation formula
$$\pi(z) = \sum_{k=0}^{N-1} t_k \cdot \prod_{j\neq k} \frac{z - z_j}{z_k - z_j}\,. \tag{7}$$
Given the defining property $\pi(z)\chi(z) \equiv 1 \bmod \Delta(z)$ as per step 2, we may substitute all the N roots of Δ for z and obtain an equivalent set of defining conditions for our polynomial π having the previous form with now
$$z_k = -\delta_k^{-1}, \quad t_k = \chi(z_k)^{-1} = \prod_{j=0}^{N-1}(1+\delta_k^{-1}\chi_j)^{-1}\,. \tag{8}$$
Lagrange's formula provides then an explicit expression for the multiplicative inverse:
$$\pi(z) \quad = \sum_{k=0}^{N-1} \frac{\prod_{j\neq k}(z+\delta_j^{-1})}{\prod_j(1+\delta_k^{-1}\chi_j)\cdot\prod_{j\neq k}(-\delta_k^{-1}+\delta_j^{-1})}$$
from which one gets
$$\pi(z) \quad = \sum_{k=0}^{N-1} \frac{\prod_{j\neq k}(1+\delta_j z)}{\prod_j(1+\delta_k^{-1}\delta_j)\cdot\prod_{j\neq k}(1+\delta_k^{-1}\chi_j)}\,.$$
It is now plain by setting $z = 0$ in the last expression that the value of $\pi(0)$ is just the number provided by equation (4), which concludes the proof.

5 Some special cases

We shall provide here additional insight into what happens to the general expression of $P(U < V)$ in two special cases.

The fist one corresponds to all U and V elementary ingredients having identical variances χ and δ respectively. We will provide a fully constructive expression in this case, which proves useful to analyze the behaviour of a diverse scheme having N independent signals available with the same SNR for each branch.

The second special case corresponds to arbitrary χ_i but all δ_i identical. This specialization is useful in the asymptotic analysis of a diverse scheme with N inputs with different SNR on each branch.

Our first result regarding the polynomial $\pi(z)$ defined as the multiplicative inverse of $(1-\chi z)^N$ modulo $(1+\delta z)^N$ reads:

$$\pi(z) = \sum_{j=0}^{N-1} \binom{N-1+j}{N-1} \left(\frac{\delta}{\chi+\delta}\right)^{N+j} (1+\delta z)^j . \tag{9}$$

The proof starts with the obvious identity

$$1 = (1-u)^N (1-u)^{-N} = (1-u)^N \sum_{i=0}^{\infty} \binom{N-1+i}{N-1} u^i . \tag{10}$$

Introducing another variable y, changing u into u/y and reducing modulo u^N yields:

$$(y-u)^N \sum_{i=0}^{N-1} u^i y^{N-1-i} \equiv y^{2N-1} \bmod u^N . \tag{11}$$

Substituting $y = \chi+\delta$ and $u = \chi(1+\delta z)$ in the previous identity then provides the claimed result. We are mainly interested in the special case $z = 0$, and the combinatorial identity

$$\sum_{i=0}^{j} \binom{N-1+i}{i} \binom{N-1-i}{j-i} = \binom{2N-1}{j} \tag{12}$$

allows us to put the desired number into the following alternate form, which at least better highlights the behaviour for large χ to δ ratio:

$$\pi(0) = \left(\frac{\delta}{\chi+\delta}\right)^{2N-1} \left(\sum_{j=0}^{N-1} \binom{2N-1}{j} \left(\frac{\chi}{\delta}\right)^j \right) . \tag{13}$$

The second special case of interest we shall investigate now is when we allow arbitrary values for χ_i, while still keeping all the δ identical. Since we are now dealing with a more general case, we shall restrict our ambitions and just prove that for fixed χ's and δ getting small, we have

$$\pi(0) \sim \binom{2N-1}{N-1} \delta^N \left(\prod_{i=0}^{N-1} \frac{1}{\chi_i} \right) . \tag{14}$$

For the sake of streamlining the notations, we shall fix δ to 1 and just let all χ_i grow, which does not change our problem since the end result only depends on the δ to χ ratios. We have to assess the behaviour of $\pi(0)$ as a function of the χ's, where

$$\pi(z) \cdot \prod_{i=0}^{N-1} (1-\chi_i z) \equiv 1 \bmod (1+z)^N . \tag{15}$$

Shifting the z variable by setting $y = z+1$ and defining $\tau_i = \chi_i(1+\chi_i)^{-1}$, we observe that we have $1-\chi_i z = (1+\chi_i)(1-\tau_i y)$, so that if one defines the polynomial θ by setting

$$\theta(y) \cdot \prod_{i=0}^{N-1} (1-\tau_i y) \equiv 1 \bmod y^N , \tag{16}$$

one then has the following relationship between the previously defined quantities:

$$\pi(0) = \frac{\theta(1)}{\prod_{i=0}^{N-1}(1+\chi_i)}. \tag{17}$$

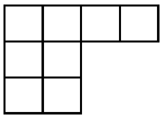
Therfore, it is sufficient to show that for large χ_i, we have $\theta(1) \sim \binom{2N-1}{N-1}$. But since our numbers τ_i are close to 1 in this case, it is sufficient to inspect the following consequence of (11) above:

$$\theta(y) \sim (1-y)^{-N} \equiv \sum_{i=0}^{N-1} \binom{N-1+i}{N-1} y^i \bmod y^N \tag{18}$$

and then to substitute $y = 1$, using (12) with $j = N-1$ to get the result.

Appendix: partitions, tableaux and Schur functions

A *partition* is a finite weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0)$. Graphically, a partition can be represented by a diagram of boxes, called its Ferrers graph, whose ith row contains λ_i boxes. A filling of the boxes with positive integers is called a *Young tableau* of *shape* λ whenever the numbers are weakly increasing along rows and strictly increasing along columns. For example,



1	1	1	4
2	2		
3	5		

are respectively the Ferrers diagram of the partition $\lambda = (4,2,2)$ and a Young tableau of shape $(4,2,2)$. The *conjugate partition* λ' is formed by the heights of the columns of the diagram. Here, $\lambda' = (3,3,1,1)$. If $X = \{x_1,\ldots,x_n\}$ is a set of n variables, one associates to any Young tableau T a monomial X^T defined as the product of the factors x_i for each entry i of T. For the tableau T above, $X^T = x_1^3 x_2^2 x_3 x_4 x_5$. The *Schur function* $s_\lambda(X)$ associated to the partition λ can be defined a the sum of the monomials X^T, for T running over all Young tableaux of shape λ filled with numbers not greater than n (for more information, see [6]).

In formula (6), the notation $\lambda \subseteq (N^{N-1})$ means that the sum runs over all partitions λ whose Ferrers graph is contained into the rectangle (N^{N-1}) with $N-1$ rows of length N, *i.e.* such that $\lambda_i \leq N$ for all i. For such a λ, $(N,\lambda) = (N,\lambda_1,\ldots,\lambda_{N-1})$ is again a partition. Finally, $\lambda^\vee$ denotes the conjugate of the partition ν whose Ferrers graph is the complement of of λ in (N^{N-1}), *e.g.* for $N = 6$ and $\lambda = (3,2,1,1)$, we have $\nu = (6,5,5,4,3)$ and $\lambda^\vee = (5,5,5,4,3,1)$.

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