On Witt Vectors and Symmetric Functions

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Abstract

We present some remarks on certain symmetric functions involved in the definition of universal Witt rings. In particular, we prove Reutenauer's conjecture on their positivity, and relate them to the Dress congruences.

1 Introduction

The universal ring $W(A)$ of Witt vectors associated to a commutative ring A [18] can be characterized by the following properties [1] (see also [10, 3]):

(W1) As a set, $W(A) = \{\mathbf{a} = (a_n)_{n\geq 1}\,,\, a_n \in A\}$, and for any ring homomorphism $f : A \to B$, the map $W(f) :\ \mathbf{a} \mapsto (f(a_n))_{n\geq 1}$ is a ring homomorphism.

(W2) The maps $w_n :\ \mathbf{a} \mapsto \sum_{d|n} d a_d^{n/d}$ are ring homomorphisms $W(A) \to A$.

Operations on Witt vectors are better understood in terms of symmetric functions. Let $X = \{x_n\,,\, n \geq 1\}$ be an infinite set of commuting indeterminates (called here an *alphabet*), and following [15], define symmetric functions $q_n(X)$ by

$$\prod_{n\geq 1} \frac{1}{1 - t^n q_n(X)} = \sigma_t(X) := \sum_{n\geq 1} t^n h_n(X) \tag{1}$$

where the complete homogeneous functions $h_n(X)$ are defined by $\sigma_t(X) = \prod_{n\geq 1}(1-tx_n)^{-1}$ (*cf.* [12]).
The q_n's are connected to the power-sums $p_n(X) = \sum_i x_i^n$ by

$$p_n = \sum_{d|n} d q_d^{n/d} \tag{2}$$

and condition (W2) can be regarded as expressing the familiar properties of power sums

$$p_n(X + Y) = p_n(X) + p_n(Y)$$

$$p_n(XY) = p_n(X)p_n(Y)$$

where we use the λ-ring notation (an alphabet is identified with the formal sum of its elements). Reutenauer conceived the idea of studying the symmetric functions $q_n(X)$

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from a representation theoretical point of view, and investigated their expansions on the Schur basis [15]. He conjectured that for $n \geq 2$, $(-q_n)$ is Schur positive, proved it in a number of special cases, and obtained on the way various algebraic identities.
In this note, we generalize certain results of [15], and relate them to other works. In particular, we prove the positivity conjecture, as well as a noncommutative version implying that the $(-q_n)$ are positive sums of ribbon Schur functions. We study the adjoint basis r_λ of the Reutenauer basis $q^\lambda = q_{\lambda_1} \cdots q_{\lambda_r}$, and relate it to the Dress congruences. We also give a combinatorial description of the r_λ's in terms of generalized ribbon tableaux.
Dress derived his congruences from the theory of (restricted) Burnside rings. That this not a mere coincidence was shown by Dress and Siebeneicher who related Witt vectors to the Burnside ring of the infinite cyclic group [6] as well as to the algebra of necklaces [14].
In the following our notations will be those of [12] and [8].

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2 The Dress congruences and Reutenauer's basis

In [5], A. Dress gives a series of congruences constituting a test to determine whether a central function on $\mathfrak{S}_n$ is a virtual character, *i.e.* belongs to the $\mathbb{Z}$-module generated by the irreducible characters. Equivalently, this gives a characterization of the $\mathbb{Z}$-lattice generated by Schur functions in the algebra *Sym* of symmetric functions with rational coefficients.

Theorem 2.1 [5] *A mapping $\chi : \mathrm{Part}\,(n) \to \mathbb{Z}$, where $\mathrm{Part}\,(n)$ is the set of partitions of n, is a virtual character of $\mathfrak{S}_n$ if and only if for any partition $\pi = (1^{k_1} 2^{k_2} \cdots n^{k_n})$ of n one has*

$$\sum_{\pi^1,\ldots,\pi^n} \prod_{i=1}^{n} \frac{k_i!}{\prod_j j^{h^i_j} h^i_j!} \cdot \chi(1^{h^1_1} 2^{h^1_2 + h^2_1} \cdots n^{\sum_{ab=n} h^a_b}) \equiv 0 \left[\prod_{i=1}^{n} k_i! \right] , \tag{3}$$

where the sum is taken over all systems $(\pi^1, \ldots, \pi^n)$ of partitions $\pi^i = (1^{h^i_1} 2^{h^i_2} \cdots)$ of k_i, $i = 1, \ldots, n$.

For a partition $\pi = (1^{k_1} 2^{k_2} \cdots n^{k_n})$, define

$$r_\pi = h_{k_1}[p_1] h_{k_2}[p_2] \cdots h_{k_n}[p_n] , \tag{4}$$

where for a symmetric function F, the notation $F[p_k]$ means the plethysm of F by p_k, *i.e.* $F(x_1^k, x_2^k, \ldots)$.

Proposition 2.2 *The Dress congruences for a symmetric function F are equivalent to the following property:*

$$\text{For any partition } \pi \qquad \langle F \, , \, r_\pi \rangle \in \mathbb{Z} \ . \tag{5}$$

Proof — The congruences (3) applied to the central function $\chi : \pi \mapsto \langle F, p_\pi \rangle$ are equivalent to

$$\sum_i \sum_{\pi^i \vdash k_i} \left\langle F\,,\, \frac{p_1^{h_1^1} p_2^{h_2^1+h_1^2} \cdots p_n^{\sum_{ab=n} h_b^a}}{z_{\pi^1} z_{\pi^2} \cdots z_{\pi^n}} \right\rangle \in \mathbb{Z}\ .$$

But this expression is equal to

$$\left\langle F\,,\, \sum_i \sum_{\pi^i \vdash k_i} \frac{p_{\pi^1}[p_1]}{z_{\pi^1}} \cdot \frac{p_{\pi^2}[p_2]}{z_{\pi^2}} \cdots \frac{p_{\pi^n}[p_n]}{z_{\pi^n}} \right\rangle$$

$$= \langle F\,,\, h_{k_1}[p_1] h_{k_2}[p_2] \cdots h_{k_n}[p_n] \rangle = \langle F\,,\, r_\pi \rangle$$

□

It is not difficult to check that the r_λ form a basis of *Sym*, and property (5) means that F is a $\mathbb{Z}$-linear combination of the adjoint basis r_λ^*. This last one appears implicitely in [15]. In this paper, the author defines symmetric functions $q_n(X)$ by (1) From this, one can define a basis of *Sym* by setting

$$q^\mu = q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$$

The linear independence of the q^λ (λ a partition) follows from the equation

$$h_n = \sum_{\mu \vdash n} q^\mu \tag{6}$$

which shows that the q_n generate *Sym* as an algebra. It is also clear that the q^μ generate the same lattice as the s_λ, so that Proposition 2.2 provides an alternative proof of Theorem 2.1.

Proposition 2.3 *The basis* (q^λ) *is adjoint to the basis* (r_λ), i.e. $\langle q^\lambda\,,\, r_\mu \rangle = \delta_{\lambda\mu}$.

Proof — One knows (*cf.* [12]) that two bases (u_λ), (v_μ) of *Sym* are adjoint to each other iff

$$\sum_\lambda u_\lambda(X) v_\lambda(Y) = \sigma_1(XY)\ . \tag{7}$$

Hence, one computes

$$\sum_\lambda q^\lambda(X) r_\lambda(Y) = \sum_{\mathbf{m}} \prod_{i \geq 1} q_i(X)^{m_i} h_{m_i}[p_i(Y)]$$

$$= \prod_{i\geq 1} \sigma_{q_i(X)}[p_i(Y)] = \prod_{i \geq 1} \prod_{j \geq 1} \frac{1}{1 - y_j^i q_i(X)}$$

$$= \prod_{j \geq 1} \sigma_{y_j}(X) = \sigma_1(XY)\ .$$

□

The following proposition has been conjectured by Reutenauer in [15]. It has also been proven independently by Bill Doran [4].

Proposition 2.4 *The functions $q_1, -q_m, m \geq 2$ are Schur positive, i.e. a linear combination of Schur functions with non-negative integer coefficients.*

Proof — Set for $n \geq 1$,

$$F(n) := 1 - \prod_{i \geq n} (1 - q_i)^{-1} = -q_n + \text{terms of higher degree.}$$

Then

$$F(1) = -\sum_{i>0} h_i, \quad F(2) = \sum_{i \geq 0} s_{(i+1,1)},$$

$$F(n+1) = q_n + F(n) - q_n F(n).$$

We will show by induction that the $F(n)$ and the $-q_n, n \geq 2$ are Schur positive (the case $q_1 = h_1$ is trivial). Now, the claim is obvious for $n = 2$. Suppose that the claim is true for n and consider

$$F(n+1) = q_n + F(n) - q_n F(n).$$

By induction, $-q_n F(n)$ is Schur positive and also $q_n + F(n)$. For the summand q_n cancels with the first term $F_n(n)$ of $F(n)$. This gives the proposition. $\square$

The q_n possess some interesting properties. For example, for n a prime, $-q_n$ coincides with the well-known cyclic character [7]

$$-q_n = \ell_n = \frac{1}{n}(p_1^n - p_n).$$

The ℓ_n and the q_n are related by a plethystic equation [15]. We can give a slightly more general statement, using the other cyclic characters. Define

$$\ell_n^{(k)} := \frac{1}{n} \sum_{d|n} c(k,d) p_d^{n/d},$$

where $c(k,d)$ is von Sterneck's function, sometimes referred to as a Ramanujan sum (*i.e.* $c(k,d)$ is the sum of the k-th powers of the *primitive* d-th roots of unity). The symmetric function $\ell_n^{(k)}$ is the characteristic of the representation of $\mathfrak{S}_n$ induced by the representation ρ_k of a transitive cyclic subgroup Γ, defined by $\rho_k(\gamma) = \omega^k$, where $\omega = e^{2\pi i/n}$ and γ is a generator of Γ [7].

Proposition 2.5 *For $n \geq 1$ we have*

$$\sum_{d|n} \ell_d^{(k)}[q_{n/d}] = \begin{cases} p_n & \textit{if } n \mid k \\ 0 & \textit{otherwise.} \end{cases}$$

Proof — Let

$$H := \sigma_1(X) = \sum_{n \geq 0} h_n, \quad L := \sum_{n \geq 1} \ell_n^{(k)}.$$

We now imitate the argument of [15], starting from the identity [16]

$$H[L] = \prod_{d|k} \frac{1}{1 - p_d}.$$

As $(1-p_d)^{-1}[q_m] = (1-q_m[p_d])^{-1} = (1-q_m)^{-1}[p_d]$, we get

$$\prod_m (1-p_d)^{-1}[q_m] = \prod_m (1-q_m)^{-1}[p_d] = H[p_d].$$

Hence

$$H[\sum_m L[q_m]] = \prod_m H[L[q_m]] = \prod_m \prod_{d|k} \frac{1}{(1-p_d)}[q_m] = \prod_{d|k} H[p_d] = H[\sum_{d|k} p_d].$$

The transformation $F \mapsto H[F]$ being invertible, we obtain

$$\sum_m L[q_m] = \sum_{d|k} p_d \ ,$$

as required. □

3 A combinatorial description of r_λ

The scalar products $\langle s_\mu\,,\, r_\lambda\rangle$ admit a simple description in terms of general ribbon tableaux, such as those introduced in [17].
Let $\lambda = (1^{a_1} 2^{a_2} \ldots s^{a_s})$. Then, using the λ-ring notation,

$$r_\lambda(-X) = h_{a_1}[-p_1(X)] h_{a_2}[-p_2(X)] \cdots h_{a_s}[-p_s(X)]$$
$$= (-1)^{a_1+a_2+\cdots a_s} e_{a_1}[p_1] e_{a_2}[p_2] \cdots e_{a_s}[p_s](X)$$
$$= (-1)^{\ell(\lambda)} m_{(1^{a_1})} m_{(2^{a_2})} \cdots m_{(s^{a_s})}(X) \ .$$

By Muir's formula and a standard argument argument as in the proof of the Murnaghan-Nakayama rule (*cf.* [2] or the new edition of [12] for details), this implies the following description.

Proposition 3.1 *Let* $\mathrm{RTab}\,(\lambda)$ *be the set of ribbon tableaux having exactly* a_k *ribbons of lenght* k*, each carrying the label* k*. For such a tableau* T*, set* $d(T) = \sum_R (h(R)-1)$*, where the sum runs over the ribbons* R *composing* T*, and* $h(R)$ *is the height of the ribbon* R*. Let also* $\mathrm{sh}\,(T)$ *denote the shape of* T *and* $s(T) = s_{\mathrm{sh}\,(T)}$*. Then, notation,*

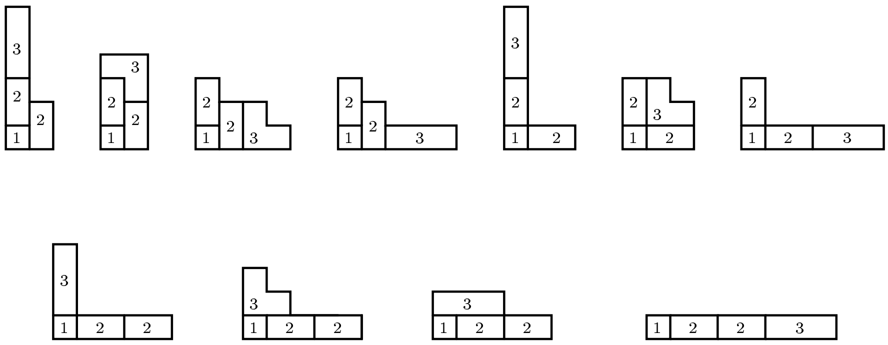
$$r_\lambda = \sum_{T \in \mathrm{RTab}\,(\lambda)} (-1)^{d(T)}\, s(T) \ .$$

□

For example,

$$r_{3221} = s_8 - s_{611} + s_{53} + s_{5111} - s_{431} + s_{332} - s_{311111} - s_{2222} + s_{221111} \ ,$$

which corresponds to the following tableaux:



the only cancellation being between the terms associated to the two tableaux of shape $(5, 2, 1)$.
The same reasoning gives a similar description of the Schur expansion of any product of plethysms of the type $h_i[p_j]$.

4 Non-commutative analogues

Noncommutative symmetric functions are defined in [8]. Noncommutative analogues of the cyclic characters $\ell_n^{(k)}$ have been defined in [11]. In the same spirit, the recurrence relations used in the proof of 2.4 suggest the following definition of an analogue of the q_n in the algebra of non-commutative symmetric functions.
We recall that the algebra **Sym** of noncommutative symmetric functions can be defined as the free associative algebra $K\langle S_1, S_2, \ldots, \rangle$ generated by an infinite sequence of non-commutative variables S_n, graded by the weight function $w(S_n) = n$. A basis of the homogeneous component $\mathbf{Sym}_n$ is given by the products $S^I = S_{i_1} \cdots S_{i_r}$, where I is a composition of n. Another natural basis consists of the noncommutative ribbon Schur functions, which can be defined by $R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J$, where $\preceq$ is the reverse refinement order on compositions, and $\ell(I) = r$ is the lenght of the composition I.
Let $F(1) := -\sum_{i \geq 1} R_i, Q_1 := R_1$, and define recursively

$$F(n+1) := Q_n + F(n) - Q_n F(n),$$

and let Q_{n+1} be the term of degree $n+1$ in F_{n+1}.
For convenience, we set $\tilde{Q}_1 := Q_1, \tilde{Q}_n := -Q_n, n \geq 2$.
We note that $F(2) = \sum_{i \geq 1} V_i$, where $V_i := R_{(1,i-1)}$. Hence each $\tilde{Q}_n$ is a positive sum of monomials

$$V^I := V_{i_1} \cdots V_{i_r},$$

where I are certain compositions of n into integers greater than 1. These compositions can be defined recursively (in fact, it is a reformulation of the recurrence relation) by a certain class $\mathcal{T}_n$ of labelled binary trees:

1. The tree consisting of the root only which has label n belongs to $\mathcal{T}_n$.

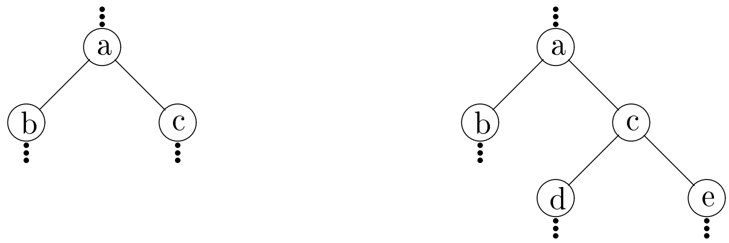
2. Let $t_a \in \mathcal{T}_a, t_b \in \mathcal{T}_b$ with $2 \leq a \leq b, a + b = n$, and consider the following tree. Its root is labelled by n, the left subtree is t_a and the right subtree t_b. This tree belongs to $\mathcal{T}_{a+b}$, iff the label c of the left son of the root of t_b satisfies $a > c$.

3. Each tree in $\mathcal{T}_n$ can be obtained in this way.

The leaves of these trees yield when written from left to right a composition $I(t)$ and

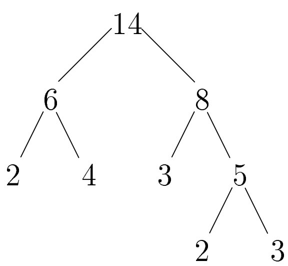
$$\tilde{Q}_n = \sum_{t \in \mathcal{T}_n} k_{t,n} V^{I(t)}.$$

(In fact, we will show $k_{t,n} = 1$.)
The construction in point 2. can be better seen on a picture. A tree is in some $\mathcal{T}_n$, if each subtree is of the following form



where $\quad a = b + c,\ b \leq c, \qquad c = d + e,\ d \leq e, \qquad b > d, \qquad 2 \leq a, b, c, d, e.$

The mapping from $\mathcal{T}_n$ to the set of compositions of n is injective. Indeed, let I be a composition corresponding to some tree and consider the smallest number s appearing in I and its leftmost occurence in I. If $I = (n)$, everything is clear. If not, then s has a father b and this father must have a right son a. This a must be a leaf as each left son of a must be smaller than s which is impossible. Note that $s \leq a$. This is true for any tree yielding I. If we consider I' where s, a is replaced by $s + a$, then I' comes again from a tree in $\mathcal{T}_n$. Just delete the corresponding leaves in any tree for I. Proceeding by induction on the number of elements in I we get that there is a unique tree $t \in \mathcal{T}_n$ for I.
For example, consider $I := (2, 4, 3, 2, 3)$. The smallest element is 2 and $I' := (6, 3, 2, 3)$. Continuing we get $(6, 3, 5), (6, 8), (14)$ and the corresponding tree in $\mathcal{T}_{14}$ is



On the other hand, for $I := (4, 2, 3, 2, 2)$ we get $(4, 5, 2, 2), (4, 5, 4), (9, 4)$. But the last composition does not come from a tree. So I does not come from one either.
Translating this in terms of the $\tilde{Q}_n$ yields

Proposition 4.1 *Each $\tilde{Q}_n$ can be expressed*

a) as a positive and multiplicity-free sum of monomials V^I, *b) as a positive and multiplicity-free sum of ribbons R_I*

c) as a multiplicity-free sum of S^I: If S^I appears in this sum, then then its coefficient is $(-1)^{l(I)}$.

Proof — a) follows from the above reasoning. As the compositions appearing in a) do not contain the part 1, it is easy to see how to recover b) and c) form a). □

Let us give some examples:

$$\begin{array}{rcl}
\tilde{Q}_1 &=& V^{(1)} = R_{(1)} = S^{(1)}, \\
\tilde{Q}_2 &=& V^{(2)} = R_{(1,1)} = S^{(1,1)} - S^{(2)}, \\
\tilde{Q}_3 &=& V^{(3)} = R_{(1,2)} = S^{(1,2)} - S^{(3)}, \\
\tilde{Q}_4 &=& V^{(4)} + V^{(2,2)} \\
&=& R_{(1,3)} + R_{(1,2,1)} + R_{(1,1,1,1)} \\
&=& -S^{(4)} + S^{(2,2)} + S^{(1,3)} - S^{(1,1,2)} - S^{(2,1,1)} + S^{(1,1,1,1)}.
\end{array}$$

Furthermore,

$$\begin{array}{rcl}
\tilde{Q}_5 &=& V^{(5)} + V^{(2,3)}, \\
\tilde{Q}_6 &=& V^{(6)} + V^{(2,4)} + V^{(3,3)}, \\
\tilde{Q}_7 &=& V^{(7)} + V^{(2,5)} + V^{(3,4)} + V^{(3,2,2)}, \\
\tilde{Q}_8 &=& V^{(8)} + V^{(2,6)} + V^{(3,5)} + V^{(4,4)} + V^{(3,2,3)} \\
&& +V^{(4,2,2)} + V^{(2,2,4)} + V^{(2,2,2,2)}.
\end{array}$$

The Hopf-algebra dual of the non-commutative symmetric functions is not isomorphic to the algebra itself, but to Gessel's Hopf algebra of quasisymmetric functions [9, 13]. The dual basis of (S^I) is the basis of quasi-monomial functions, defined for a commutative totally ordered infinite alphabet $X = \{x_i, i \geq 1\}$ by

$$M_I(X) = \sum_{j_1<j_2<\ldots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \cdots x_{j_r}^{i_r} \; .$$

The dual basis of (R_I) is the quasi-ribbon basis $(F_I(X)$.
In this setting the dual basis of (Q^I) can be easily described.

Proposition 4.2 *Let I be a composition of n and $D(I) := \{d_1 < d_2 < \ldots < d_r\}$ be the descent set of I. Denote by $K := (k_1, \ldots, k_{r+1})$ the descent decomposition associated to I considered as a sequence, i.e. $k_l := \sum_{d_{l-1}<j\leq d_l} i_j$, where $d_0 := 0$. Define the quasi-symmetric function*

$$P_I := \sum_{K \preceq J \preceq I} M_J,$$

where $\preceq$ denotes the reverse refinement order.
Then, (P_I) is the dual basis of (Q^J).

Proof — The proof is analogous to the commutative case, this time using the non-commutative analogue of the Cauchy identity.
First we know (as the bases (M_I) and (S^J) are dual of each other)

$$\sigma(XA; 1) = \sum_I M_I(X) S^I(A) \; .$$

Taking into account that

$$\sum_n S_n = (1-Q_1)^{-1}(1-Q_2)^{-1}\cdots = \sum_{J \text{ non-decreasing}} Q^J$$

we see that Q^I appears in S^J, iff the *sequence* I has descent composition K which is coarser than J, i.e. $K \preceq J$, and J is coarser than I. Hence, collecting terms gives the result. $\square$

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