Symmetrized squares and cubes of the fundamental unirreps of $Sp(2n,\mathbf{R})$

Jean-Yves Thibon†, Frédéric Toumazet† and Brian G. Wybourne‡

†Institut Gaspard Monge, Université de Marne-la-Vallée, 2 rue de la Butte-Verte, 93166 Noisy-le-Grand cedex, France

‡Instytut Fizyki, Uniwersitet Mikołaja Kopernika, ul. Grudziądzka 5/7, 87-100 Toruń, Poland

Abstract. We establish the complete decompositions of the symmetrized Kronecker squares of the basic harmonic representations of $Sp(2n,\mathbf{R})$ conjectured by Grudzinski and Wybourne [J. Phys. A: Math. Gen. **29** (1996) 6631-6641] and give analogous complete results for the symmetrized cubes. Equivalence relationships are shown to exist between certain pairs of plethysms.

1. Introduction

The non-compact symplectic group $Sp(2n,\mathbf{R})$ is well-known as the dynamical group of the isotropic $n-$dimensional harmonic oscillator [12] and finds significant applications in symplectic models of nuclei [1][9] and in the mesoscopic properties of quantum dots [3][4][13]. A central problem in making applications is the resolution of Kronecker powers of the two fundamental irreducible unitary representations (unirreps) which, following King and Wybourne [6] we shall designate as $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$, into their various symmetry types. This problem has recently been studied by Grudzinski and Wybourne [2] whom, on the basis of numerical calculations, conjectured the explicit decompositions

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{2\} = \sum_{i\geq 0}\langle 1\,(4i)\rangle \tag{1}$$

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \sum_{i\geq 0}\langle 1\,(2+4i)\rangle \tag{2}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i\geq 0}\langle 1\,(2+4i)\rangle \tag{3}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{1^2\} = \langle 1\,(1^2)\rangle + \sum_{i\geq 1}\langle 1\,(4i)\rangle\ . \tag{4}$$

It was shown that the validity of such conjectures implies some unusual $S-$function identities which they duly established but without establishing a rigorous proof of the conjectured equivalences. Herein we supply a formal proof of (1) to (4). Moreover, the methods introduced lead to the establishment of a complete resolution of the Kronecker cubes of the fundamental irreps of $Sp(2n,\mathbf{R})$. These results are then used to demonstrate the existence of a symmetry relationship linking plethysms of the $\langle\frac{1}{2}(0)\rangle$ unirrep to those of $\langle\frac{1}{2}(1)\rangle$.

2. Background and notations

Our notation for characters are as in [10, 6], and for symmetric functions, as in [11]. The $Sp(2n,\mathbf{R})\downarrow U(n)$ branching rule for the harmonic series representations $\langle\frac{1}{2}k(\lambda)\rangle$ is given by the formula ([10] Eq.(3.9), [6] Eq.(5.6))

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \epsilon^{\frac{k}{2}}\{\{\lambda_s\}_N^k D_N\}_N \tag{5}$$

where $N=\min(n,k)$, and $\{\lambda_s\}_N^k$ is the signed sequence of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules of $O(k)$ (see [10]). For example, the terms $\langle 1(m)\rangle$ arising in the right-hand sides of (1)-(4) have $U(n)$ content for $n\geq 2$

$$\langle 1(m)\rangle \to \epsilon\,\{(\{m\}-\{m,2\})_2\cdot D_2\}_2 \tag{6}$$

while

$$\langle 1(1)\rangle \to \epsilon\,\{\{1\}D_2\}_2 \tag{7}$$

and

$$\langle 1(1^2)\rangle \to \epsilon\left\{\{1^2\}D_2\right\}_2 \tag{8}$$

We will also need formula (8.18) of [6], giving the tensor product of two harmonic series representations of $Sp(2n,\mathbf{R})$ as

$$\langle\tfrac{1}{2}k(\mu)\rangle\times\langle\tfrac{1}{2}l(\lambda)\rangle = \langle\tfrac{1}{2}(k+l)\left(\left(\{\mu_s\}^k\cdot\{\lambda_s\}^l\cdot D\right)\right)_{k+l,n}\rangle \tag{9}$$

where $((\lambda))_{k+l,n}$ is defined to be zero unless $\lambda'_1\leq n$ and $\lambda'_1+\lambda'_2\leq k+l$ in which case (λ) is retained. For example

$$\langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(0)\rangle = \sum_{i\geq 0}\langle 1(2i)\rangle \tag{10}$$

$$\langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle = \sum_{i\geq 0}\langle 1(2i+1)\rangle \tag{11}$$

$$\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(1)\rangle = \langle 1(1^2)\rangle + \sum_{i\geq 1}\langle 1(2i)\rangle \tag{12}$$

Finally, the branching rule $Sp(2nk,\mathbf{R}) \downarrow Sp(2n,\mathbf{R}) \times O(k)$ ([6] Eq.(4.3)) is given by

$$\tilde{\Delta} \to \sum_{\lambda} \langle \tfrac{1}{2} k(\lambda) \rangle \times [\lambda] \tag{13}$$

when λ runs over partitions satisfying $\lambda'_1 + \lambda'_2 \leq k$ and $\lambda'_1 \leq n$. The harmonic representation $\tilde{\Delta}_{2nk}$ of $Sp(2nk,\mathbf{R})$ has the following restrictions under the chain $Sp(2nk,\mathbf{R}) \supset Sp(2n,\mathbf{R}) \times O(k) \supset Sp(2n,\mathbf{R}) \times 1$

$$\tilde{\Delta}_{2nk} \to \sum_{\lambda} \langle \tfrac{1}{2} k(\lambda) \rangle \times [\lambda] \to \tilde{\Delta}_{2n}^{\times k}$$

As pointed out by the second referee, the Kronecker products can also be easily obtained by means of theorem 3.1 of [7].

3. Decomposition of symmetrized squares

In this section we give a proof of Eqs. (1)-(4).

Let us first look at the $U(n)$ content of the left-hand-side (l.h.s.) of (1) and (3). We know that under $Sp(2n,\mathbf{R}) \downarrow U(n)$ [2],

$$\langle \tfrac{1}{2}(0) \rangle \otimes \{2\} \to \left(\epsilon^{\frac{1}{2}} \sum_{m \geq 0} \{2m\} \right) \otimes \{2\} = \epsilon h_2 \left[\sum_{m \geq 0} h_{2m} \right]$$

and

$$\langle \tfrac{1}{2}(1) \rangle \otimes \{2\} \to \left(\epsilon^{\frac{1}{2}} \sum_{m \geq 0} \{2m+1\} \right) \otimes \{2\} = \epsilon h_2 \left[\sum_{m \geq 0} h_{2m+1} \right]$$

where for later convenience the r.h.s. has been rewritten in terms of symmetric functions following the notation of [8]. Hence, the sum of the left-hand sides correspond to the terms of even degree in the series

$$h_2[\sigma_1] = h_2 \left[\sum_{m \geq 0} h_{2m} + \sum_{m \geq 0} h_{2m+1} \right]$$

$$= h_2 \left[\sum_{m \geq 0} h_{2m} \right] + \left(\sum_{m \geq 0} h_{2m} \right) \cdot \left(\sum_{m \geq 0} h_{2m+1} \right) + h_2 \left[\sum_{m \geq 0} h_{2m+1} \right] ,$$

which (up to the factor ϵ), describes the $U(n)$ content of $\tilde{\Delta} \otimes \{2\}$. Therefore, to compute the sum

$$\langle \tfrac{1}{2}(0) \rangle \otimes \{2\} + \langle \tfrac{1}{2}(1) \rangle \otimes \{2\}$$

it is enough to evaluate

$$\tilde{\Delta} \otimes \{2\} = \langle \tfrac{1}{2}(0) \rangle \otimes \{2\} + \langle \tfrac{1}{2}(1) \rangle \otimes \{2\} + \langle \tfrac{1}{2}(0) \rangle \times \langle \tfrac{1}{2}(1) \rangle \ . \tag{14}$$

Now, we can resolve $\tilde{\Delta}_{2n}^2$ into its symmetric and antisymmetric part by regarding it as the restriction of $\tilde{\Delta}_{4n}$ under $Sp(4n,\mathbf{R}) \downarrow Sp(2n,\mathbf{R})$ and introducing the chain of groups

$$Sp(4n,\mathbf{R}) \supset Sp(2n,\mathbf{R}) \times O(2) \supset Sp(2n,\mathbf{R}) \times S_2 \ .$$

Indeed, under $Sp(4n,\mathbf{R}) \downarrow Sp(2n,\mathbf{R}) \times S_2$,we have

$$\tilde{\Delta}_{4n} \to \left(\tilde{\Delta}_{2n} \otimes \{2\}\right) \times (2) + \left(\tilde{\Delta}_{2n} \otimes \{1^2\}\right) \times (1^2) \ .$$

Under $Sp(4n,\mathbf{R}) \downarrow Sp(2n,\mathbf{R}) \times O(2)$, the branching rule (13) gives

$$\tilde{\Delta}_{2n} \to \langle 1(1^2)\rangle \times [1^2] + \sum_{i\geq 0} \langle 1(i)\rangle \times [i]$$

The restriction $O(k) \downarrow S_k$ is given by [5, 11]

$$[\lambda] \to \langle 1\rangle \otimes \{\lambda/G\} \tag{15}$$

where $G = \{0\} + \{1\} - \{21\} \dots$ involves only self conjugate partitions, so that

$$[1^2] \to \langle 1\rangle \otimes \left(\{1^2\} + \{1\}\right) = \langle 1\rangle$$

and for $k = 2$, $[1^2] \to (\langle 1\rangle)_{S_2} = (1^2)$ reduces to the alternating representation of S_2. For $i \geq 1$,

$$[i] \downarrow \langle 1\rangle \otimes (\{i\} + \{i-1\})$$

where for S_2, $\langle 1\rangle \otimes \{j\}$ reduces to (2) for j even and to (1^2) for j odd. Therefore, the restriction

$$\tilde{\Delta}_{2n} \downarrow Sp(2n,\mathbf{R}) \times S_2$$

is equal to

$$\langle 1(1^2)\rangle \times (1^2) + \langle 1(0)\rangle \times (2) + \sum_{i\geq 1} \langle 1(i)\rangle \times \left[(2) + (1^2)\right]$$

so that

$$\tilde{\Delta} \otimes \{2\} \ = \sum_{i\geq 0} \langle 1(i)\rangle \tag{16}$$

$$\tilde{\Delta} \otimes \{1^2\} = \langle 1(1^2)\rangle + \sum_{i\geq 1} \langle 1(i)\rangle \tag{17}$$

If we consider the restriction of (14) to $U(n)$, $\langle \frac{1}{2}(0)\rangle \otimes \{2\}$ yields symmetric functions of degree $d \equiv 0$ mod [4], $\langle \frac{1}{2}(1)\rangle \otimes \{2\}$ give terms of degree $d \equiv 2$ mod [4] and $\langle \frac{1}{2}(0)\rangle \times \langle \frac{1}{2}(1)\rangle$ the terms of odd degree. From (16) and (11) we have

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i\geq 0} \langle 1(2i)\rangle \ . \tag{18}$$

Comparing (18) and (10), we obtain

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(1)\rangle \otimes \{2\}\ , \tag{19}$$

(since $\langle \frac{1}{2}(0)\rangle^2 = \langle \frac{1}{2}(0)\rangle \otimes \{2\} + \langle \frac{1}{2}(0)\rangle \otimes \{1^2\}$).

To conclude, consider the $U(n)$ content of the difference

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{2\} - \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \langle \tfrac{1}{2}(0)\rangle \otimes p_2\ , \tag{20}$$

where p_2 is the second power sum. Under restriction to $U(n)$, the r.h.s reads in terms of symmetric functions

$$p_2\left[\epsilon^{\frac{1}{2}} \sum_{n\geq 0} h_{2n}\right] = \epsilon \sum_{n\geq 0} p_2\left[h_{2n}\right]\ .$$

This expression can be expanded in the basis of Schur functions

$$\sum_{n\geq 0} p_2\left[h_{2n}\right] = \sum_{n\geq 0} \sum_{k=0}^{2n} (-1)^k s_{4n-k,k} \tag{21}$$

Under restriction to $U(n)$, the left-hand-side of (20) corresponds to the even terms in the series obtained by taking the difference of the two expressions

$$\sum_{i\geq 0} \langle 1(4i)\rangle \downarrow \left(\epsilon\left(\sum_{i\geq 0} h_{4i}\right)\sigma_1[h_2] - \epsilon\left(\sum_{i\geq 1} s_{4i,2}\right)\sigma_1[h_2]\right)_{d\equiv 0[4]}$$

and

$$\sum_{i\geq 0} \langle 1(4i)\rangle \downarrow \left(\epsilon\left(\sum_{i\geq 0} h_{4i}\right)\sigma_1[h_2] - \epsilon\left(\sum_{i\geq 0} s_{4i+2,2}\right)\sigma_1[h_2]\right)_{d\equiv 2[4]}$$

We can extract these terms by taking the real part of

$$\sigma_1[h_2]\left[\sum_{n\geq 0} (\mathrm{i}t)^n h_n - \sum_{n\geq 2} (\mathrm{i}t)^n s_{n,2}\right]$$

which is equal for $t=1$ to

$$\sigma_1[h_2]\left[\sigma_{\mathrm{i}} - \sigma_{\mathrm{i}}\left(s_2 - \frac{1}{\mathrm{i}} s_1\right) + \frac{1}{\mathrm{i}} s_1 + s_{1^2}\right] \tag{22}$$

We want to show that in two variables $\{x,y\}$, the symmetric functions coincide with the r.h.s of (21).
Indeed,

$$p_2\left[\sum_{n\geq 0} h_{2n}\right] = \frac{p_2}{2}\left[\sigma_1 + \sigma_{-1}\right]$$

$$= \frac{p_2}{2}\left[\frac{1}{(1-x)(1-y)} + \frac{1}{(1+x)(1+y)}\right]$$
$$= \frac{1}{2}\left[\frac{1}{(1-x^2)(1-y^2)} + \frac{1}{(1+x^2)(1+y^2)}\right]$$

and using the specialization of

$$\sigma_1\left[h_2(x,y)\right] = \sigma_1\left[x^2+xy+y^2\right] = \frac{1}{(1-x^2)(1-xy)(1-y^2)} \ ,$$
$$\sigma_{\mathrm{i}}(x,y) = \frac{1}{(1-\mathrm{i}x)(1-\mathrm{i}y)}$$

and so on, we see that the specialization S of (22) is equal to

$$S = \frac{1}{2}\left[\frac{1}{(1-x^2)(1-y^2)} + \frac{1}{(1+x^2)(1+y^2)}\right] + \mathrm{i}\frac{y+x+x^3y^2+x^2y^3}{(1+x^2)(1+y^2)(1-x^2)(1-xy)(1-y^2)}$$

and we have now the result that

$$\langle\tfrac{1}{2}(0)\rangle\otimes\{2\} - \langle\tfrac{1}{2}(0)\rangle\otimes\{1^2\} = \sum_{i\geq 0}\langle 1(4i)\rangle - \sum_{i\geq 0}\langle 1(4i+2)\rangle \ . \tag{23}$$

From (23) and (10) we obtain (1) and (2), and taking into account (19) and (12), we obtain (3) and (4).

4. Symmetrized cubes

We shall now establish similar decompositions for the symmetrized cubes of the fundamental representations $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$. As above, we obtain $\tilde\Delta_{2n}^{\times 3}$ by restriction of $\tilde\Delta_{6n}$ through the chain

$$Sp(6n,\mathbf{R}) \supset Sp(2n,\mathbf{R})\times O(3) \supset Sp(2n,\mathbf{R})\times S_3 \supset Sp(2n,\mathbf{R}) \ .$$

To obtain the plethysm $\tilde\Delta_{2n}\otimes\{\mu\}$, we need to determine the restriction from $O(3)$ to S_3. We first remark that in the restriction $O(3)\downarrow S_3$ only the partitions $[1^3]$, $[m]$ and $[m,1]$ are retained. Since $[m,1]=[m]^*$ and $[m]^*\downarrow S_3 = \mathrm{sgn}\times[m]\downarrow S_3$ (see [10]), it is enough to determine $[m]\downarrow S_3$ (clearly, $[1^3]\to\mathrm{sgn}$). Applying (15), we find

$$[m]\downarrow\langle 1\rangle\otimes\{m/G\} = \langle 1\rangle\otimes\{m\} + \langle 1\rangle\otimes\{m-1\} \ .$$

The value of $\langle 1\rangle\otimes\{m\}$ for S_k is equal to the coefficient of z^m in the series

$$(1-z)h_n\left(\frac{X}{1-z}\right) = \prod_{j=2}^{n}\frac{1}{1-z^j}\cdot\sum_{\lambda\vdash n}\tilde K_{\lambda,1^n}(z)s_\lambda(X)$$

(cf. [11]) where $\tilde{K}_{\lambda,1^k}(z)$ are the Kostka-Foulkes polynomials. In particular, for $n=3$, $[m]\downarrow S_3$ is the sum of the coefficients of z^m and z^{m-1} in

$$F(z)=\frac{1}{(1-z^2)(1-z^3)}\left[z^3(1^3)+(z^2+z)(21)+(3)\right] \ . \tag{24}$$

Writing

$$\frac{1}{(1-z^2)(1-z^3)}=\sum_{n\geq 0}a_n z^n \tag{25}$$

we obtain

$$F(z)=\left(\sum_{n\geq 0}a_n z^n\right)(3)+\left(\sum_{n\geq 1}a_{n-2}z^n+a_{n-1}z^n\right)(21)+\left(\sum_{n\geq 3}a_{n-3}z^n\right)(1^3)\ ,$$

and after simplification, one finds that

$$[m]\downarrow S_3=(a_m+a_{m-1})(3)+(a_{m-1}+2a_{m-2}+a_{m-3})(21)+(a_{m-3}+a_{m-4})(1^3)\ ,$$

$$[m,1]\downarrow S_3=(a_{m-3}+a_{m-4})(3)+(a_{m-1}+2a_{m-2}+a_{m-3})(21)+(a_m+a_{m-1})(1^3)\ ,$$

and

$$[1^3]\downarrow S_3=\mathrm{sgn}=(1^3)$$

Hence,

$$\tilde{\Delta}_{3n}\downarrow Sp(2n)\times S_3=\sum_{m\geq 0}\langle\tfrac{3}{2}(m)\rangle E_1(m)+\sum_{m\geq 1}\langle\tfrac{3}{2}(m,1)\rangle E_2(m)+\langle\tfrac{3}{2}(1^3)\rangle(1^3)$$

where

$$E_1(m)=(a_m+a_{m-1})(3)+(a_{m-1}+2a_{m-2}+a_{m-3})(21)+(a_{m-3}+a_{m-4})(1^3)\ ,$$

and

$$E_2(m)=(a_{m-3}+a_{m-4})(3)+(a_{m-1}+2a_{m-2}+a_{m-3})(21)+(a_m+a_{m-1})(1^3)$$

On the other hand, since $\tilde{\Delta}=\langle\frac{1}{2}(0)\rangle+\langle\frac{1}{2}(1)\rangle$,

$$\begin{aligned}\tilde{\Delta}\otimes\{3\} &= \langle\tfrac{1}{2}(0)\rangle\otimes\{3\}+\langle\tfrac{1}{2}(1)\rangle\otimes\{3\}\\ &+ \langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle\otimes\{2\}+\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(0)\rangle\otimes\{2\}\end{aligned}\tag{26}$$

$$\begin{aligned}\tilde{\Delta}\otimes\{21\} &= \langle\tfrac{1}{2}(1)\rangle\otimes\{21\}+\langle\tfrac{1}{2}(0)\rangle\otimes\{21\}\\ &+ \langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle\otimes\{2\}+\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(0)\rangle\otimes\{2\}\\ &+ \langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle\otimes\{1^2\}+\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(0)\rangle\otimes\{1^2\}\end{aligned}\tag{27}$$

and

$$\begin{array}{rcl}\tilde{\Delta}\otimes\{1^3\} & = & \langle\frac{1}{2}(1)\rangle\otimes\{1^3\}+\langle\frac{1}{2}(0)\rangle\otimes\{1^3\} \\ & + & \langle\frac{1}{2}(0)\rangle\times\langle\frac{1}{2}(1)\rangle\otimes\{1^2\}+\langle\frac{1}{2}(1)\rangle\times\langle\frac{1}{2}(0)\rangle\otimes\{1^2\}\end{array} \tag{28}$$

To extract the plethysms $\langle\frac{1}{2}(i)\rangle\otimes\{\lambda\}$ from these expressions, we rewrite the left-hand-sides as (26),(27) and (28), and we manage to subtract the contribution of the crossed products $\langle\frac{1}{2}(i)\rangle\times\langle\frac{1}{2}(i-1)\rangle\otimes\{\mu\}$, $(|\mu|=2)$.
From Eqs. (1)-(4), we obtain

$$\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(0)\rangle\otimes\{2\}=\langle\tfrac{1}{2}(1)\rangle\times\sum_{i\geq 0}\langle 1(4i)\rangle\ , \tag{29}$$

$$\langle\tfrac{1}{2}(1)\rangle\times\langle\tfrac{1}{2}(0)\rangle\otimes\{1^2\}=\langle\tfrac{1}{2}(1)\rangle\times\sum_{i\geq 0}\langle 1(4i+2)\rangle\ , \tag{30}$$

$$\langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle\otimes\{2\}=\langle\tfrac{1}{2}(0)\rangle\times\sum_{i\geq 0}\langle 1(4i+2)\rangle\ , \tag{31}$$

and

$$\langle\tfrac{1}{2}(0)\rangle\times\langle\tfrac{1}{2}(1)\rangle\otimes\{1^2\}=\langle\tfrac{1}{2}(0)\rangle\times\left(\langle 1(1^2)\rangle+\sum_{i\geq 1}\langle 1(4i)\rangle\right)\ . \tag{32}$$

From (9) we find

$$\langle\tfrac{1}{2}(1)\rangle\times\langle 1(0)\rangle=\sum_{i\geq 0}\langle\tfrac{3}{2}(2i+1)\rangle \tag{33}$$

and for $m\neq 0$

$$\langle\tfrac{1}{2}(1)\rangle\times\langle 1(m)\rangle=\sum_{i\geq 0}\langle\tfrac{3}{2}(m+2i,1)\rangle+\langle\tfrac{3}{2}(m+2i+1)\rangle \tag{34}$$

Let $p_2(m)=\lceil\frac{m+1}{2}\rceil$, the number of partitions of m into parts not greater that 2, given by the generating function

$$\sum_{m\geq 0}p_2(m)z^m=\frac{1}{(1-z)(1-z^2)}$$

Expanding the r.h.s of (29) by means of (33) and (34), we get

$$\begin{array}{rcl}\langle\frac{1}{2}(1)\rangle\times\langle\frac{1}{2}(0)\rangle\otimes\{2\} & = & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1)\rangle \\ & + & \sum_{\substack{i\geq 1\\ j\geq 0}}\langle\frac{3}{2}(4i+2j,1)\rangle+\langle\frac{3}{2}(4i+2j+1)\rangle \\ & = & \sum_{i\geq 0}\langle\frac{3}{2}(2i+1)\rangle \\ & + & \sum_{m\geq 1}p_2(m-2)\left(\langle\frac{3}{2}(2m,1)\rangle+\langle\frac{3}{2}(2m+1)\rangle\right)\end{array}$$

where $4i+2j=2m$ and $p_2(m-2)$ comes from the rearrangement

$$\sum_{\substack{i\geq 1,\ j\geq 0\\ 2i+j=m}}1=\sum_{\substack{k\geq 0,\ j\geq 0\\ 2k+j=m-2}}1=p_2(m-2)$$

For equation (31) we have a similar expression. For $m \neq 0$

$$\langle \tfrac{1}{2}(0)\rangle \times \langle 1(m)\rangle = \sum_{j\geq 0} \langle \tfrac{3}{2}(m+2j+1,1)\rangle + \langle \tfrac{3}{2}(m+2j)\rangle$$

then

$$\begin{array}{rcl} \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} & = & \sum_{i\geq 0}\sum_{j\geq 1} \langle \tfrac{3}{2}(4i+2j+1,1)\rangle + \langle \tfrac{3}{2}(4i+2j)\rangle \\ & = & \sum_{m\geq 1} p_2(m-1)\left(\langle \tfrac{3}{2}(2m+1,1)\rangle + \langle \tfrac{3}{2}(2m)\rangle\right) \end{array}$$

Now it is possible to give a closed expression for

$$\begin{array}{l} \langle \tfrac{1}{2}(0)\rangle \otimes \{3\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{3\} \\ = \tilde{\Delta} \otimes \{3\} - \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} - \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{2\} \end{array}$$

that is,

$$\begin{array}{rcl} \langle \tfrac{1}{2}(0)\rangle \otimes \{3\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{3\} & = & \sum_{m\geq 0}(a_m + a_{m-1})\langle \tfrac{3}{2}(m)\rangle \\ & + & \sum_{m\geq 3}(a_{m-3} + a_{m-4})\langle \tfrac{3}{2}(m,1)\rangle \\ & - & \sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1)\rangle \\ & - & \sum_{k\geq 1} p_2(k-2)\left(\langle \tfrac{3}{2}(2k,1)\rangle + \langle \tfrac{3}{2}(2k+1)\rangle\right) \\ & - & \sum_{k\geq 1} p_2(k-1)\left(\langle \tfrac{3}{2}(2k+1,1)\rangle + \langle \tfrac{3}{2}(2k)\rangle\right) \end{array} \qquad (35)$$

We can now give the coefficients of $\langle \tfrac{3}{2}(m)\rangle$ and $\langle \tfrac{3}{2}(m,1)\rangle$ in (35). The coefficient of $\langle \tfrac{3}{2}(m)\rangle$ is equal to

$$\begin{cases} a_m + a_{m-1} - p_2(2k-1) = a_m + a_{m-1} - a_{2k-1} - a_{2k-2} - a_{2k-3} & \text{for } m = 4k, \\ a_m + a_{m-1} - p_2(2k) = a_m + a_{m-1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3} & \text{for } m = 4k+2, \\ a_m + a_{m-1} - 1 - p_2(k-2)) = a_m + a_{m-1} - 1 - a_{k-2} - a_{k-3} - a_{k-4} & \text{for } m = 2k+1, \end{cases}$$

and the coefficient of $\langle \tfrac{3}{2}(m,1)\rangle$ is equal to

$$\begin{cases} a_{m-3} + a_{m-4} - p_2(k-2) = a_{m-3} + a_{m-4} - a_{k-2} - a_{k-3} - a_{k-4} & \text{for } m = 2k, \\ a_{m-3} + a_{m-4} - p_2(k-1)) = a_{m-3} + a_{m-4} - a_{k-1} - a_{k-2} - a_{k-3} & \text{for } m = 2k+1, \end{cases}$$

To separate the components of $\langle \tfrac{1}{2}(0)\rangle \otimes \{3\}$ and $\langle \tfrac{1}{2}(1)\rangle \otimes \{3\}$, we consider their $U(n)$ contents. It is enough to see that the terms of $\langle \tfrac{1}{2}(0)\rangle \otimes \{3\}$ are those $\langle \tfrac{3}{2}(\lambda)\rangle$ with $|\lambda|$ even and the terms of $\langle \tfrac{1}{2}(1)\rangle \otimes \{3\}$ are those with $|\lambda|$ odd.

$$\begin{array}{rcl} \langle \tfrac{1}{2}(0)\rangle \otimes \{3\} & = & \sum_{k\geq 0}(a_{4k} + a_{4k-1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3})\langle \tfrac{3}{2}(4k)\rangle \\ & + & \sum_{k\geq 0}(a_{4k+2} + a_{4k+1} - 1 - a_{2k-1} - a_{2k-2} - a_{2k-3})\langle \tfrac{3}{2}(4k+2)\rangle \\ & + & \sum_{k\geq 0}(a_{2k-2} + a_{2k-3} - a_{k-1} - a_{k-2} - a_{k-3})\langle \frac{3}{2}(2k+1,1)\rangle \end{array} \qquad (36)$$

and

$$\begin{array}{rcl} \langle \tfrac{1}{2}(1)\rangle \otimes \{3\} & = & \sum_{k\geq 0}(a_{2k+1} + a_{2k} - 1 - a_{k-2} - a_{k-3} - a_{k-4})\langle \tfrac{3}{2}(2k+1)\rangle \\ & + & \sum_{k\geq 0}(a_{2k-3} + a_{2k-4} - a_{k-2} - a_{k-3} - a_{k-4})\langle \frac{3}{2}(2k,1)\rangle \end{array} \qquad (37)$$

The derivation of the other plethysms is similar. The right-hand-side of (32) evaluates to

$$\begin{aligned}
\langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} &= \textstyle\sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1,1)\rangle + \displaystyle\sum_{i\geq 1}\langle \tfrac{3}{2}(4i)\rangle \\
&+ \textstyle\sum_{i\geq 1}\sum_{j\geq 1}\langle \tfrac{3}{2}(4i+2j-1,1)\rangle + \langle \tfrac{3}{2}(4i+2j)\rangle \\
&= \textstyle\sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1,1)\rangle + \displaystyle\sum_{i\geq 1}\langle \tfrac{3}{2}(4i)\rangle \\
&+ \textstyle\sum_{m\geq 3} p_2(m-3)\left(\langle \tfrac{3}{2}(2m-1,1)\rangle + \langle \tfrac{3}{2}(2m)\rangle\right)
\end{aligned}$$

and for (30) we obtain

$$\begin{aligned}
\langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} &= \textstyle\sum_{i\geq 0}\sum_{j\geq 1}\langle \tfrac{3}{2}(4i+2j+1)\rangle + \langle \tfrac{3}{2}(4i+2j,1)\rangle \\
&= \textstyle\sum_{m\geq 1} p_2(m-1)\left(\langle \tfrac{3}{2}(2m,1)\rangle + \langle \tfrac{3}{2}(2m+1)\rangle\right)
\end{aligned}$$

Subtracting the contributions of the crossed products in (28) we obtain

$$\begin{aligned}
&\langle \tfrac{1}{2}(1)\rangle \otimes \{1^3\} + \langle \tfrac{1}{2}(0)\rangle \otimes \{1^3\} \\
&= \tilde{\Delta} \otimes \{1^3\} - \langle \tfrac{1}{2}(0)\rangle \times \langle \tfrac{1}{2}(1)\rangle \otimes \{1^2\} - \langle \tfrac{1}{2}(1)\rangle \times \langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\}
\end{aligned}$$

and this gives

$$\begin{aligned}
&\langle \tfrac{1}{2}(0)\rangle \otimes \{1^3\} + \langle \tfrac{1}{2}(1)\rangle \otimes \{1^3\} \\
&= \textstyle\sum_{m\geq 0}(a_{m-3}+a_{m-4})\langle \tfrac{3}{2}(m)\rangle + \displaystyle\sum_{m\geq 4}(a_m + a_{m-1})\langle \tfrac{3}{2}(m,1)\rangle \\
&+\langle \tfrac{3}{2}(1^3)\rangle \\
&- \textstyle\sum_{i\geq 0}\langle \tfrac{3}{2}(2i+1,1)\rangle - \displaystyle\sum_{i\geq 1}\langle \tfrac{3}{2}(4i)\rangle \\
&- \textstyle\sum_{m\geq 3} p_2(m-3)\left(\langle \tfrac{3}{2}(2m-1,1)\rangle + \langle \tfrac{3}{2}(2m)\rangle\right) \\
&- \textstyle\sum_{m\geq 1} p_2(m-1)\left(\langle \tfrac{3}{2}(2m,1)\rangle + \langle \tfrac{3}{2}(2m+1)\rangle\right)
\end{aligned}$$

We can now write down the coefficients of $\langle \tfrac{3}{2}(m)\rangle$ and $\langle \tfrac{3}{2}(m,1)\rangle$. We give first the coefficient of $\langle \tfrac{3}{2}(m)\rangle$

$$\begin{cases}
a_{m-3}+a_{m-4}-1-p_2(2k-3) & \\
= a_{m-3}+a_{m-4}-1-a_{2k-3}-a_{2k-4}-a_{2k-5} & \text{for } m=4k, \\
a_{m-3}+a_{m-4}-p_2(2k-2) & \\
= a_{m-3}+a_{m-4}-a_{2k-2}-a_{2k-3}-a_{2k-4} & \text{for } m=4k+2, \\
a_{m-3}+a_{m-4}-p_2(k-1) & \\
= a_{m-3}+a_{m-4}-a_{k-1}-a_{k-2}-a_{k-3} & \text{for } m=2k+1,
\end{cases}$$

and then the coefficient of $\langle \tfrac{3}{2}(m,1)\rangle$

$$\begin{cases}
a_m+a_{m-1}-p_2(k-1) & \\
= a_m+a_{m-1}-a_{k-1}-a_{k-2}-a_{k-3} & \text{for } m=2k, \\
a_m+a_{m-1}-1-p_2(k-2) & \\
= a_m+a_{m-1}-1-a_{k-2}-a_{k-3}-a_{k-4} & \text{for } m=2k+1.
\end{cases}$$

Separating the terms as above, we obtain

$$\begin{array}{rcl}\langle\frac{1}{2}(0)\rangle\otimes\{1^3\} &=& \sum_{k\geq 0}(a_{4k-3}+a_{4k-4}-1-a_{2k-3}-a_{2k-4}-a_{2k-5})\langle\frac{3}{2}(4k)\rangle\\ &+& \sum_{k\geq 0}(a_{4k-1}+a_{4k-2}-a_{2k-2}-a_{2k-3}-a_{2k-4})\langle\frac{3}{2}(4k+2)\rangle\\ &+& \sum_{k\geq 0}(a_{2k+1}+a_{2k}-1-a_{k-2}-a_{k-3}-a_{k-4})\langle\frac{3}{2}(2k+1,1)\rangle\end{array}\qquad(38)$$

and

$$\begin{array}{rcl}\langle\frac{1}{2}(1)\rangle\otimes\{1^3\} &=& \sum_{k\geq 0}(a_{2k-2}+a_{2k-3}-a_{k-1}-a_{k-2}-a_{k-3})\langle\frac{3}{2}(2k+1)\rangle\\ &+& \sum_{k\geq 1}(a_{2k}+a_{2k-1}-a_{k-1}-a_{k-2}-a_{k-3})\langle\frac{3}{2}(2k,1)\rangle\\ &+& \langle\frac{3}{2}(1^3)\rangle\end{array}\qquad(39)$$

Finally

$$\begin{array}{l}\langle\frac{1}{2}(1)\rangle\otimes\{21\}+\langle\frac{1}{2}(0)\rangle\otimes\{21\}\\ =\tilde{\Delta}\otimes\{21\}\\ -\langle\frac{1}{2}(0)\rangle\times\langle\frac{1}{2}(1)\rangle\otimes\{2\}-\langle\frac{1}{2}(1)\rangle\times\langle\frac{1}{2}(0)\rangle\otimes\{2\}\\ -\langle\frac{1}{2}(0)\rangle\times\langle\frac{1}{2}(1)\rangle\otimes\{1^2\}-\langle\frac{1}{2}(1)\rangle\times\langle\frac{1}{2}(0)\rangle\otimes\{1^2\}\end{array}$$

which yields

$$\begin{array}{rcl}\langle\frac{1}{2}(0)\rangle\otimes\{21\}+\langle\frac{1}{2}(1)\rangle\otimes\{21\} &=& \sum_{m\geq 0}\langle\frac{3}{2}(m)\rangle(a_{m-1}+2a_{m-2}+a_{m-3})\\ &+& \sum_{m\geq 1}\langle\frac{3}{2}(m,1)\rangle(a_{m-1}+2a_{m-2}+a_{m-3})\\ &-& \sum_{i\geq 0}\langle\frac{3}{2}(2i+1)\rangle\\ &-& \sum_{k\geq 1}p_2(k-2)\left(\langle\frac{3}{2}(2k,1)\rangle+\langle\frac{3}{2}(2k+1)\rangle\right)\\ &-& \sum_{k\geq 1}p_2(k-1)\left(\langle\frac{3}{2}(4i)\rangle\right)\\ &-& \sum_{m\geq 3}p_2(m-3)\left(\langle\frac{3}{2}(2m-1,1)\rangle+\langle\frac{3}{2}(2m)\rangle\right)\\ &-& \sum_{m\geq 1}p_2(m-1)\left(\langle\frac{3}{2}(2m,1)\rangle+\langle\frac{3}{2}(2m+1)\rangle\right)\end{array}$$

and hence the coefficient of $\langle\frac{3}{2}(m)\rangle$ is
for $m=4k$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-1-p_2(2k-1)-p_2(2k-3)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-1-a_{2k-1}-a_{2k-2}-2a_{2k-3}-a_{2k-4}-a_{2k-5}\end{array}\qquad(40)$$

for $m=4k+2$,

$$\begin{array}{l}a_{m-1}+2a_{m-2}+a_{m-3}-p_2(2k)-p_2(2k-2)\\ =a_{m-1}+2a_{m-2}+a_{m-3}-a_{2k-1}-2a_{2k-2}-2a_{2k-3}-a_{2k-4}\end{array}\qquad(41)$$

for $m = 2k+1$,

$$a_{m-1} + 2a_{m-2} + a_{m-3} - p_2(k-1) - p_2(k-2) \\ = a_{m-1} + 2a_{m-2} + a_{m-3} - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4} \tag{42}$$

and for the coefficient of $\langle \frac{3}{2}(m,1)\rangle$, for $m = 2k$,

$$a_{m-1} + 2a_{m-2} + a_{m-3} - p_2(k-1) - p_2(k-2) \\ = a_{m-1} + 2a_{m-2} + a_{m-3} - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4} \tag{43}$$

for $m = 2k+1$,

$$a_{m-1} + 2a_{m-2} + a_{m-3} - 1 - p_2(k-1) - p_2(k-2) \\ = a_{m-1} + 2a_{m-2} + a_{m-3} - 1 - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4} \tag{44}$$

To separate the terms of $\langle \frac{1}{2}(0)\rangle \otimes \{21\}$ and $\langle \frac{1}{2}(1)\rangle \otimes \{21\}$ we again make use of the $U(n)$ content of $\langle \frac{1}{2}(0)\rangle \otimes \{21\} + \langle \frac{1}{2}(1)\rangle \otimes \{21\}$, which yields

$$\begin{aligned} &\langle \tfrac{1}{2}(0)\rangle \otimes \{21\} \\ &= \textstyle\sum_{k\geq 0}(a_{4k-1} + 2a_{4k-2} + a_{4k-3} - 1 - a_{2k-1} - a_{2k-2} - 2a_{2k-3} - a_{2k-4} - a_{2k-5})\langle \tfrac{3}{2}(4k)\rangle \\ &+ \textstyle\sum_{k\geq 0}(a_{4k+1} + 2a_{4k} + a_{4k-1} - a_{2k-1} - 2a_{2k-2} - 2a_{2k-3} - a_{2k-4})\langle \tfrac{3}{2}(4k+2)\rangle \\ &+ \textstyle\sum_{k\geq 0}(a_{2k} + 2a_{2k-1} + a_{2k-2} - 1 - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4})\langle \tfrac{3}{2}(2k+1,1)\rangle \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\langle \tfrac{1}{2}(1)\rangle \otimes \{21\} \\ &= \textstyle\sum_{k\geq 0}(a_{2k} + 2a_{2k-1} + a_{2k-2} - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4})\langle \tfrac{3}{2}(2k+1)\rangle \\ &+ \textstyle\sum_{k\geq 0}(a_{2k-1} + 2a_{2k-2} + a_{2k-3} - a_{k-1} - 2a_{k-2} - 2a_{k-3} - a_{k-4})\langle \tfrac{3}{2}(2k,1)\rangle \end{aligned} \tag{46}$$

5. Star equivalent $Sp(2n,\mathbf{R})$ unirreps and plethysm

Recall the modification rule for $O(k)$

$$[\lambda] = (-1)^{c-1}[\lambda - h]^*, \qquad h = 2\lambda_1{}' - k \geq 0 \tag{47}$$

where a continuous strip of boxes of length h, starting at the foot of the first column and working up along the right edge and ending in column c is removed from the Ferrers graph of (λ). The constraints on partitions (λ) corresponding to unirreps $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n,\mathbf{R})$ allows us to restrict our attention to those cases corresponding to removing boxes from just the first column. Thus for the unirreps $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n,\mathbf{R})$, we have

$$(\lambda) \to (\lambda - h)^*, \qquad h = 2\lambda_1{}' - k \geq 0 \tag{48}$$

and conversely we may add boxes to the first column of a standard partition (λ) for $O(k)$ to produce a $(\lambda + h)$ that is non-standard for $O(k)$ but standard for $Sp(2n,\mathbf{R})$

$$(\lambda + h) \to (\lambda)^*, \qquad h = k - 2\lambda_1{}' \geq 0 \tag{49}$$

Unirreps of $Sp(2n,\mathbf{R})$ will be said to be *star equivalent* if

$$\langle \tfrac{1}{2}k(\lambda)\rangle^* = \langle \tfrac{1}{2}k(\lambda)^*\rangle = \langle \tfrac{1}{2}k(\lambda \pm h)\rangle \tag{50}$$

which for brevity, we shall write as

$$\langle \tfrac{1}{2}k(\lambda)\rangle \stackrel{*}{\equiv} \langle \tfrac{1}{2}k(\lambda \pm h)\rangle \tag{51}$$

where it is assumed that $k \le n$. Thus $\langle \frac{1}{2}(0)\rangle \stackrel{*}{\equiv} \langle \frac{1}{2}(1)\rangle$ is a pair of star equivalent unirreps of $Sp(2n,\mathbf{R})$.

Bearing in mind the preceding remarks and definitions, consider the leading terms in the plethysm $\langle \frac{1}{2}(0)\rangle \otimes \{3\}$

$$\begin{array}{lllll}
\langle \frac{3}{2}(0)\rangle & + \langle \frac{3}{2}(4)\rangle & + \langle \frac{3}{2}(6)\rangle & + \langle \frac{3}{2}(8)\rangle & + \langle \frac{3}{2}(91)\rangle \\
+ \langle \frac{3}{2}(10\ )\rangle & + 2\langle \frac{3}{2}(12\ )\rangle & + \langle \frac{3}{2}(13\ 1)\rangle & + \langle \frac{3}{2}(14\ )\rangle & + \langle \frac{3}{2}(15\ 1)\rangle \\
+ 2\langle \frac{3}{2}(16\ )\rangle & + \langle \frac{3}{2}(17\ 1)\rangle & + 2\langle \frac{3}{2}(18\ )\rangle & + \langle \frac{3}{2}(19\ 1)\rangle & + 2\langle \frac{3}{2}(20\ )\rangle \\
+ 2\langle \frac{3}{2}(21\ 1)\rangle & + 2\langle \frac{3}{2}(22\ )\rangle & + \langle \frac{3}{2}(23\ 1)\rangle & &
\end{array}$$

Replacing each unirrep by its star equivalent gives the leading terms of $(\langle \frac{1}{2}(0)\rangle \otimes \{3\})^*$ as

$$\begin{array}{lllll}
\langle \frac{3}{2}(1^3)\rangle & + \langle \frac{3}{2}(41)\rangle & + \langle \frac{3}{2}(61)\rangle & + \langle \frac{3}{2}(81)\rangle & + \langle \frac{3}{2}(9)\rangle \\
+ \langle \frac{3}{2}(10\ 1)\rangle & + 2\langle \frac{3}{2}(12\ 1)\rangle & + \langle \frac{3}{2}(13\ )\rangle & + \langle \frac{3}{2}(14\ 1)\rangle & + \langle \frac{3}{2}(15\ )\rangle \\
+ 2\langle \frac{3}{2}(16\ 1)\rangle & + \langle \frac{3}{2}(17\ )\rangle & + 2\langle \frac{3}{2}(18\ 1)\rangle & + \langle \frac{3}{2}(19\ )\rangle & + 2\langle \frac{3}{2}(20\ 1)\rangle \\
+ 2\langle \frac{3}{2}(21\ )\rangle & + 2\langle \frac{3}{2}(22\ 1)\rangle & + \langle \frac{3}{2}(23\ )\rangle & &
\end{array}$$

which are precisely the leading terms in $\langle \frac{1}{2}(0)\rangle \otimes \{1^3\}$

$$\begin{array}{lllll}
\langle \frac{3}{2}(1^3)\rangle & + \langle \frac{3}{2}(41)\rangle & + \langle \frac{3}{2}(61)\rangle & + \langle \frac{3}{2}(81)\rangle & + \langle \frac{3}{2}(9)\rangle \\
+ \langle \frac{3}{2}(10\ 1)\rangle & + 2\langle \frac{3}{2}(12\ 1)\rangle & + \langle \frac{3}{2}(13\ )\rangle & + \langle \frac{3}{2}(14\ 1)\rangle & + \langle \frac{3}{2}(15\ )\rangle \\
+ 2\langle \frac{3}{2}(16\ 1)\rangle & + \langle \frac{3}{2}(17\ )\rangle & + 2\langle \frac{3}{2}(18\ 1)\rangle & + \langle \frac{3}{2}(19\ )\rangle & + 2\langle \frac{3}{2}(20\ 1)\rangle \\
+ 2\langle \frac{3}{2}(21\ )\rangle & + 2\langle \frac{3}{2}(22\ 1)\rangle & + \langle \frac{3}{2}(23\ )\rangle & &
\end{array}$$

Similarly, the leading terms in $(\langle \frac{1}{2}(0)\rangle \otimes \{21\})^*$ are

$$\begin{array}{lllll}
\langle \frac{3}{2}(2)\rangle & + \langle \frac{3}{2}(4)\rangle & + \langle \frac{3}{2}(51)\rangle & + \langle \frac{3}{2}(6)\rangle & + \langle \frac{3}{2}(71)\rangle \\
+ 2\langle \frac{3}{2}(8)\rangle & + \langle \frac{3}{2}(91)\rangle & + 2\langle \frac{3}{2}(10\ )\rangle & + 2\langle \frac{3}{2}(11\ 1)\rangle & + 2\langle \frac{3}{2}(12\ )\rangle \\
+ 2\langle \frac{3}{2}(13\ 1)\rangle & + 3\langle \frac{3}{2}(14\ )\rangle & + 2\langle \frac{3}{2}(15\ 1)\rangle & + 3\langle \frac{3}{2}(16\ )\rangle & + 3\langle \frac{3}{2}(17\ 1)\rangle \\
+ 3\langle \frac{3}{2}(18\ )\rangle & + 3\langle \frac{3}{2}(19\ 1)\rangle & + 4\langle \frac{3}{2}(20\ )\rangle & + 3\langle \frac{3}{2}(21\ 1)\rangle & + 4\langle \frac{3}{2}(22\ )\rangle \\
+ 4\langle \frac{3}{2}(23\ 1)\rangle & & & &
\end{array}$$

which are identical to the leading terms in $\langle \frac{1}{2}(0)\rangle \otimes \{21\}$

$$\begin{array}{lllll}
\langle \frac{3}{2}(2)\rangle & + \langle \frac{3}{2}(4)\rangle & + \langle \frac{3}{2}(51)\rangle & + \langle \frac{3}{2}(6)\rangle & + \langle \frac{3}{2}(71)\rangle \\
+ 2\langle \frac{3}{2}(8)\rangle & + \langle \frac{3}{2}(91)\rangle & + 2\langle \frac{3}{2}(10\ )\rangle & + 2\langle \frac{3}{2}(11\ 1)\rangle & + 2\langle \frac{3}{2}(12\ )\rangle \\
+ 2\langle \frac{3}{2}(13\ 1)\rangle & + 3\langle \frac{3}{2}(14\ )\rangle & + 2\langle \frac{3}{2}(15\ 1)\rangle & + 3\langle \frac{3}{2}(16\ )\rangle & + 3\langle \frac{3}{2}(17\ 1)\rangle \\
+ 3\langle \frac{3}{2}(18\ )\rangle & + 3\langle \frac{3}{2}(19\ 1)\rangle & + 4\langle \frac{3}{2}(20\ )\rangle & + 3\langle \frac{3}{2}(21\ 1)\rangle & + 4\langle \frac{3}{2}(22\ )\rangle \\
+ 4\langle \frac{3}{2}(23\ 1)\rangle & & & &
\end{array}$$

The above results lead us to conjecture that

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{\lambda\} \overset{*}{\equiv} \langle \tfrac{1}{2}(1)\rangle \otimes \{\lambda'\} \tag{52}$$

That the conjecture holds for $\lambda \vdash 2$ is evident from (1) to (4).That the conjecture holds for $\lambda \vdash 3$ follows by noting that

$$\langle \tfrac{3}{2}(m)\rangle \overset{*}{\equiv} \langle \tfrac{3}{2}(m,1)\rangle$$

and

$$\langle \tfrac{3}{2}(0)\rangle \overset{*}{\equiv} \langle \tfrac{3}{2}(1^3)\rangle$$

Application of these star equivalences to the results obtained for the symmetrization of the cubes of the star equivalent pair $\langle \frac{1}{2}(0)\rangle \overset{*}{\equiv} \langle \frac{1}{2}(1)\rangle$ together with inspection of the given coefficients, noting in particular (40) to (44), completes the verification of (52) for all $(\lambda) \vdash 2, 3$. Proof that the conjecture (52) holds for all (λ) will be given in a later paper (with R.C. King) together with a proof of a more general star equivalence

$$\begin{aligned} \langle \tfrac{1}{2}k(\mu)\rangle &\overset{*}{\equiv} \langle \tfrac{1}{2}k(\mu \pm h)\rangle \otimes \{\lambda'\} \quad (\ k \ \ \text{odd}) \\ &\overset{*}{\equiv} \langle \tfrac{1}{2}k(\mu \pm h)\rangle \otimes \{\lambda\} \quad (\ k \ \ \text{even}) \end{aligned} \tag{53}$$

6. Concluding remarks

It is now possible to establish results for higher powers by applying the same methods to obtain the plethysms. But because longer partitions are involved the calculations rapidly become cumbersome. For example, the branching rule $O(4) \downarrow S_4$ is obtained by examination of partitions of the form $[1^4]$, $[m,1,1]$, $[m,2]$, $[m,1]$ and $[m]$. One remarks that $[m,1,1] = [m,1]^*$ and $[1^4] = [0]^*$ and one needs to examine only the partitions of the form $[m]$, $[m,1]$, $[m,2]$. One applies the formula to give $[m] \downarrow S_4$, $[m,1] \downarrow S_4$, $[m,2] \downarrow S_4$ which is equivalent to

$$[m] \downarrow \langle 1\rangle \otimes \{m/G\}$$

$$[m,1] \downarrow \langle 1\rangle \otimes \{m1/G\}$$

and

$$[m,2] \downarrow \langle 1\rangle \otimes \{m2/G\}$$

Now, to obtain the result, one writes that $\{m,j\} = \{m\}\{j\} - \{m+1\}\{j-1\}$ and finally

$$\langle 1\rangle \otimes \{m,j\} = (\langle 1\rangle \otimes \{m\}) * (\langle 1\rangle \otimes \{j\}) - (\langle 1\rangle \otimes \{m+1\}) * (\langle 1\rangle \otimes \{j-1\})$$

These remarks are sufficient to give a method to compute the fourth power. The existence of the star equivalence operation for unirreps of $Sp(2n,\mathbf{R})$ implies certain one-to-one mappings between the states of N-particles in even parity orbitals and those of N-particles in odd parity orbitals.

Acknowledgements

The work of BGW has been partially supported by a Polish KBN Grant. BGW is grateful to JYT for facilitating a visit to Institut Gaspard Monge where most of this work was done and to Professor R.C. King (University of Southampton) for stimulating discussions and hospitality at University of Southampton.

References

[1] Carvalho M J 1990*J. Phys. A: Math. Gen.* **23** 1909
[2] Grudzinski K and Wybourne B G 1996 *J. Phys. A: Math. Gen.* **29** 6631
[3] Haase R W and Johnson N F 1993 *J. Phys. A: Math. Gen.* **26** 1663
[4] Haase R W and Johnson N F 1993 *Phys. Rev.* **B48** 1583
[5] King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509
[6] King R C and Wybourne B G 1985 *J. Phys. A: Math. Gen.* **18** 3113
[7] Leung E Y and Ton-That T 1995 *Proc. Amer. Math. Soc.* **123** **4** 1281
[8] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials, second edition*, (Oxford science publications)
[9] Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419
[10] Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A: Math. Gen.* **18** 939
[11] Scharf T, Thibon J-Y, Wybourne B G 1993 *J. Phys. A: Math. Gen.* **26** 7461-78
[12] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[13] Wybourne B G 1994 *Rept. Math. Phys.* **34** 9