A crystalizable version of $U_q(gl_n)$ *

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Dedicated to the memory of Marcel–Paul Schützenberger

Abstract

We interpret quasi-symmetric functions and noncommutative symmetric functions as characters of a degenerate quantum group obtained by putting $q = 0$ in a variant of $U_q(gl_N)$.

1 Introduction

This paper concludes a series of investigations (*cf.* [6, 4, 16, 24]) devoted to the representation theoretical interpretations of the algebras of quasi-symmetric functions (see [10]) and of noncommutative symmetric functions (see [9]), two different generalizations of the classical theory of symmetric functions which eventually turned out to be dual to each other (*cf.* [22]).

The new theories are very similar to the classical one, the main difference being that the rôle of partitions is in both cases played by compositions. Both algebras possess distinguished bases which can be regarded as analogues of Schur functions. In particular, the structure constants in these bases are nonnegative integers. This suggests the existence of representation-theoretical interpretations similar to the relation between symmetric functions, symmetric groups and general linear groups.

The analogue of the relation between Schur functions and representations of the symmetric group was eventually found to be provided by the 0-Hecke algebra $H_n(0)$ of type A (see [6, 4, 16]). The known q-analogue of the Schur-Weyl duality between generic Hecke algebras $H_n(q)$ and the quantum groups associated with general linear groups (*cf.* [11]) immediately suggested the existence of an interpretation of quasi-symmetric functions and noncommutative symmetric functions as characters of a quantum group at $q = 0$.

However one has to be careful since the standard quantum groups are not defined for $q = 0$. Indeed, their presentations are generally symmetric in q and q^{-1}, and have no limit for $q \to 0$. The theory of crystal bases (*cf.* [12]) which is sometimes loosely referred to as "the theory of quantum groups at $q = 0$" allows only to set $q = 0$ in certain formulas, obtained from the generic case, and involving only polynomials after some normalization. That is to say, the theory of crystal bases does not deal with a degenerate case, but rather describes the combinatorial aspects of the generic case (e.g. the combinatorics of Young tableaux in the case of gl_N). The relation between quantum groups at $q = 0$ and classical combinatorics was first pointed out in [2].

One can associate two kinds of quantum algebras to a classical group G. First a q analogue $F_q(G)$ of the algebra of polynomial functions on the group (*cf.* [7]) whose comodules are in duality with polynomial representations of the group. Second a q-analogue $U_q(g)$ of the universal enveloping algebra of its Lie algebra (*cf.* [11]).

There exists a non-standard version of $F_q(GL_N)$, due to Dipper and Donkin (see [3]) in which the specialization $q = 0$ is possible. This allowed us to intepret quasi-symmetric functions and noncommutative symmetric functions as characters of $F_0(GL_N)$ comodules, an exact analogue of the relation between

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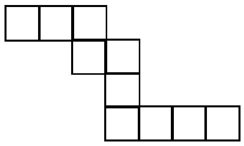
Schur functions and polynomial representations of GL_N (see [16]). Moreover this point of view leads to an intepretation of quasi-symmetric functions similar to the plactic interpretation of symmetric functions introduced by Lascoux and Schützenberger in [19].

However, to the best of our knowledge, the corresponding q-analogue of $U(gl_N)$ was not available. It is this algebra that we present in this paper. We give its basic properties, including the quantum Schur-Weyl duality in the generic case, and study the degeneracy at $q = 0$. We can then prove that distinguished bases of quasi-symmetric functions and of noncommutative symmetric functions are actually characters of irreducible and of indecomposable polynomial modules over the degenerate algebra and we give the classification of such modules.

2 Notations and background

2.1 Compositions

A composition of n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers whose sum is equal to n. The number r is called the length $\ell(I)$ of I. One represents usually a composition I by a skew Young diagram called a *ribbon diagram* of shape I. For example, the ribbon diagram of shape $I = (3, 2, 1, 4)$ is



The *mirror image* of I is the composition $\overline{I} = (i_r, \ldots, i_1)$. The *conjugate* composition $I^\sim$ of I is the composition obtained by reading from right to left the heights of the columns of the ribbon diagram of I. For example, $(3, 2, 1, 4)^\sim = (1, 1, 1, 3, 2, 1, 1)$.

One associates with a composition $I = (i_1, \ldots, i_r)$ of n the subset $D(I)$ of $[1, n-1]$ defined by $D(I) = \{\, i_1, i_1 + i_2, \ldots, i_1 + \ldots + i_{r-1} \,\}$. The *reverse refinement order*, denoted $\prec$, on the set of all compositions of the same integer n is defined by $I \prec J$ iff $D(I) \subset D(J)$. For example $(3, 2, 6) \prec (2, 1, 2, 3, 1, 2)$. One associates to a permutation σ of $\mathfrak{S}_n$ a composition $C(\sigma)$ of n called the *descent composition* of σ, and whose ribbon diagram is called the shape of σ. The descent composition of σ encodes its *descent set* $\mathcal{D}(\sigma) = \{i \in [1, n-1] \mid \sigma(i) > \sigma(i+1)\}$. That is, $D(C(\sigma) = \mathcal{D}(\sigma)$. For example, $C(245316) = (3, 1, 2)$ and $\mathcal{D}(245316) = \{3, 4\}$..

The set D_I of all permutations $\sigma \in \mathfrak{S}_n$ having I as descent composition is an interval $[\alpha(I), \omega(I)]$ for the weak order (permutohedron) of $\mathfrak{S}_n$, where

- $\alpha(I)$ is the permutation obtained by filling the columns of the ribbon diagram of shape I from bottom to top and from left to right with the numbers $1, 2, \ldots, n$;
- $\omega(I)$ is the permutation obtained by filling the rows of the ribbon diagram of shape I from left to right and from bottom to top with the numbers $1, 2, \ldots, n$.

One has for instance $\alpha(2, 2, 1, 1, 3) = 132765489$ and $\omega(2, 2, 1, 1, 3) = 896754123$.

2.2 Ribbon shaped tableaux and quasi-tableaux

Let $(A, <)$ be a totally ordered alphabet. Let also I be a composition. A *tableau* of ribbon shape I is a skew Young tableau obtained by filling a ribbon diagram r of shape I by letters of A in such a way that

- each row of r is nondecreasing from left to right;
- each column of r is strictly increasing from bottom to top.

A word w is said to be of *ribbon shape* I if it can be obtained by reading from top to bottom and from left to right the columns of a tableau of ribbon shape I, which amounts to say that the descent composition of w is I. For example, the ribbon shape of $w = aacbacdcd$ is $(3, 1, 3, 2)$.

A *quasi-tableau* of ribbon shape I is an object obtained by filling a ribbon diagram r of shape I by letters of A in such a way that

- each row of r is nondecreasing from left to right;
- each column of r is strictly increasing from *top to bottom*.

A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-tableau of shape I. Observe that this convention allows again to read the shape of a quasi-ribbon word on the word itself. For example, the word $u = aacbabbac$ is not a quasi-ribbon word since the planar representation of u obtained by writing its decreasing factors as columns is not a quasi-tableau. On the other hand, the word $v = aacbacdcd$ is a quasi-ribbon word of shape $(3, 1, 3, 2)$.

We will sometimes refer to a tableau of ribbon shape I as a ribbon tableau of shape I, but this notion should not be confused with the ribbon (or rim-hook) tableaux obtained by tiling a Young diagram with ribbons, each of them carrying only one label (this notion is not used at all in this paper).

2.3 The hypoplactic algebra

The *plactic algebra* [19] on the totally ordered alphabet A is the $\mathbb{C}$-algebra $Pl(A)$ defined as the quotient of $\mathbb{C}\langle A\rangle$ by the relations

$$\begin{cases} aba = baa \quad , \quad bba = bab & \text{for } a < b \ , \\ acb = cab \quad , \quad bca = bac & \text{for } a < b < c \ . \end{cases}$$

These relations, due to Knuth [17], generate the equivalence relation that identifies two words which have the same P-symbol under the Robinson-Schensted correspondence. The associated monoid structure on the set of tableaux was studied by Lascoux and Schützenberger [19] under the name "plactic monoid". Their point of view is now explained by Kashiwara's theory of crystal bases (*cf.* [12, 13]) which also leads to the definition of plactic algebras associated with all classical simple Lie algebras (*cf.* [18, 21]). Other interpretations of the plactic relations in terms of quantum groups can be found in [2, 20].

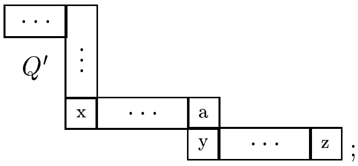
The hypoplactic algebra is a quotient of the plactic algebra where ribbon quasi-tableaux play the same rôle as Young tableaux in the plactic case. This algebra was introduced in [16] where it is shown that a remarkable commutative subalgebra of the hypoplactic algebra is isomorphic to the character ring of a degenerate quantum group and provides a hypoplactic realization of the algebra of quasi-symmetric functions analogous to the plactic interpretation of ordinary symmetric functions discovered by Lascoux and Schützenberger [19]. We will see in Section 6.3 that the same result holds when one rather considers polynomial modules over the crystalization of the version of $\mathcal{U}_q(gl_N)$ introduced in Section 3.

Definition 2.1 *The* hypoplactic algebra $HPl(A)$ *is the quotient of the plactic algebra* $Pl(A)$ *by the quartic relations*

$$\begin{cases} baba = abab \quad , \quad baca = abac & \text{for } a < b < c \ , \\ cacb = acbc \quad , \quad cbab = bacb & \text{for } a < b < c \ , \\ badc = dbca \quad , \quad acbd = cdab & \text{for } a < b < c < d \ . \end{cases}$$

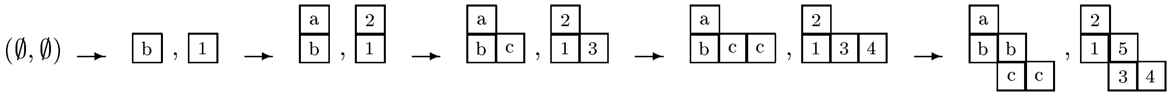
The hypoplactic monoid has the same properties as the plactic monoid with respect to a Robinson-Schensted type correspondence where tableaux are replaced by ribbon quasi-tableaux [16]. The insertion algorithm is as follows. Let Q be a ribbon quasi-tableau and let $a \in A$. Denote by Q' be the quasi-tableau obtained from Q by deleting its last row and let x (resp. z) be the first (resp. last) letter of the last row of Q. The result $\mathcal{Q}$ of the insertion of a in Q is defined by the following rules:

- if $z \leq a$, $\mathcal{Q}$ is obtained by adding a box containing a at the end of the last row of Q;
- if $x \leq a < z$, let y be the first letter of the last row of Q which is strictly greater than a. The quasi-tableau $\mathcal{Q}$ is then



- if $a < x$, Q is obtained by inserting a in Q' and glueing under the quasi-ribbon tableau obtained in this way the last row of Q.

Let now $w = a_1 \dots a_n$ be a word of length n. One associates with w a pair (Q, R), where Q is a quasi-tableau of ribbon shape I and where R is a standard ribbon tableau of the same shape. The quasi-tableau Q is obtained by inserting successively with the previous algorithm all letters of w (from left to right) starting from an empty diagram. The standard ribbon tableau R is iteratively constructed by putting at each step $i \in [1, n]$ of the insertion algorithm the number i in the box that contains at this moment in Q the letter a_i inserted at this step. Let us illustrate this process on $w = baccb$.



The correspondence $w \to (Q, R)$ is a bijection, and we have the following quasi-ribbon analogue of Knuth's theorem (*cf.* [17]).

Proposition 2.2 [16] *Two words $u, v \in \mathbb{A}^*$ are mapped to the same quasi-tableau Q by the above insertion process iff $u \equiv v$ with respect to the hypoplactic congruence.*

We also recall the quasi-ribbon version of the cross-section theorem of [19].

Theorem 2.3 [16] *The classes of quasi-ribbon words form a linear basis of the hypoplactic algebra.*

2.4 Noncommutative symmetric functions and quasi-symmetric functions

The algebra of *noncommutative symmetric functions* (*cf.* [9]) is the free associative algebra $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \dots \rangle$ generated by an infinite sequence of noncommutative indeterminates S_k, called *complete* symmetric functions. For a composition $I = (i_1, i_2, \dots, i_r)$, one sets

$$S^I = S_{i_1} \, S_{i_2} \, \dots \, S_{i_r} \ .$$

The family (S^I) is a linear basis of $\mathbf{Sym}$. Although it is convenient to define $\mathbf{Sym}$ as an abstract algebra, a useful realisation can be obtained by taking an infinite alphabet $A = \{ a_1, a_2, \dots \}$ and defining its complete homogeneous symmetric functions by the generating function

$$\sum_{n \geq 0} t^n \, S_n(A) = \overrightarrow{\prod_{i \geq 1}} \, (1 - t \, a_i)^{-1} \ . \tag{1}$$

The noncommutative *ribbon Schur functions* R_I are defined by

$$R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} \, S^J \ .$$

The family (R_I) is another homogenous basis of $\mathbf{Sym}$. One should notice that R_I reduces to the sum of all words of ribbon shape I in the realization of $\mathbf{Sym}$ given by equation (1) (see [9]).

The algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions introduced by Gessel in [10] (*cf.* [9, 22]). Let $X = \{ x_1, x_2, \dots \}$ be an infinite totally ordered set of commutative indeterminates. An element $f \in \mathbb{C}[X]$ is said to be a *quasi-symmetric function* iff for each composition $K = (k_1, \dots, k_r)$ all monomials $x_{i_1}^{k_1} \, x_{i_2}^{k_2} \cdots x_{i_r}^{k_r}$ with $i_1 < i_2 < \dots < i_r$ have the same coefficient in f. The quasi-symmetric functions form a subalgebra of $\mathbb{C}[X]$ denoted by $QSym$. The *quasi-monomial functions*

$$M_I = \sum_{j_1<\ldots<j_r} x_{j_1}^{i_1} \ldots x_{j_r}^{i_r}$$

(labelled by compositions I) form clearly a basis of $QSym$. Another important basis of $QSym$ is given by the *quasi-ribbon functions*

$$F_I = \sum_{I \preceq J} M_J \ ,$$

e.g., $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$. It is important to note that F_I is the commutative image of the sum of all quasi-ribbon words of shape I. The pairing $\langle \cdot, \cdot \rangle$ between **Sym** and $QSym$ is defined by $\langle S^I, M_J \rangle = \delta_{IJ}$ or equivalently $\langle R_I, F_J \rangle = \delta_{IJ}$ (*cf.* [22, 9]). This duality can also be interpreted as the canonical duality between the Grothendieck groups respectively associated with finite dimensional and projective modules over 0-Hecke algebras (*cf.* [6, 16]).

3 The quantum enveloping algebra $\mathcal{U}_q(gl_N)$

In this section, we introduce a variant of the quantized enveloping algebra $U_q(gl_n)$ which will display the same behaviour for generic q, but can also be specialized at $q = 0$. However, to allow this specialization, we have to define it only as a bialgebra (no antipode).

3.1 The bialgebra $\mathcal{U}_q(gl_N)$

Definition 3.1 *Let $q \in \mathbb{C}$ be an indeterminate (or a complex parameter) and let $N \geq 1$ be an integer. The* crystalizable quantum analogue *$\mathcal{U}_q(gl_N)$ of the universal enveloping algebra of the Lie algebra $\mathcal{G}l_N$ is the algebra over $\mathbb{C}(q)$ generated by the elements $(e_i)_{1\leq i\leq N-1}$, $(f_i)_{1\leq i\leq N-1}$ and $(k_i)_{1\leq i\leq N}$ with relations:*

$$k_i\,k_j = k_j\,k_i \qquad \text{for } 1 \leq i,j \leq N \ , \tag{2}$$

$$\begin{cases} q\,k_i\,e_{i-1} = e_{i-1}\,k_i & \text{for } 2 \leq i \leq N-1 \ , \\ k_i\,e_i = q\,e_i\,k_i & \text{for } 1 \leq i \leq N-1 \ , \\ k_i\,e_j = e_j\,k_i & \text{for } j \neq i-1, i \ , \end{cases} \tag{3}$$

$$\begin{cases} k_i\,f_{i-1} = q\,f_{i-1}\,k_i & \text{for } 2 \leq i \leq N-1 \ , \\ q\,k_i\,f_i = f_i\,k_i & \text{for } 1 \leq i \leq N-1 \ , \\ k_i\,f_j = f_j\,k_i & \text{for } j \neq i-1, i \ , \end{cases} \tag{4}$$

$$[e_i, f_j] = \delta_{i,j}\,\frac{k_i - k_{i+1}}{q-1} \qquad \text{for } 1 \leq i,j \leq N \ , \tag{5}$$

$$\begin{cases} q\,e_{i+1}\,e_i^2 - (1+q)\,e_i\,e_{i+1}\,e_i + e_i^2\,e_{i+1} = 0 & \text{for } 1 \leq i \leq N-2 \ , \\ q\,e_{i+1}^2\,e_i - (1+q)\,e_{i+1}\,e_i\,e_{i+1} + e_i\,e_{i+1}^2 = 0 & \text{for } 1 \leq i \leq N-2 \ , \end{cases} \tag{6}$$

$$\begin{cases} f_{i+1}\,f_i^2 - (1+q)\,f_i\,f_{i+1}\,f_i + q\,f_i^2\,f_{i+1} = 0 & \text{for } 1 \leq i \leq N-2 \ , \\ f_{i+1}^2\,f_i - (1+q)\,f_{i+1}\,f_i\,f_{i+1} + q\,f_i\,f_{i+1}^2 = 0 & \text{for } 1 \leq i \leq N-2 \ , \end{cases} \tag{7}$$

$$\begin{cases} [e_i, e_j] = 0 & \text{for } |i-j| > 1 \ , \\ [f_i, f_j] = 0 & \text{for } |i-j| > 1 \ , \end{cases} \tag{8}$$

$$\begin{cases} e_i\,e_{i+1}\,k_{i+1} = k_{i+1}\,e_i\,e_{i+1} & \text{for } 1 \leq i \leq N-2 \ , \\ f_{i+1}\,f_i\,k_{i+1} = k_{i+1}\,f_{i+1}\,f_i & \text{for } 1 \leq i \leq N-2 \ . \end{cases} \tag{9}$$

Note 3.2 Relations (9) are obvious consequences of the other relations when $q \neq 0$. Thus they do not play any rôle in the generic case. One must however add them in order to endow $\mathcal{U}_q(gl_N)$ with a bialgebra structure in the $q = 0$ case.

Proposition 3.3 *Let $q \in \mathbb{C}$. Then $\mathcal{U}_q(gl_N)$ is a $\mathbb{C}$-bialgebra for the comultiplication Δ, with counity ε defined by*

$$\begin{cases} \Delta(e_i) &= 1 \otimes e_i + e_i \otimes k_i & \text{for } 1 \le i \le N-1 \ , \\ \Delta(f_i) &= k_{i+1} \otimes f_i + f_i \otimes 1 & \text{for } 1 \le i \le N-1 \ , \\ \Delta(k_i) &= k_i \otimes k_i & \text{for } 1 \le i \le N \ , \end{cases}$$

$$\begin{cases} \varepsilon(e_i) &= 0 & \text{for } 1 \le i \le N-1 \ , \\ \varepsilon(f_i) &= 0 & \text{for } 1 \le i \le N-1 \ , \\ \varepsilon(k_i) &= 1 & \text{for } 1 \le i \le N \ . \end{cases}$$

3.2 The fundamental representation of $\mathcal{U}_q(gl_N)$

Let now V be a $\mathbb{C}$-vector space of dimension N with basis $(\xi_i)_{1 \le i \le N}$, i.e.

$$V = \bigoplus_{i=1}^{N} \mathbb{C}\xi_i \ .$$

Let us denote by $(E_{ij})_{1 \le i,j \le N}$ the canonical basis of $End_{\mathbb{C}}(V)$ relative to the basis (ξ_i) of V. That is $E_{ij}(\xi_k) = \delta_{jk}\, \xi_i$. It is easy to check that one can define an algebra morphism ρ_V of $\mathcal{U}_q(gl_N)$ in $End_{\mathbb{C}}(V)$ by setting

$$\begin{cases} \rho_V(e_i) &= E_{i,i+1} & \text{for } 1 \le i \le N-1 \ , \\ \rho_V(f_i) &= E_{i+1,i} & \text{for } 1 \le i \le N-1 \ , \\ \rho_V(k_i) &= q\, E_{i,i} + \sum_{j \ne i} E_{j,j} & \text{for } 1 \le i \le N \ . \end{cases}$$

The pair (ρ_V, V) is then called the *fundamental representation* (or *vector representation*) of $\mathcal{U}_q(gl_N)$. Since $\mathcal{U}_q(gl_N)$ is a bialgebra, one can define for every $n \ge 2$ its n-th tensor power $(\rho_{n,N}, V^{\otimes n})$ by

$$\rho_{n,N} = \rho_V^{\otimes n} \circ \Delta^{(n)} \ : \ \mathcal{U}_q(gl_N) \longrightarrow End_{\mathbb{C}}(V^{\otimes n})$$

where $\Delta^{(n)}$ denotes the n-fold comultiplication $\mathcal{U}_q(gl_N) \to \mathcal{U}_q(gl_N)^{\otimes n}$ inductively defined by:

$$\begin{cases} \Delta^{(2)} &= \Delta \ , \\ \Delta^{(n)} &= \Big((Id_{\mathcal{U}_q(gl_N)})^{\otimes(n-2)} \otimes \Delta \Big) \circ \Delta^{(n-1)} & \text{for } n \ge 3 \ . \end{cases}$$

4 The quantum matrix algebra $A_q(N)$

4.1 The bialgebra $A_q(N)$

We denote by $A_q(N)$ the $\mathbb{C}(q)$-algebra generated by n^2 elements $(x_{ij})_{1 \le i,j \le N}$ subject to the defining relations:

$$\begin{cases} x_{il}\, x_{jk} &= q\, x_{jk}\, x_{il} & \text{for } 1 \le i \le j \le N, \ 1 \le k < l \le N \ , \\ x_{ik}\, x_{jk} &= x_{jk}\, x_{ik} & \text{for } 1 \le i, j, k \le N \ , \\ x_{jl}\, x_{ik} - x_{ik}\, x_{jl} &= (q-1)\, x_{jk}\, x_{il} & \text{for } 1 \le i < j \le N, \ 1 \le k < l \le N \ . \end{cases} \tag{10}$$

This algebra is a quantization of the bialgebra of polynomial functions on the variety of $N \times N$ matrices over $\mathbb{C}$. Up to the symmetry $x_{ij} \leftrightarrow x_{ji}$, $A_q(N)$ is the quantization introduced by Dipper and Donkin in [3]. It is not isomorphic to the more familiar quantization of Faddev-Reshetikhin-Takhtadzhyan [7]. Although for generic values of q both versions play essentially the same rôle, an essential difference is that $A_q(N)$ is defined for $q = 0$.

One can endow $A_q(N)$ with a structure of bialgebra whose comultiplication $\overline{\Delta}$ and counity η are defined by

$$\overline{\Delta}(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj} \qquad \text{and} \qquad \eta(x_{ij}) = \delta_{ij} \ .$$

4.2 The quantum diagonal algebra

The quantum diagonal algebra (or quantum torus) $\Delta_q(N)$ is a q-analogue of the algebra of polynomial functions on a maximal torus of $GL(n)$. As such, it occurs in the character theory of Hecke algebras, of $A_q(N)$ and of $\mathcal{U}_q(gl_N)$ (*cf.* Section 6.2 and [16]).

Definition 4.1 *The* quantum diagonal algebra $\Delta_q(N)$ *is the subalgebra of* $A_q(N)$ *generated by the diagonal elements* $x_{11}, \ldots, x_{NN}$.

We propose a conjecture that would give a presentation of $\Delta_q(N)$ in the generic case. We first introduce the following definition.

Definition 4.2 *Let* $(A, <)$ *be a totally ordered alphabet. The* quantum pseudoplactic algebra $PPl_q(A)$ *is the quotient of* $\mathbb{C}(q)\langle A\rangle$ *by the relations*

$$\left\{ \begin{array}{rcll} q\,a\,a\,b - (q+1)\,a\,b\,a + b\,a\,a & = & 0 & \text{for } a < b\,, \\ q\,a\,b\,b - (q+1)\,b\,a\,a + b\,b\,a & = & 0 & \text{for } a < b\,, \\ c\,a\,b - a\,c\,b - b\,c\,a + b\,a\,c & = & 0 & \text{for } a < b < c\,. \end{array} \right.$$

Conjecture 4.3 [16] *Let* $A = \{ a_1 < \ldots < a_N \}$ *be a totally ordered alphabet of cardinality* N. *For every complex number* q *distinct from* 0 *and* 1, *the mapping* $\varphi : a_i \to x_{ii}$ *induces an isomorphism between* $PPl_q(A)$ *and the quantum diagonal algebra* $\Delta_q(N)$.

The conjecture is clearly not true for $q = 1$, since the quantum diagonal algebra $\Delta_1(N)$ is just an algebra of commutative polynomials. The case $q = 0$ is more interesting:

Theorem 4.4 [16] *The ring homomorphism defined by* $\varphi : a_i \longrightarrow x_{ii}$ *is an isomorphism between the hypoplactic algebra* $HPl(\{a_1, \ldots, a_N\})$ *and the quantum diagonal algebra* $\Delta_0(N)$ *at* $q = 0$.

4.3 Duality between $A_q(N)$ and $\mathcal{U}_q(gl_N)$

For every $q \in \mathbb{C}$, the two $\mathbb{C}$-bialgebras $A_q(N)$ and $\mathcal{U}_q(gl_N)$ are in duality, in a sense to be made precise below. Before constructing explicitely this duality, let us recall that according to the general theory of bialgebras (see for instance [14]), the dual $\mathcal{U}_q(gl_N)^*$ of $\mathcal{U}_q(gl_N)$ has a canonical algebra structure, defined by the convolution product

$$\varphi \cdot \psi = \mu \circ (\varphi \otimes \psi) \circ \Delta$$

for $\varphi, \psi \in \mathcal{U}_q(gl_N)^*$, where μ is the multiplication map $\mu(g \otimes h) = g\,h$, $g, h \in \mathcal{U}_q(gl_N)$.

Let us now introduce the linear functionals $(a_{ij})_{1 \le i,j \le N} \in \mathcal{U}_q(gl_N)^*$, defined as the matrix coefficients of the vector representation :

$$\rho_V(g) = \begin{pmatrix} a_{11}(g) & \ldots & a_{1N}(g) \\ \vdots & \ddots & \vdots \\ a_{N1}(g) & \ldots & a_{NN}(g) \end{pmatrix}$$

for $g \in \mathcal{U}_q(gl_N)$. The relation between the two quantum groups $A_q(N)$ and $\mathcal{U}_q(gl_N)$ is then given by the following result.

Theorem 4.5 *The* N^2 *linear functionals* $(a_{ij})_{1 \le i,j \le N}$ *of* $\mathcal{U}_q(gl_N)^*$ *satisfy the quantum relations (10).*

Theorem 4.5 allows us to define an algebra homomorphism $\Psi : \ A_q(N) \longrightarrow \mathcal{U}_q(gl_N)^*$ by $\Psi(x_{ij}) = a_{ij}$ for $1 \le i, j \le N$. This morphism can then be used to define the duality of bialgebras between $A_q(N)$ and $\mathcal{U}_q(gl_N)$ by setting $\langle x, g\rangle = \Psi(x)(g)$ for every $x \in A_q(N)$ and $g \in \mathcal{U}_q(gl_N)$.

5 The quantum Schur-Weyl duality

A fundamental property of the standard version of $U_q(gl_N)$ is the existence of a q-analogue of the classical Schur-Weyl duality involving the Hecke algebra instead of the symmetric group [11]. Such a duality can also be worked out for $\mathcal{U}_q(gl_N)$. The double commutant theorem will break down at $q=0$, but one can still give a weaker statement, which will allow us to interpret quasi-symmetric functions and noncommutative symmetric functions as characters of $\mathcal{U}_0(gl_N)$.

The (Iwahori) Hecke algebra $H_n(q)$ of type A_{n-1} is the $\mathbb{C}(q)$-algebra generated by the $n-1$ elements $(T_i)_{1\le i\le n-1}$ with relations

$$\left\{ \begin{array}{rcll} T_i^2 & = & (q-1)\,T_i + q & \text{for } 1\le i\le n-1 \ , \\ T_i\,T_{i+1}\,T_i & = & T_{i+1}\,T_i\,T_{i+1} & \text{for } 1\le i\le n-2 \ , \\ T_i\,T_j & = & T_j\,T_i & \text{for } |i-j|>1 \ . \end{array} \right.$$

For generic complex values of q, the Hecke algebra $H_n(q)$ is isomorphic to $\mathbb{C}[\mathfrak{S}_n]$ (the standard presentation is obtained for $q=1$) and hence semi-simple, except when $q=0$ or when q is a root of unity.

Following [3, 5], we now introduce a variant of Jimbo's right action of $H_n(q)$ on $V^{\otimes n}$ (*cf.* [11]) which is defined by setting

$$\left\{ \begin{array}{rcll} \mathbf{v}\cdot T_i & = & \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} \ , \\ \mathbf{v}\cdot T_i & = & q\,\mathbf{v} & \text{if } k_i = k_{i+1} \ , \\ \mathbf{v}\cdot T_i & = & q\,\mathbf{v}^{\sigma_i} + (q-1)\,\mathbf{v} & \text{if } k_i > k_{i+1} \ , \end{array} \right.$$

for $\mathbf{v} = \xi_{k_1}\otimes\cdots\otimes\xi_{k_n}$ in $V^{\otimes n}$ (where $\mathbf{v}^{\sigma_i}$ denotes the tensor obtained from $\mathbf{v}$ by permuting the i and $i+1$-th factors). We denote by $\pi_{n,N}$ the representation of $H_n(q)$ in $End_{\mathbb{C}}(V^{\otimes n})$ defined by this action:

$$\pi_{n,N}(h) : \mathbf{v}\in V^{\otimes n} \longrightarrow \mathbf{v}\cdot h \in V^{\otimes n} \ .$$

One can check that $\rho_{n,N}$ commutes with $\pi_{n,N}$. In fact, one has much more when q is a non zero complex number which is not a non trivial (*i.e.* distinct from 1) k-th root of unity for some $k\le n$.

Theorem 5.1 *Let q be a nonzero complex number which is not a non trivial k-th root of unity for some $k\le n$. The two subalgebras $\pi_{n,N}(H_n(q))$ and $\rho_{n,N}(\mathcal{U}_q(gl_N))$ of $End_{\mathbb{C}}(V^{\otimes n})$ are then the commutant of each other.*

6 Polynomial $\mathcal{U}_q(gl_N)$-modules

6.1 The generic case

Definition 6.1 *A* polynomial $\mathcal{U}_q(gl_N)$-module *of degree n is a sub-$\mathcal{U}_q(gl_N)$-module of the n-th tensor power $V^{\otimes n}$ of the fundamental representation V of $\mathcal{U}_q(gl_N)$.*

In this section we assume that q is a complex number which is neither zero nor a non trivial k-th root of unity for some $k\le n$. For a partition λ of n, let $\mathcal{Y}_\lambda(q)\in H_n(q)$ be a q-Young idempotent. It will be convenient to use the version introduced in [5]. The family $(H_n(q)\,\mathcal{Y}_\lambda(q))_{\lambda\vdash n}$ is a complete system of $H_n(q)$-irreducible modules. Since $H_n(q)$ is semi-simple, it follows from Theorem 5.1 (by a standard argument, see e.g. Lemma 6.22 of [8]) that the family of $\mathcal{U}_q(gl_N)$-modules

$$\mathbb{S}_\lambda(V;q) = V^{\otimes n}\cdot\mathcal{Y}_\lambda(q) \ ,$$

where λ runs over all partitions of n, is a complete system of irreducible polynomial $\mathcal{U}_q(gl_N)$-modules of degree n.

6.2 Characters of polynomial $\mathcal{U}_q(gl_N)$-modules

Let M be a polynomial $\mathcal{U}_q(gl_N)$-module of degree n and let $(m_i)_{1\le i\le m}$ be a $\mathbb{C}(q)$-linear basis of M. Since M is a $\mathcal{U}_q(gl_N)$-module, there exists a family $(\mu_{i,j})_{1\le i,j\le m}$ of linear functionals of $\mathcal{U}_q(gl_N)^*$ such that

$$g\cdot m_i = \sum_{j=1}^{m} \mu_{i,j}(g)\, m_j$$

for $g\in\mathcal{U}_q(gl_N)$ and $i\in[1,m]$. One can check that the trace $c(M)$ of the action of $\mathcal{U}_q(gl_N)$ on M, *i.e.* the element

$$c(M) = \sum_{i=1}^{m} \mu_{i,i}$$

of $\mathcal{U}_q(gl_N)^*$ does not depend on the choice of the basis. Moreover $c(M)$ belongs to the subalgebra generated by the $(a_{i,j})_{1\le i,j\le N}$ in $\mathcal{U}_q(gl_N)^*$. To see this, it suffices to notice that for $I\in[1,N]^n$,

$$g\cdot\xi^I = \sum_{J\in[1,N]^n} a_{JI}(g)\,\xi^J$$

where a_{JI} stands for $a_{j_1i_1}\,\ldots\,a_{j_ni_n}$ if $I=(i_1,\ldots,i_n)$ and $J=(j_1,\ldots,j_n)$. It follows that there exists elements $p_{IJ}(q)$ of $\mathbb{C}(q)$ such that

$$c(M) = \sum_{I,J\in[1,N]^n} p_{IJ}(q)\, a_{IJ}\ .$$

Since the map Ψ (defined at the end of Section 4.3) is injective on the linear subspace of $A_q(N)$ spanned by homogeneous elements of fixed length, we can lift this formula to $A_q(N)$ (according to Theorem 4.5). Let now x_{IJ} be the monomial $x_{i_1j_1}\,\ldots\,x_{i_nj_n}\in A_q(N)$. The element

$$\chi(M) = \sum_{I,J\in[1,N]^n} p_{IJ}(q)\, x_{IJ}$$

of $A_q(N)$ will be called the *character* of the polynomial $\mathcal{U}_q(gl_N)$-module M. It turns out that these characters are not arbitrary elements of $A_q(N)$.

Theorem 6.2 [16] *Let q be a complex number distinct from* 1 *and let M be a polynomial $\mathcal{U}_q(gl_N)$-module. Then the character $\chi(M)$ belongs to the diagonal algebra $\Delta_q(n)$.*

It follows in particular from Theorem 6.2 and from Theorem 4.4 that the character of a polynomial $\mathcal{U}_0(gl_N)$-module is an element of the hypoplactic algebra.

Note 6.3 The proof of Theorem 6.2 relies on an interesting formula which expresses the quantum determinant in $A_q(N)$ in terms of the diagonal generators x_{ii} of $A_q(N)$:

$$\left|\begin{matrix} x_{11} & x_{12} & \ldots & x_{1N} \\ x_{21} & x_{22} & \ldots & x_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{n2} & \ldots & x_{NN}\end{matrix}\right|_q \stackrel{\text{def}}{=} \sum_{\sigma\in\mathfrak{S}_N} \varepsilon(\sigma)\, x_{\sigma(1)1}\,\ldots\,x_{\sigma(N)N} = \frac{1}{(1-q)^{N-1}}\,[x_{NN},[\ldots,[x_{22},x_{11}]_q\ldots]_q]_q\ ,$$

where $[a,b]_q = ab-qba$.

Note 6.4 Using Weyl's technique (see Section 6 of [8]), one can show that the commutative character of $\mathbb{S}_\lambda(V;q)$ is the Schur function of weight λ. Hence the characters (in our sense) of the irreducible polynomials $\mathcal{U}_q(gl_N)$-modules are quantum analogues of Schur functions. One can check that

$$\chi(\mathbb{S}_n(V;q)) = \sum_{i_1+\ldots+i_N=n} x_{11}^{i_1}\,\ldots\,x_{NN}^{i_N}\ ,$$

$$\chi(\mathbb{S}_{1^n}(V;q)) = \frac{1}{(1-q)^{n-1}} \sum_{1\le i_1<\ldots<i_n\le N} [x_{i_n,i_n},[\ldots,[x_{i_2,i_2},x_{i_1,i_1}]_q\ldots]_q]_q$$

for all n. These quantum Schur functions form a linear basis of a commutative subalgebra of $\Delta_q(N)$ which is isomorphic to the algebra of ordinary symmetric functions (*cf.* [16]).

6.3 Polynomial $\mathcal{U}_0(gl_N)$-modules

From now on, we set $q = 0$. For $i \in [1, n-1]$, let $\Box_i = 1 + T_i$ considered here as an element of $H_n(0)$. These elements satisfy the relations

$$\begin{cases} \Box_i^2 &= \Box_i & \text{for } 1 \le i \le n-1 \ , \\ \Box_i \, \Box_{i+1} \, \Box_i &= \Box_{i+1} \, \Box_i \, \Box_{i+1} & \text{for } 1 \le i \le n-2 \ , \\ \Box_i \, \Box_j &= \Box_j \, \Box_i & \text{for } |i-j| > 1 \ . \end{cases}$$

One can therefore associate with every permutation σ of $\mathfrak{S}_n$ the element $\Box_\sigma$ of $H_n(0)$ defined by $\Box_\sigma = \Box_{i_1} \ldots \Box_{i_r}$ where $\sigma_{i_1} \ldots \sigma_{i_r}$ is any reduced decomposition of σ.

We are now in position to analyse the structure of the polynomial $\mathcal{U}_0(gl_N)$-modules. Since the algebra $\mathcal{U}_0(gl_N)$ is not semi-simple as we will see below, we are led to study two kinds of modules:

1. the irreducible polynomial $\mathcal{U}_0(gl_N)$-modules;
2. the indecomposable polynomial $\mathcal{U}_0(gl_N)$-modules which are direct summands of some $V^{\otimes n}$.

Let us now see how to construct a complete family of irreducible polynomial $\mathcal{U}_0(gl_N)$-modules. To this purpose, let I be a composition of n and let us introduce the element

$$\eta_I = T_{\omega(\overline{I})} \, \Box_{\alpha(I^\sim)}$$

of $H_n(0)$. Norton [23] and Carter [1] showed that the $H_n(0)\,\eta_I$ form a complete family of irreducible $H_n(0)$-modules (see also [16]). As the action of $\mathcal{U}_0(gl_N)$ on $V^{\otimes n}$ still commutes with the right action of $H_n(0)$, one can use η_I to construct the $\mathcal{U}_0(gl_N)$-module

$$\mathbf{D}_I = V^{\otimes N} \cdot \eta_I \ .$$

It appears that $\mathbf{D}_I$ is an irreducible polynomial $\mathcal{U}_0(gl_N)$-module. The argument is similar to the proof of its irreduciblility as an $A_0(N)$-comodule [16]. Moreover one can also show that the characters of these modules are noncommutative analogues of the quasi-ribbon quasi-symmetric functions F_I.

Proposition 6.5 *Let I be a composition of n. The character $\chi(\mathbf{D}_I)$ of the polynomial $\mathcal{U}_0(gl_N)$-module $\mathbf{D}_I$ is equal to the sum, denoted by $F_I(x_{11}, \ldots, x_{nn})$, of all quasi-ribbon words over the ordered alphabet $\{ x_{11} < \ldots < x_{NN} \}$.*

To proceed further, we introduce another family of polynomial $\mathcal{U}_0(gl_N)$-modules. Let again I be a composition of n and consider the element

$$\nu_I = T_{\alpha(I)} \, \Box_{\alpha(\overline{I}^\sim)}$$

of $H_n(0)$. These elements generate a complete family $(H_n(0)\,\nu_I)$ of indecomposable projective left $H_n(0)$-modules (*cf.* [1, 23]). We can construct the polynomial $\mathcal{U}_0(gl_N)$-module

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I \ .$$

The characters of these modules are non commutative analogues of ribbon Schur functions.

Proposition 6.6 *Let I be a composition of n. The character $\chi(\mathbf{N}_I)$ of the polynomial $\mathcal{U}_0(gl_N)$-module $\mathbf{N}_I$ is equal to the sum, denoted by $R_I(x_{11}, \ldots, x_{nn})$, of all words of ribbon shape I over the ordered alphabet $\{ x_{11} < \ldots < x_{NN} \}$.*

We have the following decomposition

$$V^{\otimes n} = \bigoplus_{I \vdash n} \mathbf{N}_I \tag{11}$$

as $\mathcal{U}_0(gl_N)$-modules. This follows easily from results of Carter [1] and Norton [23] (*cf.* [16] for a similar argument). Using this decomposition and the previous result, we can give the complete classification of irreductible polynomial $\mathcal{U}_0(gl_N)$-modules and of indecomposable polynomial $\mathcal{U}_0(gl_N)$-modules which are direct summands of some $V^{\otimes n}$.

Proposition 6.7 *The* $\mathbf{D}_I$ *form a complete family of irreducible polynomial* $\mathcal{U}_0(gl_N)$*-modules.*

The noncommutative analogues of quasi-ribbon functions arising in Proposition 6.7 form a linear basis of a commutative subalgebra of the hypoplactic algebra. This subalgebra is isomorphic to the algebra of quasi-symmetric functions and plays therefore the rôle of the character ring of polynomial $\mathcal{U}_0(gl_N)$-modules.

Proposition 6.8 *The* $\mathbf{N}_I$ *form a complete family of indecomposable polynomial* $\mathcal{U}_0(gl_N)$*-modules which are direct summands of some* $V^{\otimes n}$.

The noncommutative analogues of ribbon functions in Proposition 6.6 form a linear basis of a noncommutative subalgebra of the hypoplactic algebra. This realization of the algebra of noncommutative symmetric functions is therefore a kind character ring associated with "projective" polynomial $\mathcal{U}_0(gl_N)$-modules (*i.e.* direct summands of some $V^{\otimes n}$). The noncommutativity of this ring reflects the fact that at $q = 0$, the tensor products $M \otimes N$ and $N \otimes M$ are actually not isomorphic.

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