Stable Dynamics of Sand Automata*

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Abstract. In this paper, we study different notions of stability of sand automata, dynamical systems inspired by sandpile models and cellular automata. First, we study the topological stability properties of equicontinuity and ultimate periodicity, proving that they are equivalent. Then, we deal with nilpotency. The classical definition for cellular automata being meaningless in that setting, we define a more suitable one. Finally, we prove that this simple dynamical behavior is undecidable.

1 Introduction

Self-organized criticality (SOC [2]) is a common phenomenon observed in a huge variety of processes in physics, biology and computer science. A SOC system evolves to a "critical state" after some finite transient. Examples of SOC systems are: sandpiles, snow avalanches, star clusters in the outer space, earthquakes, forest fires, load balance in operating systems [1]. Among them, sandpile models are a paradigmatic formal model for SOC systems [10].

In [3], the authors introduced sand automata as a generalization of sandpile models and transposed them in the setting of discrete dynamical systems. A key-point of [3] was to introduce a (locally compact) metric topology to study the dynamical behavior of sand automata. A first and important result was a fundamental representation theorem similar to the well-known theorem of Hedlund for cellular automata [11, 3]. In [4, 5], the authors investigate sand automata by dealing with some basic set properties and decidability issues. Then, in [8], a new compact topology is introduced, inspired by the strong relation between sand automata and cellular automata. It is proved that with

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this new topology, the representation theorem still holds, while the compactness provides new opportunities for further topological studies of the model.

In this paper we continue the study of sand automata dynamics of [4, 5], using the topological framework from [8]. More precisely, we study different types of stability. Indeed, stability over time is a major issue for isolating the realistic sandpile models satisfying the SOC principles. First, we deal with the dynamical stability, i.e., the equicontinuity and ultimate periodicity properties. We prove that they are equivalent. Then, the insignificance of expansivity, a form of strong instability, suggests that the topological classification for cellular automata from [13] cannot be easily generalized to sand automata.

Finally, we study nilpotency, a very strong form of dynamical stability. The classical definition of nilpotency for cellular automata [7, 12] is no more meaningful here, since it would prevent any sand automaton from being nilpotent. Therefore, we introduce a new definition which captures the intuitive idea that a nilpotent automaton destroys all configurations: a sand automaton is nilpotent if all configurations get closer and closer to a uniform configuration, not necessarily reaching it. Finally, we prove that this behavior is undecidable, using the undecidability of the nilpotency of spreading cellular automata.

The paper is structured as follows. In Section 2, we recall basic definitions and results on cellular automata and sand automata. Then, in Section 3, results on the topological stability of sand automata are proved and discussed. Nilpotency of sand automata is then defined and proved undecidable in Section 4.

2 Definitions

For all $a, b \in \mathbb{Z}$ with $a \leq b$, let $[a, b] = \{a, a+1, \ldots, b\}$ and $\widetilde{[a, b]} = [a, b] \cup \{+\infty, -\infty\}$. Let $\mathbb{N}_+$ be the set of positive integers.

For a vector $i \in \mathbb{Z}^d$, denote by $|i|$ the infinite norm of i. Let A a (possibly infinite) alphabet, and $r, d \in \mathbb{N}^*$. Denote by $\mathcal{M}_r^d$ the set of all the d-dimensional vectors of the hyper-rectangle $[-r, r]^d$, with values in A.

2.1 Cellular Automata

Let A be a finite alphabet. A *CA configuration* of dimension d is a function from $\mathbb{Z}^d$ to A. The set $A^{\mathbb{Z}^d}$ of all the CA configurations is called the *CA configuration space*. This space is usually equipped with the Tychonoff metric d_T defined by

$$\forall x, y \in A^{\mathbb{Z}^d}, \quad \mathsf{d}_T(x, y) = 2^{-k} \quad \text{where} \quad k = \min\left\{|j| : j \in \mathbb{Z}^d, x_j \neq y_j\right\} \ .$$

The topology induced by d_T coincides with the product topology induced by the discrete topology on A. It makes the CA configuration space is a Cantor space: it is compact, perfect (i.e., it has no isolated points) and totally disconnected.

A *cellular automaton* (CA) is a quadruple $\langle A, d, r, g\rangle$, where A is the alphabet, also called the *state set*, $d \in \mathbb{N}$ is the dimension, $r \in \mathbb{N}$ is the *radius* and

$g : \mathcal{M}_r^d \to A$ is the *local rule* of the automaton. The local rule g induces a *global rule* $G : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined as follows,

$$\forall x \in A^{\mathbb{Z}^d}, \forall i \in \mathbb{Z}^d, \quad G(x)_i = g\big(M_r^i(x)\big) \enspace ,$$

where $M_r^i(x) \in \mathcal{M}_r^d$ is the *finite portion* of x of reference position $i \in \mathbb{Z}^d$ and radius r defined by $\forall k \in [-r, r]^d$, $M_r^i(x)_k = x_{i+k}$.

For any $k \in \mathbb{Z}^d$ the *shift map* $\sigma^k : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is defined by $\forall x \in A^{\mathbb{Z}^d}, \forall i \in \mathbb{Z}^d$, $\sigma^k(x)_i = x_{i+k}$. A function $F : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is said to be *shift-commuting* if $\forall k \in \mathbb{Z}^d$, $F \circ \sigma^k = \sigma^k \circ F$. Note that CA are exactly the class of all shift-commuting functions which are (uniformly) continuous with respect to the Tychonoff metric (Hedlund's theorem from [11]). For the sake of simplicity, we will make no distinction between a CA and its global rule G.

For a given CA, a state $s \in A$ is *quiescent* (resp., *spreading*) if for all matrices $U \in \mathcal{M}_r^d$ such that $\forall k \in [-r, r]^d$, (resp., $\exists k \in [-r, r]^d$) $U_k = s$, it holds that $g(U) = s$. Remark that a spreading state is also quiescent. A CA is said to be spreading if it has a spreading state. In the sequel, the spreading state of any spreading CA will be denoted $0 \in A$.

2.2 SA Configurations

A *configuration* is a set of sand grains organized in piles and distributed all over the d-dimensional lattice $\mathbb{Z}^d$. A *pile* is an element of $\widetilde{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$ which represents a number of grains. One pile is positioned in each point of the lattice $\mathbb{Z}^d$. Formally, a configuration x is a function from $\mathbb{Z}^d$ to $\widetilde{\mathbb{Z}}$ which associates any vector $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ with the number $x_i \in \widetilde{\mathbb{Z}}$ of grains in the pile of position i. When the dimension d id known without ambiguity, we note 0 the null vector of Z^d. Denote by $\mathcal{C} = \widetilde{\mathbb{Z}}^{\mathbb{Z}^d}$ the set of all configurations.

A configuration $x \in \mathcal{C}$ is said to be *constant* if there is an integer $c \in \mathbb{Z}$ such that for any vector $i \in \mathbb{Z}^d$, $x_i = c$. In that case we write $x = \underline{c}$. A configuration $x \in \mathcal{C}$ is said to be *bounded* if there exist two integers $m_1, m_2 \in \mathbb{Z}$ such that for all vectors $i \in \mathbb{Z}^d$, $m_1 \le x_i \le m_2$. Denote by $\mathcal{B}$ the set of all bounded configurations.

A *measuring device* β_r^m of precision $r \in \mathbb{N}$ and reference height $m \in \mathbb{Z}$ is a function from $\widetilde{\mathbb{Z}}$ to $\widetilde{[-r, r]}$ defined as follows

$$\forall n \in \widetilde{\mathbb{Z}}, \quad \beta_r^m(n) = \begin{cases} +\infty & \text{if } n > m + r \enspace , \\ -\infty & \text{if } n < m - r \enspace , \\ n - m & \text{otherwise.} \end{cases}$$

A measuring device is used to evaluate the relative height of two piles, with a bounded precision. This is the technical basis of the definition of cylinders, distance and ranges which are used all along this article.

In [8], a topology, inspired by the topology on CA configurations, is defined as follows.

Definition 1 (cylinder). *For any configuration $x \in \mathcal{C}$, for any $r \in \mathbb{N}$, and for any $i \in \mathbb{Z}^d$, the* cylinder *of x centered on i and of radius r is the d-dimensional matrix $C_r^i(x) \in \mathcal{M}_r^d$ defined on the finite alphabet $\widetilde{[-r,r]}$ by*

$$\forall k \in [-r,r]^d, \quad \left(C_r^i(x)\right)_k = \beta_r^0(x_{i+k}) \ .$$

Definition 2. *For any pair of configurations $x, y \in \mathcal{C}$, we define*

$$\mathsf{d}(x,y) = 2^{-k} \quad \textit{where} \quad k = \min\left\{r \in \mathbb{N} : C_r^0(x) \neq C_r^0(y)\right\} \ .$$

As a consequence, two configurations x, y are compared by putting boxes (the cylinders) at height 0 around the corresponding piles indexed by 0. The integer k is the size of the smallest cylinders in which a difference appears between x and y.

That topology makes the SA configuration space perfect, totally disconnected, and, unlike the original topology used in [11, 3], compact (see [8]).

2.3 Sand Automata

For any integer $r \in \mathbb{N}$, for any configuration $x \in \mathcal{C}$ and any index $i \in \mathbb{Z}^d$ with $x_i \neq \pm\infty$, the *range* of center i and radius r is the d-dimensional matrix $R_r^i(x) \in \mathcal{M}_r^d$ on the finite alphabet $A = \widetilde{[-r,r]} \cup \perp$ such that

$$\forall k \in [-r,r]^d, \quad \left(R_r^i(x)\right)_k = \begin{cases} \perp & \text{if } k = 0 \ , \\ \beta_r^{x_i}(x_{i+k}) & \text{otherwise.} \end{cases}$$

The range is used to define a sand automaton. It is a kind of cylinder, where the observer is always located on the top of the pile x_i (called the *reference*). It represents what the automaton is able to see at position i. Sometimes the central $\perp$ symbol may be omitted for simplicity sake. The set of all possible ranges of radius r, in dimension d, is denoted by $\mathcal{R}_r^d$.

A *sand automaton* (SA) is a deterministic finite automaton working on configurations. Each pile is updated synchronously, according to a local rule which computes the variation of the pile by means of the range. Formally, a SA is a triple $\langle d, r, f\rangle$, where d is the dimension, r is the *radius* and $f : \mathcal{R}_r^d \to [-r,r]$ is the *local rule* of the automaton. The *global rule* $F : \mathcal{C} \to \mathcal{C}$ is defined by

$$\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d, \quad F(x)_i = \begin{cases} x_i & \text{if } x_i = \pm\infty \ , \\ x_i + f(R_r^i(x)) & \text{otherwise.} \end{cases}$$

The following example illustrates a sand automaton whose behavior will be studied in Section 4. For more examples, we refer to [5].

Example 1 (the automaton $\mathcal{N}$). This automaton destroys a configuration by collapsing all piles towards the lowest one. It decreases a pile when there is a lower pile in the neighborhood. Let $\mathcal{N} = \langle 1, 1, f_{\mathcal{N}}\rangle$ of global rule $F_{\mathcal{N}}$ where

$$\forall a, b \in \widetilde{[-1,1]}, \quad f_{\mathcal{N}}(a,b) = \begin{cases} -1 & \text{if } a < 0 \text{ or } b < 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

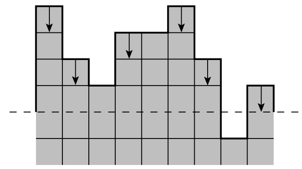

Fig. 1. Illustration of the behavior of $\mathcal{N}$.

When no misunderstanding is possible, we identify a SA with its global rule F. For any $k \in \mathbb{Z}^d$, we extend the definition of the *shift map* to $\mathcal{C}$, $\sigma^k : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d$, $\sigma^k(x)_i = x_{i+k}$. The *raising map* $\rho : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d$, $\rho(x)_i = x_i + 1$. A function $F : \mathcal{C} \to \mathcal{C}$ is said to be *vertical-commuting* if $F \circ \rho = \rho \circ F$. A function $F : \mathcal{C} \to \mathcal{C}$ is *infinity-preserving* if for any configuration $x \in \mathcal{C}$ and any vector $i \in \mathbb{Z}^d$, $F(x)_i = +\infty$ if and only if $x_i = +\infty$ and $F(x)_i = -\infty$ if and only if $x_i = -\infty$.

With the topology from [8], the Hedlund-like representation theorem for SA from [3] remains valid.

Theorem 1 ([8]). *A mapping $F : \mathcal{C} \to \mathcal{C}$ is a SA if and only if F is (uniformly) continuous, shift-commuting, vertical-commuting and infinity-preserving.*

3 Some Dynamical Behaviors

The concepts that first come to mind to formalize the notion of stability are inspired by the topological classifications given in [9, 13] for CA. In [13], CA are classified into four classes, from the most stable to the most unstable behavior: equicontinuous CA, non-equicontinuous CA admitting an equicontinuity configuration, sensitive but not positively expansive CA, positively expansive CA. In this section we consider these classes from the SA point of view, and we introduce the notion of ultimate periodicity, useful for the characterization of SOC systems. We prove that there exist no positively expansive SA and characterize equicontinuous SA as the ultimately periodic SA.

First, recall basic definitions. Let (X, m) a metric space and let $H : X \to X$ be a continuous application. An element $x \in X$ is an *equicontinuity point* for H if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. The map H is *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. An element $x \in X$ is *ultimately periodic* for H if there exist two integers $n \geq 0$ (the preperiod) and $p > 0$ (the period) such that $H^{n+p}(x) = H^n(x)$. H is *ultimately periodic* if there exist $n \geq 0$ and $p > 0$ such that $H^{n+p} = H^n$. If X is compact, H is equicontinuous (resp. ultimately periodic) iff all elements of X are equicontinuity points (resp. ultimately periodic). H is *sensitive* (to the initial conditions) if there is a constant $\varepsilon > 0$ such that for all points $x \in X$ and all $\delta > 0$, there is a point $y \in X$ and an integer $n \in \mathbb{N}$ such that $m(x, y) < \delta$ but $m(F^n(x), F^n(y)) > \varepsilon$. H is *positively expansive* if there is a constant $\varepsilon > 0$ such that for all distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that $m(H^n(x), H^n(y)) > \varepsilon$.

One-dimensional SA are very interesting models, which complexity lies somewhere between one-dimensional and two-dimensional CA. Indeed we have seen in the previous section that the latter can simulate SA, and it was shown in [5] that SA could simulate the former. A classification of one-dimensional cellular automata in terms of their dynamical behavior was given in [13]. Things are very different as soon as we get into the second dimension, as noted in [15, 14]. The question is now whether the complexity of the SA model is closer to that of the lower or the higher-dimensional CA.

The classification from [13] is no more relevant in the context of SA since the class of positively expansive SA is empty. This result can be related to the absence of positively expansive two-dimensional CA (see [15]), though the proof is much different.

Proposition 1. *There are no positively expansive SA.*

Proof. Let F a SA and $\delta = 2^{-k} > 0$. Take two distinct configurations $x, y \in \mathcal{C}$ such that $\forall i \in [-k, k], x_i = y_i = +\infty$. By infinity-preservingness, we get $\forall n \in \mathbb{N}, \forall i \in [-k, k], F^n(x)_i = F^n(y)_i = +\infty$, hence $d(F^n(x), F^n(y)) < \delta$. □

In a similar way as Theorem 4 of [13], the two different notions of stability, equicontinuity and ultimate periodicity, are proved to be equivalent. The proof uses the following lemma, which allows a better understanding of equicontinuity for SA.

Lemma 1. *If F is an equicontinuous SA, then the variation of a pile is bounded by the differences in an initial neighborhood, i.e., there is some integer l such that all configurations $x \in \mathcal{C}$ with $x_0 = 0$ satisfy*

$$\forall n \in \mathbb{N}, \quad |F^n(x)_0| \leq \max_{\substack{|i| \leq l \\ |x_i| < \infty}} |x_i| \ .$$

Proof. If F is equicontinuous, in particular, for $\varepsilon = 2^0$, there exists $\delta = 2^{-l}$ such that for all $x, y \in \mathcal{C}$, if $C_l^0(x) = C_l^0(y)$, then $\forall n \in \mathbb{N}, C_0^0(F^n(x)) = C_0^0(F^n(y))$. First, consider a configuration y that has *infinite l-neighborhood*, i.e., $\forall i \in [-l, l], y_i \notin [-l, l]$. Let z defined by $z_i = +\infty$ if $y_i \geq 0$ and $z_i = -\infty$ if $y_i < 0$, in such way that $C_l^0(y) = C_l^0(z)$. Then $\forall n \in \mathbb{N}, C_0^0(F^n(y)) = C_0^0(F^n(z)) = C_0^0(z)$, i.e., $F^n(y)_0 < -l \Leftrightarrow y_0 < -l$ and $F^n(y)_0 > l \Leftrightarrow y_0 > l$.

Now, let x a configuration such that $x_0 = 0$ and $m = \max_{\substack{|i| \leq l \\ |x_i| < \infty}} |x_i|$. Notice that $\rho^{l+m+1}(x)$ has infinite l-neighborhood, since $x_i \leq m$ or $x_i = +\infty$ for $|i| \leq l$. Hence, as seen before, $\forall n \in \mathbb{N}, F^n(x)_0 \leq m$. A symmetrical reasoning on $\rho^{-l-m-1}(x)$ gives $\forall n \in \mathbb{N}, |F^n(x)_0| \leq m$. □

Proposition 2. *If F is a SA, then the following statements are equivalent:*

1. *F is equicontinuous.*
2. *F is ultimately periodic.*

Proof. 2⇒1: Let F be such that with $F^{n+p} = F^n$ for some $n \geq 0$, $p > 0$. Since $F, F^2, \ldots, F^{n+p-1}$ are uniformly continuous maps, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{C}$ with $\mathsf{d}(x, y) < \delta$, it holds that $\forall q \in \mathbb{N}$, $q < n+p$, $\mathsf{d}(F^q(x), F^q(y)) < \varepsilon$. Since for any $t \in \mathbb{N}$, F^t is equal to some F^q with $q < n+p$, the map F is equicontinuous.
1⇒2: Let F an equicontinuous SA and l, as in Lemma 1, such that for all $x, y \in \mathcal{C}$, if $C_l^0(x) = C_l^0(y)$, then $\forall n \in \mathbb{N}, C_0^0(F^n(x)) = C_0^0(F^n(y))$. Let x a configuration such that x_0 is finite. Should we vertically shift it, we can assume that $x_0 = 0$. Let y the configuration defined by $y_i = \max(\min(x_i, l+1), -l-1)$ if $-l \leq i \leq l$ and $y_i = +\infty$ otherwise, in such way that $C_l^0(x) = C_l^0(y)$. Lemma 1 gives $\forall i \in [-l, l], \forall n \in \mathbb{N}, |F^n(y)_i| \leq 2l + 2$. We can thus find some preperiod q_y and some period p_y such that $\forall i \in [-l, l], F^{p_y+q_y}(y)_i = F^{q_y}(y)_i$. The other piles being infinite, hence invariant, we get $F^{p_y+q_y}(y) = F^{q_y}(y)$. As a consequence, $C_0^0(F^{p_y+q_y}(x)) = C_0^0(F^{q_y}(x))$. Define p (resp., q) as the least common multiple (resp., maximum) of all p_y (resp., q_y) for y a configuration such that $|y_i| \leq l+1$ if $|i| \leq l$ and $y_i = +\infty$ otherwise. Then, for every configuration x, $C_0^0(F^{p+q}(x)) = C_0^0(F^q(x))$; in particular for vertical and horizontal shifts of x, which give $F^{p+q}(x) = F^q(x)$. □

An important open question in the dynamical behavior of SA is the existence of non-sensitive SA without any equicontinuity configuration. An example for two-dimensional CA is given in [14], but their method can hardly be adapted for SA. However, we conjecture that such SA exist, which would lead to a classification of SA into four classes: equicontinuous, admitting an equicontinuity configuration (but not equicontinuous), non-sensitive without equicontinuity configurations, sensitive.

Another issue is the decidability of these classes. In [4], the undecidability of SA ultimate periodicity was proved on the particular subsets of finite and periodic configurations. It follows directly that equicontinuity on these subsets is undecidable. The question is still open for the whole configuration space $\mathcal{C}$.

4 Nilpotency

In this section we give a definition of nilpotency, the most stable dynamics of a dynamical system, adapted to SA. Then, we prove that this nilpotency behavior is undecidable (Theorem 3).

4.1 Nilpotency of CA

Here we recall the basic definitions and properties of nilpotent CA. Nilpotency is among the simplest dynamical behavior that an automaton may exhibit.

Intuitively, a system is nilpotent if it destroys every piece of information in any initial configuration, reaching a common constant configuration after a while. For CA, this is formalized as follows.

Definition 3 (CA nilpotency [7, 12]). *A CA G is nilpotent if*

$$\exists c \in A, \quad \exists N \in \mathbb{N} \quad \forall x \in A^{\mathbb{Z}^d}, \quad \forall n \geq N, \quad G^n(x) = \underline{c} \ .$$

Remark that, because of the compactness of the CA configuration space, a CA is nilpotent if and only if it is nilpotent for all initial configurations (i.e., all configurations eventually reach the same configuration).

Spreading CA have the following stronger characterization.

Proposition 3 ([6]). *A CA of global rule G, with spreading state 0, is nilpotent if and only if for all configurations $x \in A^{\mathbb{Z}^d}$, $\lim_{n\to\infty} \mathsf{d}_T(G^n(x), \underline{0}) = 0$.*

This equivalence is very useful since the CA nilpotency has been proved undecidable in [12], even for the restricted class of spreading CA.

Theorem 2 ([12]). *For a given state s, it is undecidable to know whether a cellular automaton with spreading state s is nilpotent.*

4.2 Nilpotency of SA

A direct adaptation of Definition 3 to SA is vain. Indeed, assume F is a SA of radius r. For any $k \in \mathbb{Z}^d$, consider the configuration $x^k \in \mathcal{B}$ defined by $x^k_0 = k$ and $x^k_i = 0$ for any $i \in \mathbb{Z}^d \setminus \{0\}$. Since the pile of height k may decrease at most by r during one step of evolution of the SA, and the other piles may increase at most by r, x^k requires at least $\lceil k/2r \rceil$ steps to reach a constant configuration. Thus, there exists no common integer n such that all configurations x^k reach a constant configuration in time n. This is a major difference with CA, which is essentially due to the unbounded set of states and to the infinity-preserving property.

Thus, we propose to label as nilpotent the SA which make every pile approach a constant value, but not necessarily reaching it ultimately. This nilpotency notion, inspired by Proposition 3, is formalized as follows for a SA F:

$$\exists c \in \mathbb{Z}, \quad \forall x \in \mathcal{C}, \quad \lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0 \ .$$

Remark that c shall not be taken in the full state set $\widetilde{\mathbb{Z}}$, because allowing infinite values for c would not correspond to the intuitive idea that a nilpotent SA “destroys” a configuration (otherwise, the raising map would be nilpotent). Anyway, this definition is not satisfying because of the vertical commutativity: two configurations which differ by a vertical shift reach two different configurations, and then no nilpotent SA may exist. A possible way to work around this issue is to make the limit configuration depend on the initial one:

$$\forall x \in \mathcal{C}, \quad \exists c \in \mathbb{Z}, \qquad \lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0 \ .$$

Again, since SA are infinity-preserving, an infinite pile cannot be destroyed (nor, for the same reason, can an infinite pile be built from a finite one). Therefore nilpotency has to involve the configurations of $\mathbb{Z}^{\mathbb{Z}^d}$, i.e., the ones without infinite piles. Moreover, every configuration $x \in \mathbb{Z}^{\mathbb{Z}^d}$ made of regular steps (i.e., in dimension 1, for all $i \in \mathbb{Z}$, $x_i - x_{i-1} = x_{i+1} - x_i$) is invariant by the SA rule (possibly composing it with the vertical shift). So it cannot reach nor approach a constant configuration. Thus, the larger reasonable set on which nilpotency might be defined is the set of bounded configurations $\mathcal{B}$. This leads to the following formal definition of nilpotency for SA.

Definition 4 (SA nilpotency).

$$\forall x \in \mathcal{B}, \quad \exists c \in \mathbb{Z}, \qquad \lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0 \ .$$

The following proposition shows that the class of nilpotent SA is nonempty. Remark that similar nilpotent SA can be constructed with any radius and in any dimension.

Proposition 4. *The SA $\mathcal{N}$ from Example 1 is nilpotent.*

Proof. Let $x \in \mathcal{B}$, let $i \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$, $x_j \geq x_i$. Clearly, after $x_{i+1} - x_i$ steps, $F_{\mathcal{N}}^{x_{i+1}-x_i}(x)_{i+1} = F_{\mathcal{N}}^{x_{i+1}-x_i}(x)_i = x_i$. By immediate induction, we obtain that for all $j \in \mathbb{Z}$ there exists $n_j \in \mathbb{N}$ such that $F_{\mathcal{N}}^{n_j}(x)_j = x_i$, hence $\lim_{n\to\infty} \mathsf{d}(F_{\mathcal{N}}^n(x), \underline{x_i}) = 0$. □

4.3 Undecidability

The main result of this section is that SA nilpotency is undecidable (Theorem 3), by reducing to it the nilpotency of spreading CA. This emphasizes the fact that the dynamical behavior of SA is very difficult to predict. We think that this result might be used as the reference undecidable problem for further questions on SA.

Problem Nil
INSTANCE: a SA $\mathcal{A} = \langle d, r, \lambda \rangle$;
QUESTION: is $\mathcal{A}$ nilpotent?

Theorem 3. *The problem **Nil** is undecidable.*

Proof. This is proved by reducing **Nil** to the nilpotency of spreading cellular automata. Remark that it is sufficient to show the result in dimension 1. Let $\mathcal{S}$ be a spreading cellular automaton $\mathcal{S} = \langle A, 1, s, g \rangle$ of global rule G, with finite set of integer states $A \subset \mathbb{N}$ containing the spreading state 0. We simulate $\mathcal{S}$ with the sand automaton $\mathcal{A} = \langle 1, r = \max(2s, \max A), f \rangle$ of global rule F using the following technique, also developed in [5]. Let $\xi : A^{\mathbb{Z}} \to \mathcal{B}$ be a function which inserts markers every two cells in the CA configuration to obtain a bounded SA configuration. These markers allow the local rule of the SA to know the absolute state of each pile and behave as the local rule of the CA. To simplify the proof, the markers are put at height 0 (see Figure 2):

$$\forall y \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad \xi(y)_i = \begin{cases} 0\,(\text{marker}) & \text{if } i \text{ is odd} \\ y_{i/2} & \text{otherwise.} \end{cases},$$

This can lead to an ambiguity when all the states in the neighborhood of size $4s+1$ are at state 0, as shown in the picture. But as in this special case the state 0 is quiescent for g, this is not a problem: the state 0 is preserved, and markers are preserved.

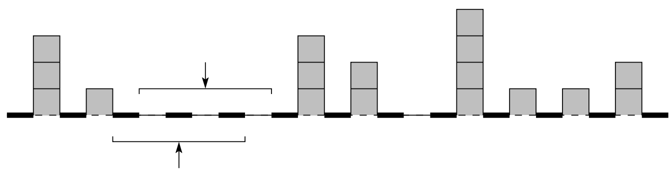

Fig. 2. Illustration of the function ξ used in the simulation of the spreading CA $\mathcal{S}$ by $\mathcal{A}$. The thick segments are the markers used to distinguish the states of the CA, put at height 0. There is an ambiguity for the two piles indicated by the arrows: with a radius 2, the neighborhoods are the same, although one of the piles is a marker and the other the state 0.

The local rule f is defined as follows, for all ranges $R \in \mathcal{R}_r^1$,

$$f(R) = \begin{cases} 0 & \text{if } R_{-2s+1}, R_{-2s+3}, \dots, R_{-1}, R_1, \dots, R_{2s-1} \in A \;, \\ g(R_{-2s}+a, R_{-2s+2}+a, \dots, R_{-2}+a, a, R_2+a, \dots, R_{2s}+a) - a & \\ \qquad \text{if } R_{-2s+1} = R_{-2s+3} = \dots = R_{2s-1} = a < 0 \text{ and } -a \in A \;. \end{cases} \tag{1}$$

The first case is for the markers (and state 0) which remain unchanged, the second case is the simulation of g in the even piles. As proved in [5], for any $y \in A^{\mathbb{Z}}$ it holds that $\xi(G(y)) = F(\xi(y))$. The images by f of the remaining ranges will be defined later on, first a few new notions need to be introduced.

A sequence of consecutive piles $(x_i, \dots, x_j)$ from a configuration $x \in \mathcal{B}$ is said to be *valid* if it is part of an encoding of a CA configuration, i.e., $x_i = x_{i+2} = \dots = x_j$ (these piles are markers) and for all $k \in \mathbb{N}$ such that $0 \leq k < (j-i)/2$, $x_{i+2k+1} - x_i \in A$ (this is a valid state). We extend this definition to configurations, when $i = -\infty$ and $j = +\infty$, i.e., $x \in \rho^c \circ \xi(A^{\mathbb{Z}})$ for a given $c \in \mathbb{Z}$ ($x \in \mathcal{B}$ is valid if it is the raised image of a CA configuration). A sequence (or a configuration) in *invalid* if it is not valid.

First we show that starting from a valid configuration, the SA $\mathcal{A}$ is nilpotent if and only if $\mathcal{S}$ is nilpotent. This is due to the fact that we chose to put the markers at height 0, hence for any valid encoding of the CA $x = \rho^c \circ \xi(y)$, with $y \in A^{\mathbb{Z}}$ and $c \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathsf{d}_T(G^n(y), \underline{0}) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0 \ .$$

It remains to prove that for any invalid configuration, $\mathcal{A}$ is also nilpotent. In order to have this behavior, we add to the local rule f the rules of the nilpotent automaton $\mathcal{N}$ for every invalid neighborhood of width $4s+1$. For all ranges $R \in \mathcal{R}_r^1$ not considered in Equation (1),

$$f(R) = \begin{cases} -1 & \text{if } R_{-r} < 0 \text{ or } R_{-r+1} < 0 \text{ or } \cdots \text{ or } R_r < 0 \ , \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Let $x \in \mathcal{B}$ be an invalid configuration. Let $k \in \mathbb{Z}$ be any index such that $\forall l \in \mathbb{Z}$, $x_l \geq x_k$. Let $i, j \in \mathbb{Z}$ be respectively the lowest and greatest indices such that $i \leq k \leq j$ and $(x_i, \ldots, x_j)$ is valid (i may equal j). Remark that for all $n \in \mathbb{N}$, $(F^n(x)_i, \ldots, F^n(x)_j)$ remains valid. Indeed, the markers are by construction the lowest piles and Equations (1) and (2) do not modify them. The piles coding for non-zero states can change their state by Equation (1), or decrease it by 1 by Equation (2), which in both cases is a valid encoding. Moreover, the piles x_{i-1} and x_{j+1} will reach a valid value after a finite number of steps: as long as they are invalid, they decrease by 1 until they reach a value which codes for a valid state. Hence, by induction, for any indices $a, b \in \mathbb{Z}$, there exists $N_{a,b}$ such that for all $n \geq N_{a,b}$ the sequence $(F^n(x)_a, \ldots, F^n(x)_b)$ is valid.

In particular, after $N_{-2Nr-1,2Nr+1}$ step, there is a valid sequence of length $4Nr+3$ centered on the origin (here, N is the number of steps needed by $\mathcal{S}$ to reach the configuration $\underline{0}$, given by Definition 3). Hence, after $N_{-2Nr,2Nr} + N$ steps, the local rule of the CA $\mathcal{S}$ applied on this valid sequence leads to 3 consecutive zeros at positions $-1, 0, 1$. All these steps are illustrated on Figure 3.

In a similar way, we prove that for all $n \geq N_{-2Nr-k,2Nr+k} + N$, the sequence $(F^n(x)_{-k}, \ldots, F^n(x)_k)$ is constant and does not evolve as n grows. Therefore, there exists $c \in \mathbb{Z}$ such that $\lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0$. We just proved that $\mathcal{A}$ is nilpotent, i.e., $\lim_{n\to\infty} \mathsf{d}(F^n(x), \underline{c}) = 0$ for all $x \in \mathcal{B}$, if and only if $\mathcal{S}$ is nilpotent (because of the equivalence of definitions given by Proposition 3), so **Nil** is undecidable (Theorem 2). □

5 Conclusion

In this article we have continued the study of sand automata, using the compact topology on the SA configuration space introduced in [8]. This topology, inspired by the topology on CA, may facilitate studies about dynamical and topological properties of SA, as for the proof of the equivalence between equicontinuity and ultimate periodicity (Proposition 2).

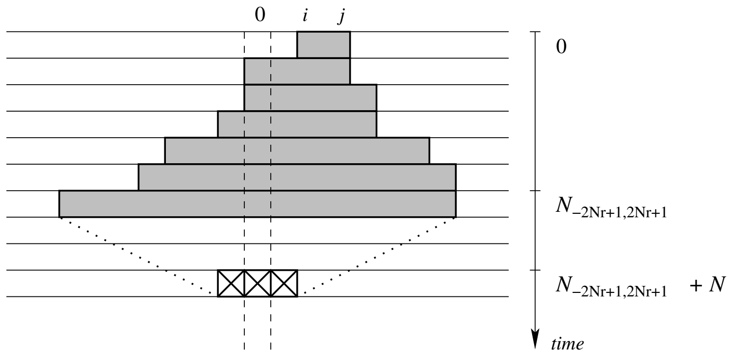

Fig. 3. Destruction of the invalid parts. The lowest valid sequence (in gray) extends until it is large enough. Then after N other steps the 3 central piles (hatched) are destroyed because the rule of the CA is applied correctly.

Then, we have given a definition of nilpotency. Although it differs from the standard one for CA, it captures the intuitive idea that a nilpotent automaton "destroys" configurations. Finally, we have proved that SA nilpotency is undecidable (Theorem 3). This fact enhances the idea that the behavior of a SA is hard to predict. We also think that this result might be used as a fundamental undecidability result, which could be reduced to other SA properties.

Besides, in the context of CA, nilpotency clearly implies ultimate periodicity. It appears that with our definitions, nilpotency of SA is not necessarily a particular case of ultimate periodicity ($\mathcal{N}$ is not ultimately periodic). However, it would be interesting to see if it could be linked to other weaker stability notions.

Moreover, the study of global properties such as injectivity and surjectivity and their corresponding dimension-dependent decidability problems could help to understand if d-dimensional SA look more like d-dimensional or $d+1$-dimensional CA. Unfortunately, deciding these dynamical properties remains a major problem. Similarly, it would be interesting to solve the open question of the dichotomy between sensitive SA and those with equicontinuous configurations. A potential counter-example would give a more precise idea of the dynamical behaviors represented by SA.

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