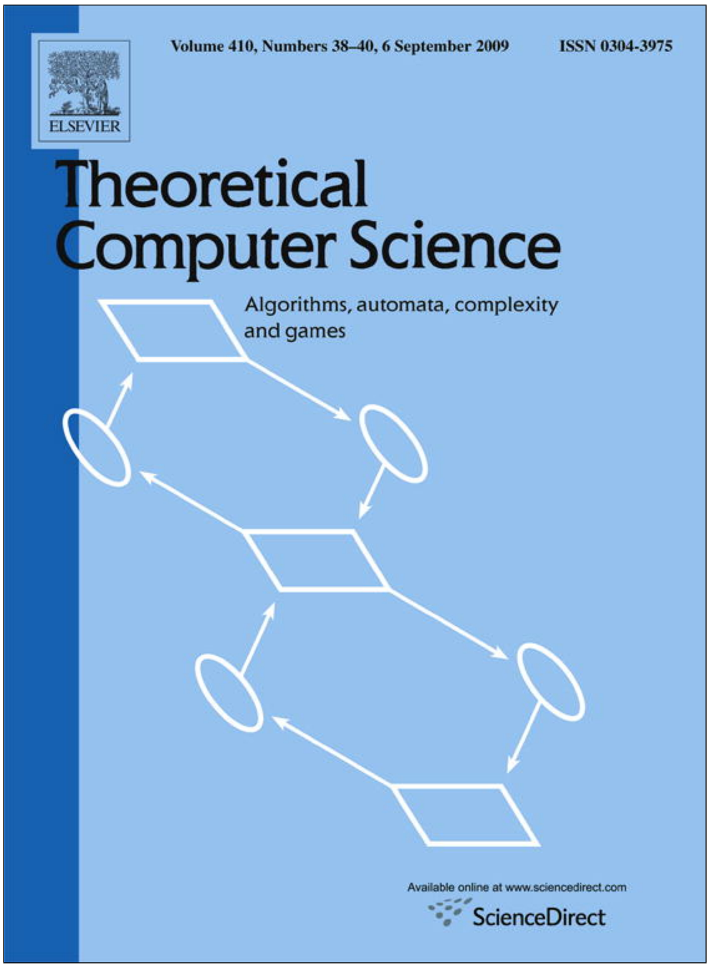

Volume 410, Numbers 38–40, 6 September 2009
ISSN 0304-3975
ELSEVIER
Theoretical Computer Science
Algorithms, automata, complexity and games
Available online at www.sciencedirect.com
ScienceDirect

Theoretical Computer Science 410 (2009) 3962–3974

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Sand automata as cellular automata[☆,☆☆]

Alberto Dennunzio [a,*], Pierre Guillon [b], Benoît Masson [c]

[a] Università degli studi di Milano-Bicocca, Dipartimento di Informatica Sistemistica e Comunicazione, viale Sarca 336, 20126 Milano, Italy
[b] Université Paris-Est, Laboratoire d'Informatique de l'Institut Gaspard Monge, UMR CNRS 8049, 5 bd Descartes, 77454 Marne la Vallée Cedex 2, France
[c] Laboratoire d'Informatique Fondamentale de Marseille (LIF)-CNRS, Aix-Marseille Université, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France

A R T I C L E I N F O

Article history:
Received 7 May 2008
Received in revised form 8 April 2009
Accepted 9 June 2009
Communicated by B. Durand

Keywords:
Sand automata
Cellular automata
Dynamical systems
Nilpotency
Undecidability

A B S T R A C T

In this paper, we exhibit a strong relation between the sand automata configuration space and the cellular automata configuration space. This relation induces a compact topology for sand automata, and a new context in which sand automata are homeomorphic to cellular automata acting on a specific subshift. We show that the existing topological results for sand automata, including the Hedlund-like representation theorem, still hold. In this context, we give a characterization of cellular automata which are sand automata, and study some dynamical behaviors such as equicontinuity. Furthermore, we deal with simple sand automata. We show that the classical definition of nilpotency is not meaningful for sand automata. Then, we introduce the suitable new notion of flattening sand automata. Finally, we prove that this simple dynamical behavior is undecidable.

1. Introduction

Self-organized criticality (SOC) is a common phenomenon observed in a huge variety of processes in physics, biology and computer science. A SOC system evolves to a "critical state" after some finite transient. Any perturbation, no matter how small, of the critical state generates a deep reorganization of the whole system. Then, after some other finite transient, the system reaches a new critical state and so on. Examples of SOC systems are: sandpiles, snow avalanches, star clusters in the outer space, earthquakes, forest fires, load balance in operating systems [6,4,5,3,28]. Among them, sandpile models are a paradigmatic formal model for SOC systems [18,19].

In [10], the authors introduced sand automata as a generalization of sandpile models and transposed them in the setting of discrete dynamical systems. A key-point of [10] was to introduce a (locally compact) metric topology to study the dynamical behavior of sand automata. A first and important result was a fundamental representation theorem similar to the well-known theorem of Hedlund for cellular automata [21,10]. In [8,9], the authors investigate sand automata by dealing with some basic set properties and decidability issues.

In this paper we continue the study of sand automata. First of all, we introduce a different metric on configurations (i.e., spatial distributions of sand grains). This metric is defined by means of the relation between sand automata and cellular automata [9,17]. With the induced topology, the configuration set turns out to be a compact (and not only locally compact), perfect and totally disconnected space. The "strict" compactness gives a better topological background to study the behavior

☆ This work has been supported by the Interlink/MIUR project "Cellular Automata: Topological Properties, Chaos and Associated Formal Languages", by the ANR Blanc "Projet Sycomore" and by the PRIN/MIUR project "Mathematical Aspects and Forthcoming Applications of Automata and Formal Languages".

☆☆ Some results of this paper have been presented at JAC 2008 [A. Dennunzio, P. Guillon, B. Masson, Topological properties of sand automata as cellular automata, in: 1st Symposium on Cellular Automata, JAC'08, 2008, pp. 216–227] and IFIP TCS 2008 [A. Dennunzio, P. Guillon, B. Masson, Stable dynamics of sand automata, in: 5th IFIP International Conference on Theoretical Computer Science, TCS'08, vol. 273, Springer, 2008, pp. 157–169] conferences.

* Corresponding author.
E-mail addresses: dennunzio@disco.unimib.it (A. Dennunzio), pierre.guillon@univ-mlv.fr (P. Guillon), benoit.masson@lif.univ-mrs.fr (B. Masson).

0304-3975/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2009.06.016

of sand automata (and in general of discrete dynamical systems). In fact, compactness provides lots of very useful results which help in the investigation of several dynamical properties [2,23]. We show that all the topological results from [10] still hold, in particular the Hedlund-like representation theorem remains valid with the compact topology [17]. Moreover, the compact topology generates a new context in which any sand automaton is homeomorphic to a cellular automaton defined on a subset of its usual domain. We show that it is possible to decide whether a given cellular automaton represents, through that homeomorphism, a sand automaton. The new context helps to prove some properties about the dynamical behavior of sand automata, such as the equivalence between equicontinuity and ultimate periodicity [16].

Then, we study the flatteningness of sand automata [16]. The classical definition of nilpotency for cellular automata [12,22] is not meaningful, since it prevents any sand automaton from being nilpotent. Therefore, we consider the new notion of flatteningness, which captures the intuitive idea that the automaton destroys all the configurations: a sand automaton is flattening if all configurations get closer and closer to some uniform configuration, not necessarily reaching it. Finally, we prove that this behavior is undecidable.

The paper is structured as follows. First, in Section 2, we recall basic definitions and results about cellular automata and sand automata. Then, in Section 3, we define a compact topology and we prove some results, in particular the representation theorem and the equivalence between equicontinuity and ultimate periodicity. Finally, in Section 4, flatteningness for sand automata is defined and proved undecidable.

2. Definitions

For all $a, b \in \mathbb{Z}$ with $a \leq b$, let $[a, b] = \{a, a+1, \ldots, b\}$ and $\widetilde{[a, b]} = [a, b] \cup \{+\infty, -\infty\}$. For $a \in \mathbb{Z}$, let $[a, +\infty) = \{n \in \mathbb{Z} : n \geq a\}$. Let $\mathbb{N}_+$ be the set of positive integers. For a vector $i \in \mathbb{Z}^d$, denote by $|i|$ the infinite norm of i. When the dimension d is known without ambiguity we may write 0 for the vector $(0, \ldots, 0)$ of $\mathbb{Z}^d$.

Let A a (possibly infinite) alphabet and $d \in \mathbb{N}_+$. A d-dimensional matrix U is a function from a finite hyper-rectangle $H \subset \mathbb{Z}^d$ taking value in A and associating any entry $i \in H$ with the symbol $U_i \in A$. Denote by $\mathcal{M}^d$ the set of all the d-dimensional matrices and by $\mathcal{M}_r^d$ the subset of the matrices on the hyper-rectangle $[-r, r]^d$.

2.1. Cellular automata and subshifts

Let A be a finite alphabet. A *CA configuration* of dimension d is a function from $\mathbb{Z}^d$ to A. The set $A^{\mathbb{Z}^d}$ of all the CA configurations is called the *CA configuration space*. This space is usually equipped with the Tychonoff metric d_T defined by

$$\forall x, y \in A^{\mathbb{Z}^d}, \quad \mathrm{d}_T(x, y) = 2^{-k} \quad \text{where } k = \min\left\{|j| : j \in \mathbb{Z}^d, x_j \neq y_j\right\}.$$

The topology induced by d_T coincides with the product topology induced by the discrete topology on A. With this topology, the CA configuration space is a Cantor space: it is compact, perfect (i.e., it has no isolated point) and totally disconnected.

For any $k \in \mathbb{Z}^d$ the *shift map* $\sigma^k : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is defined by $\forall x \in A^{\mathbb{Z}^d}, \forall i \in \mathbb{Z}^d, \sigma^k(x)_i = x_{i+k}$. A function $F : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is said to be *shift-commuting* if $\forall k \in \mathbb{Z}^d, F \circ \sigma^k = \sigma^k \circ F$.

A d-dimensional *subshift* S is a closed subset of the CA configuration space $A^{\mathbb{Z}^d}$ which is shift-invariant, i.e., for any $k \in \mathbb{Z}^d$, $\sigma^k(S) \subseteq S$. Denote by x_H the matrix obtained by restricting a configuration x on a hyper-rectangle H. Let $\mathcal{F} \subseteq \mathcal{M}^d$ and let $S_{\mathcal{F}}$ be the set of configurations $x \in A^{\mathbb{Z}^d}$ such that no restriction of any shifted configuration belongs to $\mathcal{F}$, i.e., for any hyper-rectangle H and any $k \in \mathbb{Z}^d$, $\sigma^k(x)_H \notin \mathcal{F}$. The set $S_{\mathcal{F}}$ is a subshift, and $\mathcal{F}$ is called a set of forbidden patterns. Note that for any subshift S, it is possible to find a set of forbidden patterns $\mathcal{F}$ such that $S = S_{\mathcal{F}}$. A subshift S is said to be a *subshift of finite type* (SFT) if $S = S_{\mathcal{F}}$ for some finite set $\mathcal{F}$. The language of a subshift S is $\mathcal{L}(S) = \{x_H : x \in S, H \text{ hyper-rectangle}\}$ (for more on subshifts, see [25] for instance).

A *cellular automaton* is a quadruple $\langle A, d, r, g\rangle$, where A is a finite alphabet called the *state set*, d is the dimension, $r \in \mathbb{N}$ is the *radius* and $g : \mathcal{M}_r^d \to A$ is the *local rule* of the automaton. The local rule g induces a *global rule* $G : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined as follows,

$$\forall x \in A^{\mathbb{Z}^d}, \ \forall i \in \mathbb{Z}^d, \quad G(x)_i = g\left(\sigma^i(x)_{[-r,r]^d}\right).$$

Note that CAs are exactly the class of all shift-commuting functions which are (uniformly) continuous with respect to the Tychonoff metric (Hedlund's theorem from [21]). For the sake of simplicity, we will make no distinction between a CA and its global rule G.

The local rule g can be extended naturally to all finite matrices in the following way. With a little abuse of notation, for any matrix U on a hyper-rectangle H, we define $g(U)$ as the matrix obtained by the the action of G on any configuration x with $x_H = U$.

For a given CA, a state $s \in A$ is *quiescent* (resp., *spreading*) if for all matrices $U \in \mathcal{M}_r^d$ such that $\forall k \in [-r, r]^d$, (resp., $\exists k \in [-r, r]^d$) $U_k = s$, it holds that $g(U) = s$. Remark that a spreading state is also quiescent. A CA is said to be spreading if it has a spreading state. In the sequel, we will assume that for every spreading CA, the spreading state is $0 \in A$.

2.2. SA configurations

A *SA configuration* (or simply *configuration*) is a set of sand grains organized in piles and distributed all over the d-dimensional lattice $\mathbb{Z}^d$. A *pile* is represented either by an integer from $\mathbb{Z}$ (*number of grains*), or by the value $+\infty$ (*source of grains*), or by the value $-\infty$ (*sink of grains*), i.e., it is an element of $\widetilde{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$. One pile is positioned in each point of the lattice $\mathbb{Z}^d$. Formally, a configuration x is a function from $\mathbb{Z}^d$ to $\widetilde{\mathbb{Z}}$ which associates any vector $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ with the number $x_i \in \widetilde{\mathbb{Z}}$ of grains in the pile of position i. When the dimension d is known without ambiguity we note 0 the null vector of $\mathbb{Z}^d$. Denote by $\mathcal{C} = \widetilde{\mathbb{Z}}^{\mathbb{Z}^d}$ the set of all configurations. A configuration $x \in \mathcal{C}$ is said to be *constant* if there is an integer $c \in \mathbb{Z}$ such that for any vector $i \in \mathbb{Z}^d$, $x_i = c$. In that case we write $x = \underline{c}$. A configuration $x \in \mathcal{C}$ is said to be *bounded* if there exist two integers $m_1, m_2 \in \mathbb{Z}$ such that for all vectors $i \in \mathbb{Z}^d$, $m_1 \le x_i \le m_2$. Denote by $\mathcal{B}$ the set of all bounded configurations.

A *measuring device* β_r^m of precision $r \in \mathbb{N}$ and reference height $m \in \mathbb{Z}$ is a function from $\widetilde{\mathbb{Z}}$ to $\widetilde{[-r, r]}$ defined as follows

$$\forall n \in \widetilde{\mathbb{Z}}, \quad \beta_r^m(n) = \begin{cases} +\infty & \text{if } n > m + r, \\ -\infty & \text{if } n < m - r, \\ n - m & \text{otherwise.} \end{cases}$$

A measuring device is used to evaluate the relative height of two piles, with a bounded precision. This is the technical basis of the definition of cylinders, distances and ranges which are used throughout this article.

In [10], the authors equipped $\mathcal{C}$ with a metric in such a way that two configurations are at small distance if they have the same number of grains in a finite neighborhood of the pile indexed by the null vector. The neighborhood is individuated by putting the measuring device at the top of the pile if the latter contains a finite number of grains. Otherwise the measuring device is put at height 0. In order to formalize this distance, the authors introduced the notion of *cylinder*, that we rename *top cylinder*. For any configuration $x \in \mathcal{C}$, for any $r \in \mathbb{N}$, and for any $i \in \mathbb{Z}^d$, the top cylinder of x centered in i and of radius r is the d-dimensional matrix $\Delta_r^i(x) \in \mathcal{M}_r^d$ defined on the infinite alphabet $A = \widetilde{\mathbb{Z}}$ by

$$\forall k \in [-r, r]^d, \quad \left(\Delta_r^i(x)\right)_k = \begin{cases} x_i & \text{if } k = 0, \\ \beta_r^{x_i}(x_{i+k}) & \text{if } k \neq 0 \text{ and } x_i \neq \pm\infty, \\ \beta_r^0(x_{i+k}) & \text{otherwise.} \end{cases}$$

In dimension 1 and for a configuration $x \in \mathcal{C}$, we have

$$\Delta_r^i(x) = \left(\beta_r^{x_i}(x_{i-r}), \ldots, \beta_r^{x_i}(x_{i-1}), x_i, \beta_r^{x_i}(x_{i+1}), \ldots, \beta_r^{x_i}(x_{i+r})\right)$$

if $x_i \neq \pm\infty$, while

$$\Delta_r^i(x) = \left(\beta_r^0(x_{i-r}), \ldots, \beta_r^0(x_{i-1}), 0, \beta_r^0(x_{i+1}), \ldots, \beta_r^0(x_{i+r})\right)$$

if $x_i = \pm\infty$.

By means of top cylinders, the distance $\mathrm{d}' : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ has been introduced as follows:

$$\forall x, y \in \mathcal{C}, \quad \mathrm{d}'(x, y) = 2^{-k} \quad \text{where } k = \min\left\{r \in \mathbb{N} : \Delta_r^0(x) \neq \Delta_r^0(y)\right\}.$$

Proposition 2.1 (*[10,9]*). *With the topology induced by* d', *the configuration space is locally compact, perfect and totally disconnected.*

2.3. Sand automata

For any integer $r \in \mathbb{N}$, for any configuration $x \in \mathcal{C}$ and any index $i \in \mathbb{Z}^d$ with $x_i \neq \pm\infty$, the *range* of center i and radius r is the d-dimensional matrix $R_r^i(x) \in \mathcal{M}_r^d$ on the finite alphabet $A = \widetilde{[-r, r]} \cup \perp$ such that

$$\forall k \in [-r, r]^d, \quad \left(R_r^i(x)\right)_k = \begin{cases} \perp & \text{if } k = 0, \\ \beta_r^{x_i}(x_{i+k}) & \text{otherwise.} \end{cases}$$

The range is used to define a sand automaton. It is a kind of top cylinder, where the observer is always located on the top of the pile x_i (called the *reference*). It represents what the automaton is able to see at position i. Sometimes the central $\perp$ symbol may be omitted for simplicity sake. The set of all possible ranges of radius r, in dimension d, is denoted by $\mathcal{R}_r^d$.

A *sand automaton* (SA) is a deterministic finite automaton working on configurations. Each pile is updated synchronously, according to a local rule which computes the variation of the pile by means of the range. Formally, an SA is a triple $\langle d, r, f \rangle$, where d is the dimension, r is the *radius* and $f : \mathcal{R}_r^d \to [-r, r]$ is the *local rule* of the automaton. By means of the local rule, one can define the *global rule* $F : \mathcal{C} \to \mathcal{C}$ as follows

$$\forall x \in \mathcal{C},\ \forall i \in \mathbb{Z}^d, \quad F(x)_i = \begin{cases} x_i & \text{if } x_i = \pm\infty, \\ x_i + f(R_r^i(x)) & \text{otherwise.} \end{cases}$$

Remark that the radius r of the automaton has three different meanings: it represents at the same time the number of measuring devices in every dimension of the range (number of piles in the neighborhood), the precision of the measuring

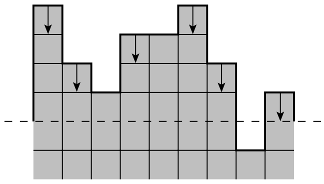


Fig. 1. Illustration of the behavior of $\mathcal{N}$.

devices in the range, and the highest return value of the local rule (variation of a pile). It guarantees that there are only a finite number of ranges and return values, so that the local rule has a finite description.

The following example illustrates a sand automaton whose behavior will be studied in Section 4. For more examples, we refer to [9].

Example 2.1 (*The Automaton $\mathcal{N}$*). This automaton destroys a configuration by collapsing all piles towards the lowest one. It decreases a pile when there is a lower pile in the neighborhood (see Fig. 1). Let $\mathcal{N} = \langle 1, 1, f_{\mathcal{N}} \rangle$ of global rule $F_{\mathcal{N}}$ where

$$\forall a, b \in [\widetilde{-1, 1}], \quad f_{\mathcal{N}}(a, b) = \begin{cases} -1 & \text{if } a < 0 \text{ or } b < 0, \\ 0 & \text{otherwise.} \end{cases}$$

When no misunderstanding is possible, we identify an SA with its global rule F. For any $k \in \mathbb{Z}^d$, we extend the definition of the *shift map* to $\mathcal{C}$, $\sigma^k : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d, \sigma^k(x)_i = x_{i+k}$. The *raising map* $\rho : \mathcal{C} \to \mathcal{C}$ is defined by $\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d, \rho(x)_i = x_i + 1$. A function $F : \mathcal{C} \to \mathcal{C}$ is said to be *vertical-commuting* if $F \circ \rho = \rho \circ F$. A function $F : \mathcal{C} \to \mathcal{C}$ is *infinity-preserving* if for any configuration $x \in \mathcal{C}$ and any vector $i \in \mathbb{Z}^d$, $F(x)_i = +\infty$ if and only if $x_i = +\infty$ and $F(x)_i = -\infty$ if and only if $x_i = -\infty$.

Remark that the raising map ρ is the sand automaton of radius 1 whose local rule always returns 1. On the opposite, the horizontal shifts σ^k are not sand automata: they destroy infinite piles by moving them, which is not permitted by the definition of the global rule.

Theorem 2.2 (*Hedlund-Like Theorem [10,9]*). *The class of SAs is exactly the class of shift and vertical-commuting, infinity-preserving functions $F : \mathcal{C} \to \mathcal{C}$ which are continuous w.r.t. the metric* d'.

3. Topology and dynamics

In this section we introduce a compact topology on the SA configuration space by means of a relation between SA and CA. With this topology, the Hedlund-like theorem still holds and each SA turns out to be homeomorphic to some CA acting on a specific subshift. We also characterize CAs whose action on this subshift represents an SA. Finally, we prove that equicontinuity is equivalent to ultimate periodicity, and that expansivity is a very strong notion: there exist no positively expansive SAs.

3.1. A compact topology for SA configurations

The set $\widetilde{\mathbb{Z}}$ can be endowed with the Alexandroff topology, i.e., the topology based on intervals $[a, b]$, where $-\infty \le a \le b \le +\infty$. Hence, $\mathcal{C}$ can be endowed with the corresponding product topology. Let us show that this topology allows a nice conjugacy of $\mathcal{C}$ into some subshift.

From [9], we know that any SA of dimension d can be simulated by a suitable CA of dimension $d+1$ (and also any CA can be simulated by an SA). In particular, a d-dimensional SA configuration can be seen as a $(d+1)$-dimensional CA configuration on the alphabet $A = \{0, 1\}$. This fact can be formalized by the functions $\zeta_0 : \widetilde{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ and $\zeta : \mathcal{C} \to \{0, 1\}^{\mathbb{Z}^{d+1}}$ defined as follows:

$$\forall l \in \widetilde{\mathbb{Z}}, \forall k \in \mathbb{Z}, \quad \zeta_0(l)_k = \begin{cases} 1 & \text{if } l \ge k, \\ 0 & \text{otherwise;} \end{cases}$$

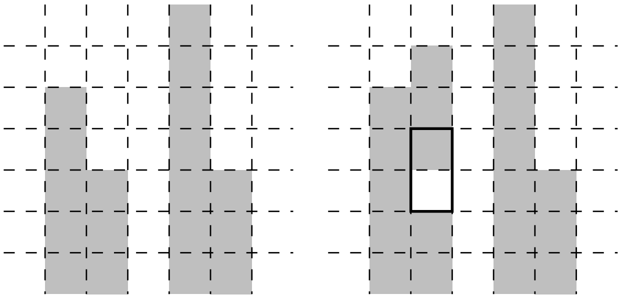
$$\forall x \in \mathcal{C}, \forall i \in \mathbb{Z}^d, \forall k \in \mathbb{Z}, \quad \zeta(x)_{(i,k)} = \zeta_0(x_i)_k\ .$$

A SA configuration $x \in \mathcal{C}$ is encoded by the CA configuration $\zeta(x) \in \{0, 1\}^{\mathbb{Z}^{d+1}}$.

Remark 3.1. Recall that $\{0, 1\}^{\mathbb{Z}}$ and $\{0, 1\}^{\mathbb{Z}^{d+1}}$ are endowed with the classical Tychonoff topology. Then ζ_0 and ζ are injective continuous functions.

Consider the matrix $K \in \mathcal{M}^{d+1}$ with 2 entries defined as $K_{(1,\dots,1,2)} = 1$ and $K_{(1,\dots,1,1)} = 0$. With a little abuse of notation, denote $S_K = S_{\{K\}}$ the subshift generated by the set of forbidden patterns $\mathcal{F} = \{K\}$.

Proposition 3.1. *The set $\zeta(\mathcal{C})$ is the subshift S_K.*



(a) Valid configuration. (b) Invalid configuration.

Fig. 2. The configuration from (a) is valid, while the configuration from (b) contains the forbidden matrix K: there is a "hole".

Proof. We can easily see that $\zeta_0(\widetilde{\mathbb{Z}})$ is the one-dimensional subshift S_K^0 of forbidden pattern $\binom{1}{0}$. By construction, we have $\zeta(\widetilde{\mathbb{Z}}^{\mathbb{Z}^d}) = \zeta_0(\widetilde{\mathbb{Z}})^{\mathbb{Z}^d}$. Hence, $\zeta(\mathcal{C})$ corresponds to the subshift $S_K = (S_K^0)^{\mathbb{Z}^d}$. $\square$

Fig. 2 illustrates the mapping ζ and the matrix $K = \binom{1}{0}$ for dimension $d = 1$. The set of SA configurations $\mathcal{C} = \widetilde{\mathbb{Z}}^{\mathbb{Z}}$ is homeomorphic to the subshift $S_K = \zeta(\mathcal{C})$ of the CA configurations set $\{0, 1\}^{\mathbb{Z}^2}$.

Definition 3.1. The distance $\mathrm{d} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is defined as follows:

$$\forall x, y \in \mathcal{C}, \quad \mathrm{d}(x, y) = \mathrm{d}_T(\zeta(x), \zeta(y)).$$

In other words, the (well-defined) distance d between two configurations $x, y \in C$ is nothing but the Tychonoff distance between configurations $\zeta(x)$ and $\zeta(y)$ in the subshift S_K. By homeomorphism, this metric corresponds to the product topology of Alexandroff topology considered above.

As an immediate consequence of the homeomorphism between $\mathcal{C}$ and S_K, the following results hold.

Proposition 3.2. *The space $\mathcal{C}$ is a compact and totally disconnected space where the open balls are clopen (i.e., closed and open) sets.*

In particular, the compactness gives that this topology does not coincide with the topology obtained as countable product of the discrete topology on $\widetilde{\mathbb{Z}}$ (not compact). Neither does it coincide with the former metric d'. The following proposition states that the new topology is a refinement of the former one.

Proposition 3.3. *Any open set O for the metric d' is open for the metric d.*

Proof. It is sufficient to show this for any open ball U of center $x \in \mathcal{C}$ and radius $\varepsilon > 0$ for metric d'. From the definition of top cylinders, such a ball can be of two different types. If x_0 is infinite, then the definition coincides with that of the open ball of center $x \in \mathcal{C}$ and radius $\varepsilon > 0$ for metric d. Now, suppose x_0 is finite. It can (nearly as) easily be seen that $U = \rho^{x_0}(V)$ where V is the open ball of center $\rho^{-x_0}(x)$ and radius ε for metric d. ρ being a homeomorphism from space $\mathcal{C}$ into itself, we can conclude U is also open for metric d. $\square$

This new metric preserves the useful property of perfectness.

Proposition 3.4. *The space $\mathcal{C}$ is perfect.*

Proof. Choose an arbitrary configuration $x \in \mathcal{C}$. For any $n \in \mathbb{N}$, let $l \in \mathbb{Z}^d$ such that $|l| = n$. We build a configuration $y \in \mathcal{C}$, equal to x except at site l, defined as follows

$$\forall j \in \mathbb{Z}^d \setminus \{l\}, \; y_j = x_j \quad \text{and} \quad y_l = \begin{cases} 1 & \text{if } x_l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 3.1, $\mathrm{d}(y, x) = 2^{-n}$. $\square$

Consider now the following notion.

Definition 3.2 (*Ground Cylinder*). For any configuration $x \in \mathcal{C}$, for any $r \in \mathbb{N}$, and for any $i \in \mathbb{Z}^d$, the *ground cylinder* of x centered on i and of radius r is the d-dimensional matrix $\Gamma_r^i(x) \in \mathcal{M}_r^d$ defined by

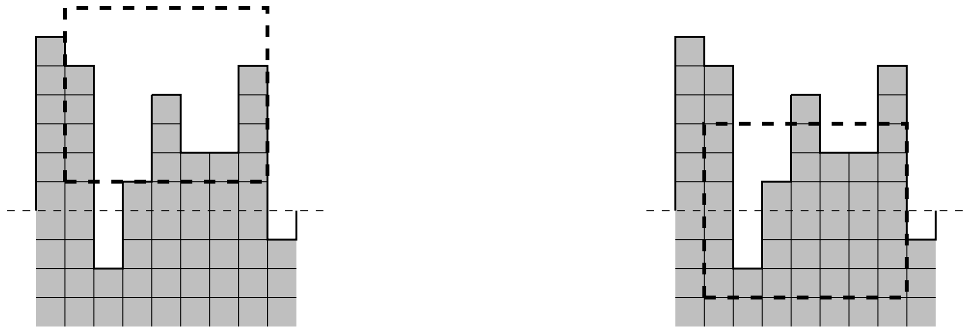
$$\forall k \in [-r, r]^d, \quad \left(\Gamma_r^i(x)\right)_k = \beta_r^0(x_{i+k}).$$

For example in dimension 1,

$$\Gamma_r^i(x) = \left(\beta_r^0(x_{i-r}), \ldots, \beta_r^0(x_i), \ldots, \beta_r^0(x_{i+r})\right).$$

Fig. 3 illustrates top cylinders and ground cylinders in dimension 1. Remark that the content of the two kinds of cylinders is totally different.

From Definition 3.1, we obtain the following expression of distance d by means of ground cylinders.



(a) Top cylinder centered on $x_i = 4$:
$\Delta_r^i(x) = (+1, -\infty, -3, 4, -2, -2, +1)$.

(b) Ground cylinder, at height 0:
$\Gamma_r^i(x) = (+\infty, -2, +1, +\infty, +2, +2, +\infty)$.

Fig. 3. Illustration of the two notions of cylinders on the same configuration, with radius 3, in dimension 1.

Remark 3.2. For any pair of configurations $x, y \in \mathcal{C}$, we have

$$\mathsf{d}(x, y) = 2^{-k} \quad \text{where } k = \min\left\{r \in \mathbb{N} : \Gamma_r^0(x) \neq \Gamma_r^0(y)\right\}.$$

As a consequence, two configurations x, y are compared by putting boxes (the ground cylinders) at height 0 around the corresponding piles indexed by 0. The integer k is the size of the smallest cylinders in which a difference appears between x and y. This way of calculating the distance d is similar to the one used for the distance d', with the difference that the measuring devices and the cylinders are now located at height 0. This is slightly less intuitive than the distance d', since it does not correspond to the definition of the local rule. However, this fact is not an issue all the more since the configuration space is compact and the representation theorem still holds with the new topology (Theorem 3.8).

For integers $d \in \mathbb{N}_+$, $r \in \mathbb{N}$, and given a matrix $U \in \mathcal{M}_r^d$ with values in $\widetilde{[-r, r]}$, denote $[U] = \left\{x \in \mathcal{C}, \Gamma_r^0(x) = U\right\}$. Note that it represents the open ball of radius 2^{-r} centered on any configuration in $[U]$. Moreover, such balls form a base of the product topology. This will facilitate reasonings on open subsets, such as in the following remark, which gives an idea on how the shift map mixes configurations.

Remark 3.3. For any $k \in \mathbb{Z}^d \backslash \{0\}$ and any ball $[U]$ of radius $r \in \mathbb{N}$, the orbit $\bigcup_{i \in \mathbb{N}} \sigma^{ik}([U])$ is dense in $\mathcal{C}$ (equivalently, it intersects all balls). Indeed, if $R \in \mathbb{N}$ and $[V]$ is a ball of radius R, then we can easily build a configuration x such that $\Gamma_R^0(x) = V$ and $\Gamma_r^{-Rk}(x) = U$. Hence, $x \in \sigma^{Rk}([U]) \cap [V]$.

3.2. SA as CA on a subshift

Let (X, m_1) and (Y, m_2) be two metric spaces. Two functions $H_1 : X \to X$, $H_2 : Y \to Y$ are (topologically) *conjugated* if there exists a homeomorphism $\eta : X \to Y$ such that $H_2 \circ \eta = \eta \circ H_1$.

Proposition 3.5. *Any d-dimensional SA F is topologically conjugated to a suitable $(d + 1)$-dimensional CA G acting on S_K.*

Proof. Let F a d-dimensional SA of radius r and local rule f. Let us define the $(d+1)$-dimensional CA G on the alphabet $\{0, 1\}$, with radius $2r$ and local rule g defined as follows (see [9] for more details). Let $M \in \mathcal{M}_r^{d+1}$ be a matrix on the finite alphabet $\{0, 1\}$ which does not contain the pattern K. If there is a $j \in [-r, r - 1]$ such that $M_{(0,\dots,0,j)} = 1$ and $M_{(0,\dots,0,j+1)} = 0$, then let $R \in \mathcal{R}_r^d$ be the range taken from M of radius r centered on $(0, \dots, 0, j)$. See Fig. 4 for an illustration of this construction in dimension $d = 1$.

The new central value depends on the height j of the central column plus its variation. Therefore, define $g(M) = 1$ if $j + f(R) \geq 0$, $g(M) = 0$ if $j + f(R) < 0$ and $g(M) = M_{(0,\dots,0)}$ (central value unchanged) if there is no such j.

By construction, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\zeta"'] & \mathcal{C} \arrow[d, "\zeta"] \\ S_K \arrow[r, "G"'] & S_K \end{tikzcd} \tag{1}$$

In other words, $G \circ \zeta = \zeta \circ F$. $\square$

Being a dynamical submodel, SA share properties with CA, some of which are proved below. However, many results which are true for CA are no longer true for SA; for instance, injectivity and bijectivity are not equivalent, as proved in [8]. Thus, SA deserve to be considered as a new model.

Corollary 3.6. *The global rule $F : \mathcal{C} \to \mathcal{C}$ of an SA is uniformly continuous w.r.t distance* d.

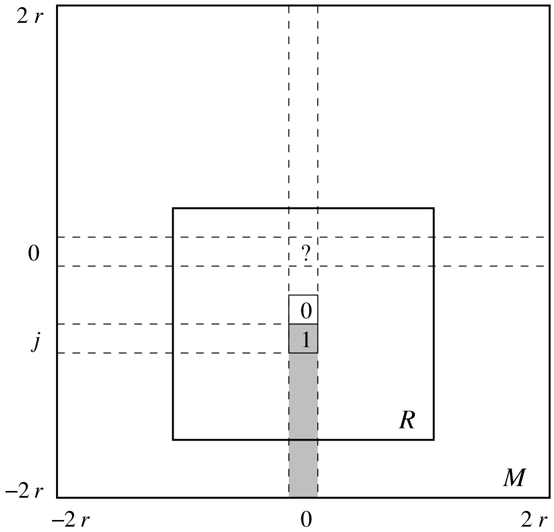


Fig. 4. Construction of the local rule g of the CA from the local rule f of the SA, in dimension 1. A range R of radius r is associated with the matrix M.

Proof. Let G be the global rule of the CA which simulates the given SA. Since the diagram (1) commutes and ζ is a homeomorphism, $F = \zeta^{-1} \circ G \circ \zeta$. Since G is a continuous map and, by Proposition 3.2, $\mathcal{C}$ is compact, hence F is uniformly continuous. $\square$

Define the projection $\pi_0 : \mathcal{C} \to \widetilde{\mathbb{Z}}$ by $\pi_0(x) = x_0$ for all $x \in \mathcal{C}$. For every $a \in \widetilde{\mathbb{Z}}$, let $P_a = \pi_0^{-1}(\{a\})$ be the clopen set of all configurations $x \in \mathcal{C}$ such that $x_0 = a$.

Lemma 3.7. *Let $F : \mathcal{C} \to \mathcal{C}$ be a continuous and infinity-preserving map. There exists $l \in \mathbb{N}$ such that for any configuration $x \in P_0$ we have $|F(x)_0| \leq l$.*

Proof. Since F is continuous and infinity-preserving, the set $F(P_0)$ is compact and included in $\pi_0^{-1}(\mathbb{Z})$. By definition of the product topology of $\mathcal{C}$, π_0 is continuous on the set $\pi_0^{-1}(\mathbb{Z})$ and in particular it is continuous on the compact $F(P_0)$. Hence $\pi_0(F(P_0))$ is a compact subset of $\widetilde{\mathbb{Z}}$ containing no infinity, and therefore it is included in some interval $[-l, l]$, where $l \in \mathbb{N}$. $\square$

Theorem 3.8. *A mapping $F : \mathcal{C} \to \mathcal{C}$ is the global transition rule of a sand automaton if and only if all the following statements hold*

(i) *F is (uniformly) continuous w.r.t the distance* d*;*
(ii) *F is shift-commuting;*
(iii) *F is vertical-commuting;*
(iv) *F is infinity-preserving.*

Proof. Let F be the global rule of an SA. By definition of SA, F is shift-commuting, vertical-commuting and infinity-preserving. From Corollary 3.6, F is also uniformly continuous.

Conversely, let F be a continuous map which is shift-commuting, vertical-commuting, and infinity-preserving. By compactness of the space $\mathcal{C}$, F is also uniformly continuous. Let $l \in \mathbb{N}$ be the integer given by Lemma 3.7. Since F is uniformly continuous, there exists an integer $r \in \mathbb{N}$ such that

$$\forall x, y \in \mathcal{C} \quad \Gamma_r^0(x) = \Gamma_r^0(y) \Rightarrow \Gamma_l^0(F(x)) = \Gamma_l^0(F(y)).$$

We now construct the local rule $f : \mathcal{R}_r^d \to [-r, r]$ of the automaton. For any input range $R \in \mathcal{R}_r^d$, set $f(R) = F(x)_0$, where x is an arbitrary configuration of P_0 such that $\forall k \in [-r, r]^d, k \neq 0, \beta_r^0(x_k) = R_k$. Note that the value of $f(R)$ does not depend on the particular choice of such a configuration $x \in P_0$. Indeed, Lemma 3.7 and uniform continuity together ensure that for any other configuration $y \in P_0$ such that $\forall k \neq 0, \beta_r^0(y_k) = R_k$, we have $F(y)_0 = F(x)_0$, since $\beta_l^0(F(x)_0) = \beta_l^0(F(y)_0)$ and $|F(y)_0| \leq l$. Thus the rule f is well-defined.

We now show that F is the global mapping of the sand automaton of radius r and local rule f. Thanks to (iv), it is sufficient to prove that for any $x \in \mathcal{C}$ and for any $i \in \mathbb{Z}^d$ with $|x_i| \neq \infty$, we have $F(x)_i = x_i + f\left(R_r^i(x)\right)$. By (ii) and (iii), for any $i \in \mathbb{Z}^d$ such that $|x_i| \neq \infty$, it holds that

$$\begin{aligned} F(x)_i &= \left[\rho^{x_i} \circ \sigma^{-i}\left(F(\sigma^i \circ \rho^{-x_i}(x))\right)\right]_i \\ &= x_i + \left[\sigma^{-i}\left(F(\sigma^i \circ \rho^{-x_i}(x))\right)\right]_i \\ &= x_i + \left[F(\sigma^i \circ \rho^{-x_i}(x))\right]_0 . \end{aligned}$$

Since $\sigma^i \circ \rho^{-x_i}(x) \in P_0$, we have by definition of f

$$F(x)_i = x_i + f\left(R_r^0(\sigma^i \circ \rho^{-x_i}(x))\right).$$

Moreover, by definition of the range, for all $k \in [-r, r]^d$,

$$R_r^0(\sigma^i \circ \rho^{-x_i}(x))_k = \beta_r^{[\sigma^i \circ \rho^{-x_i}(x)]_0}(\sigma^i \circ \rho^{-x_i}(x)_k) = \beta_r^0(x_{i+k} - x_i) = \beta_r^{x_i}(x_{i+k}),$$

hence $R_r^0(\sigma^i \circ \rho^{-x_i}(x)) = R_r^i(x)$, which leads to $F(x)_i = x_i + f\left(R_r^i(x)\right)$. □

We now deal with the following question: given a $(d+1)$-dimensional CA, does it represent a d-dimensional SA, in the sense of the conjugacy expressed by diagram (1)? In order to answer this question, we start to express the condition under which the action of a CA G can be restricted to a subshift $S_{\mathcal{F}}$, i.e., $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$ (if this fact holds, the subshift $S_{\mathcal{F}}$ is said to be G-invariant).

Lemma 3.9. *Let G and $S_{\mathcal{F}}$ be a CA and a subshift of finite type, respectively. The condition $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$ is satisfied iff for any $U \in \mathcal{L}(S_{\mathcal{F}})$ it holds that $g(U) \neq V$ for any $V \in \mathcal{F}$ defined on the same hyper-rectangle than $g(U)$.*

Proof. Suppose that $G(S_{\mathcal{F}}) \subseteq S_{\mathcal{F}}$. Choose arbitrarily $V \in \mathcal{F}$ and $U \in \mathcal{L}(S_{\mathcal{F}})$, with $g(U)$ and V defined on the same hyper-rectangle. Let $x \in S_{\mathcal{F}}$ containing the matrix U. Since $G(x) \in S_{\mathcal{F}}$, then $g(U) \in \mathcal{L}(S_{\mathcal{F}})$, and so $g(U) \neq V$. Conversely, if $x \in S_{\mathcal{F}}$ and $G(x) \notin S_{\mathcal{F}}$, then there exist $U \in \mathcal{L}(S_{\mathcal{F}})$ and $V \in \mathcal{F}$ with $g(U) = V$. □

The following lemma gives a sufficient and necessary condition under which the action of a CA G on configurations of the G-invariant subshift $S_K = \mathcal{C}$ preserves any column whose cells have the same value.

Lemma 3.10. *Let G be a $(d+1)$-dimensional CA with radius r and state set $\{0, 1\}$ and S_K be the subshift representing SA configurations. The following two statements are equivalent:*

(i) *for any $x \in S_K$ with $x_{(0,\dots,0,i)} = 1$ (resp., $x_{(0,\dots,0,i)} = 0$) for all $i \in \mathbb{Z}$, it holds that $G(x)_{(0,\dots,0,i)} = 1$ (resp., $G(x)_{(0,\dots,0,i)} = 0$) for all $i \in \mathbb{Z}$.*

(ii) *for any $U \in \mathcal{M}_r^d \cap \mathcal{L}(S_K)$ such that for all $k \in [-r, r]$, we have $U_{(0,\dots,0,k)} = 1$ (resp., $U_{(0,\dots,0,k)} = 0$), it holds that $g(U) = 1$ (resp., $g(U) = 0$).*

Proof. Suppose that (i) is true. Let $U \in \mathcal{M}_r^d \cap \mathcal{L}(S_K)$ be a matrix such that for any $k \in [-r, r]$, $U_{(0,\dots,0,k)} = 1$ and let $x \in S_K$ be a configuration such that $x_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$ and $x_{[-r,r]^d} = U$. Then $g(U) = G(x)_{(0,\dots,0,0)} = 1$.

Conversely, let $x \in S_K$ with $x_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$. Applying (ii), by shift-invariance we obtain $G(x)_{(0,\dots,0,i)} = 1$ for all $i \in \mathbb{Z}$. □

Theorem 3.8, Lemmas 3.9 and 3.10 immediately lead to the following conclusion.

Proposition 3.11. *It is decidable to check whether a given $(d+1)$-dimensional CA corresponds to a d-dimensional SA.*

3.3. Some dynamical behaviors

SAs are very interesting dynamical systems, which in some sense "lie" between d-dimensional and $(d+1)$-dimensional CAs. Indeed, we have seen in the previous section that the latter can simulate d-dimensional SAs, which can, in turn, simulate d-dimensional CAs. For the dimension $d = 1$, a classification of CAs in terms of their dynamical behavior was given in [24]. Things are very different as soon as we get into dimension $d = 2$, as noted in [27,26]. The question is now whether the complexity of the SA model is closer to that of the lower or the higher-dimensional CA.

Let (X, m) be a metric space and let $H : X \to X$ be a continuous application. An element $x \in X$ is an *equicontinuity* point for H if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. The map H is *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $m(x, y) < \delta$ implies that $\forall n \in \mathbb{N}$, $m(H^n(x), H^n(y)) < \varepsilon$. If X is compact, H is equicontinuous iff all elements of X are equicontinuity points. An element $x \in X$ is *ultimately periodic* for H if there exist two integers $n \geq 0$ (the preperiod) and $p > 0$ (the period) such that $H^{n+p}(x) = H^n(x)$. H is *ultimately periodic* if there exist $n \geq 0$ and $p > 0$ such that $H^{n+p} = H^n$. H is *sensitive* (to the initial conditions) if there is a constant $\varepsilon > 0$ such that for all points $x \in X$ and all $\delta > 0$, there is a point $y \in X$ and an integer $n \in \mathbb{N}$ such that $m(x, y) < \delta$ but $m(F^n(x), F^n(y)) > \varepsilon$. H is *positively expansive* if there is a constant $\varepsilon > 0$ such that for all distinct points $x, y \in X$, there exists $n \in \mathbb{N}$ such that $m(H^n(x), H^n(y)) > \varepsilon$.

The compactness of the space $\mathcal{C}$ and the topological conjugacy between an SA and some CA acting on S_K help to adapt some properties of CA to SA. In particular, the same characterization of equicontinuity holds.

Lemma 3.12. *If F is an equicontinuous SA, then the variation of a pile is bounded by the measuring device in an initial neighborhood, i.e., there is some integer l such that for all configurations $x \in \mathcal{C}$ and all $i \in \mathbb{Z}^d$ with $|x_i| < \infty$, we have:*

$$\forall n \in \mathbb{N}, \quad |F^n(x)_i - x_i| \leq l + \max_{|j-i| \leq 2l, |x_j| < \infty} |x_j - x_i|.$$

Proof. If F is equicontinuous, in particular, for $\varepsilon = 2^0$, there exists $\delta = 2^{-l}$ such that for all $x, y \in \mathcal{C}$, if $\Gamma_l^0(x) = \Gamma_l^0(y)$, then $\forall n \in \mathbb{N}$, $\Gamma_0^0(F^n(x)) = \Gamma_0^0(F^n(y))$. More generally, vertical and shift commutativities gives that for any $k \in \mathbb{N}$, if $\Gamma_{k+l}^0(x) = \Gamma_{k+l}^0(y)$, then $\forall n \in \mathbb{N}$, $\Gamma_k^0(F^n(x)) = \Gamma_k^0(F^n(y))$.

- First, consider a configuration y which has *infinite* $2l$*-neighborhood*, i.e., for all $i \in [-2l, 2l]^d$, $y_i \notin [-2l, 2l]$. Let z defined by $z_i = +\infty$ if $y_i \geq 0$ and $z_i = -\infty$ if $y_i < 0$, in such a way that $\Gamma^0_{2l}(y) = \Gamma^0_{2l}(z)$. Note that $F(z) = z$ since F is infinity-preserving. Then $\forall n \in \mathbb{N}$, $\Gamma^0_l(F^n(y)) = \Gamma^0_l(F^n(z)) = \Gamma^0_l(z)$. In particular, $F^n(y)_0 < -l \Leftrightarrow y_0 < -2l$ and $F^n(y)_0 > l \Leftrightarrow y_0 > 2l$.
- Now, let $x \in \mathcal{C}$ such that $x_0 = 0$, and $m = \max_{|i| \leq 2l, |x_i| < \infty} |x_i|$. Notice that $\rho^{2l+m+1}(x)$ has infinite $2l$-neighborhood, since $x_i \geq -m$ or $x_i = -\infty$, for $|i| \leq 2l$. Hence, as seen before, $\forall n \in \mathbb{N}$, $F^n(\rho^{2l+m+1}(x))_0 > l$. Using vertical commutativity, this yields $F^n(x)_0 + 2l + m + 1 > l$; hence $F^n(x)_0 \geq -m - l$. A symmetrical argument with $\rho^{-2l-m-1}(x)$ gives $\forall n \in \mathbb{N}$, $F^n(x)_0 \leq m + l$.
- For any $x \in \mathcal{C}$ such that x_0 is finite, thanks to vertical commutativity, we get $\forall n \in \mathbb{N}$, $|F^n(x)_0 - x_0| \leq l + \max_{|j| \leq 2l, |x_j| < \infty} |x_j - x_0|$, and thanks to shift commutativity, we get the stated result for any finite pile x_i. □

We state an intermediate lemma in order to simplify further proofs.

Lemma 3.13. *Any covering $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \Sigma_k$ by closed shift-invariant subsets Σ_k is such that $\mathcal{C} = \Sigma_k$ for some $k \in \mathbb{N}$.*

Proof. If $\mathcal{C} = \bigcup_{k \in \mathbb{N}} \Sigma_k$ where the Σ_k are closed, then by the Baire Theorem, some Σ_k has nonempty interior. Hence, it contains some ball $[U]$ where U is a cylinder. If it is shift-invariant, then it also contains $\overline{\bigcup_{k \in \mathbb{Z}^d} \sigma^k([U])}$, which is the whole space, thanks to Remark 3.3. □

Now we prove the equivalence of two different notions of stability, namely equicontinuity and ultimate periodicity. The proof is an extension of the one in [24] to the context of higher-dimensional CA acting on the subshift S_K. For simplicity sake, we directly use the notations of SA.

Proposition 3.14. *If F is an SA, then the following statements are equivalent:*

1. *F is equicontinuous.*
2. *F is ultimately periodic.*
3. *All configurations of $\mathcal{C}$ are ultimately periodic for F.*

Proof. 3 ⇒2: For any $n \geq 0$ and $p > 0$, let $D_{n,p}$ the closed shift-invariant subset $\{x : F^{n+p}(x) = F^n(x)\}$. Assume $\mathcal{C} = \bigcup_{n,p \in \mathbb{N}} D_{n,p}$. Then by Lemma 3.13, $\mathcal{C} = D_{n,p}$ for some $n \geq 0$ and some $p > 0$.
2 ⇒3: Obvious.
2 ⇒1: Let F be such that with $F^{n+p} = F^n$ for some $n \geq 0, p > 0$. Since $F, F^2, \dots, F^{n+p-1}$ are uniformly continuous maps, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in \mathcal{C}$ with $\mathrm{d}(x, y) < \delta$, it holds that $\forall q \in \mathbb{N}, q < n + p, \mathrm{d}(F^q(x), F^q(y)) < \varepsilon$. Since for any $t \in \mathbb{N}$, F^t is equal to some F^q with $q < n + p$, the map F is equicontinuous.
1 ⇒2: Let F be an equicontinuous SA and $l \in \mathbb{N}$ such that for all $x, y \in \mathcal{C}$, if $\Gamma^0_l(x) = \Gamma^0_l(y)$, then $\forall n \in \mathbb{N}$, $\Gamma^0_0(F^n(x)) = \Gamma^0_0(F^n(y))$. Let $x \in \mathcal{C}$, and $y \in \mathcal{C}$ defined by $y_i = \max\{\min\{x_i, l+1\}, -l-1\}$ if $|i| \leq l$ and $y_i = +\infty$ otherwise, in such a way that $\Gamma^0_l(x) = \Gamma^0_l(y)$. By Lemma 3.12, $\forall i \in [-l, l]^d$, $\forall n \in \mathbb{N}$, $|F^n(y)_i - y_i| \leq l + \max_{|j-i| \leq 2l, |y_j| < \infty} |y_j - y_i|$. By construction, we get $|F^n(y)_i| \leq 4l + 3$. Since y (and all configurations of its orbit) have a finite number of finite piles, which are bounded, we can find some preperiod q_y and some period p_y such that $F^{p_y + q_y}(y) = F^{q_y}(y)$. As a consequence and by equicontinuity, $\Gamma^0_0(F^{p_y+q_y}(x)) = \Gamma^0_0(F^{q_y}(x))$. Define p (resp., q) as the least common multiple (resp., maximum) of all p_y (resp., q_y) for $y \in \mathcal{C}$ such that $|y_i| \leq l + 1$ if $|i| \leq l$ and $y_i = +\infty$ otherwise. Then, for any $x \in \mathcal{C}$, $\Gamma^0_0(F^{p+q}(x)) = \Gamma^0_0(F^q(x))$. Thanks to vertical commutativity, we get $\forall x \in \mathcal{C}$, $F^{p+q}(x)_0 = F^q(x)_0$, and thanks to shift commutativity, we get $\forall x \in \mathcal{C}$, $F^{p+q}(x) = F^q(x)$. □

In [24], one-dimensional CAs are classified into four classes: equicontinuous CAs, non equicontinuous CAs admitting an equicontinuity configuration, sensitive but not positively expansive CAs, positively expansive CAs. This classification is no more relevant in the context of SA since the class of positively expansive SAs is empty. This result can be related to the absence of positively expansive two-dimensional CAs (see [27]), though the proof is much different.

Proposition 3.15. *There are no positively expansive SAs.*

Proof. Let F an SA and $\delta = 2^{-k} > 0$. Take two distinct configurations $x, y \in \mathcal{C}$ such that $\forall i \in [-k, k]^d$, $x_i = y_i = +\infty$. By infinity-preservingness, we get $\forall n \in \mathbb{N}$, $\forall i \in [-k, k]^d$, $F^n(x)_i = F^n(y)_i = +\infty$, hence $d(F^n(x), F^n(y)) < \delta$. □

An important open question in the dynamical behavior of SA is the existence of non-sensitive SAs without any equicontinuity configuration. An example of two-dimensional CA is given in [26], but their method can hardly be adapted for SA. This could lead to a classification of SAs into four classes: equicontinuous, admitting an equicontinuity configuration (but not equicontinuous), non-sensitive without equicontinuity configurations, sensitive.

4. The nilpotency problem

In this section we give a definition of flatteningness for SA. Then, we prove that this behavior is undecidable (Theorem 4.5).

4.1. Nilpotent CA

Here we recall the basic definitions and properties of nilpotent CA. Nilpotency is among the most stable dynamical behavior that an automaton may exhibit. Intuitively, an automaton defined by a local rule and working on configurations (either $\mathcal{C}$ or $A^{\mathbb{Z}^d}$) is nilpotent if it destroys every piece of information in any initial configuration, reaching a common constant configuration after a while. For CA, this is formalized as follows.

Definition 4.1 (*CA Nilpotency [12,22]*). A CA G is nilpotent if

$$\exists c \in A,\ \exists N \in \mathbb{N},\ \forall x \in A^{\mathbb{Z}^d},\ \forall n \geq N, \quad G^n(x) = \underline{c}.$$

Remark that in a similar way to the proof of Proposition 3.14, Definition 4.1 can be restated as follows: a CA is nilpotent if and only if it is nilpotent for all initial configurations.

Spreading CAs have the following stronger characterization.

Proposition 4.1 (*[11]*). *A CA G, with spreading state 0, is nilpotent iff for every $x \in A^{\mathbb{Z}^d}$, there exists $n \in \mathbb{N}$ and $i \in \mathbb{Z}^d$ such that $G^n(x)_i = 0$ (i.e., 0 appears in the evolution of every configuration).*

The previous result immediately leads to the following equivalence.

Corollary 4.2. *A CA of global rule G, with spreading state 0, is nilpotent if and only if for all configurations $x \in A^{\mathbb{Z}^d}$, $\lim_{n\to\infty} \mathrm{d}_T(G^n(x), \underline{0}) = 0$.*

This result has recently been extended to arbitrary CAs, in dimension 1 [20]. Recall that the CA nilpotency is undecidable [22]. Remark that the proof of this result also works for the restricted class of spreading CAs.

Theorem 4.3 (*[22]*). *For a given state s, it is undecidable to know whether a cellular automaton with spreading state s is nilpotent.*

4.2. Flattening SA

A direct adaptation of Definition 4.1 to SA is vain. For this reason, we define in this section a strong stability property, called *flatteningness*, as a refinement of CA nilpotency.

First, since SAs are infinity-preserving, an infinite pile cannot be destroyed (nor, for the same reason, can an infinite pile be built from a finite one). Therefore any stability notion has to involve the configurations of $\mathbb{Z}^{\mathbb{Z}^d}$, i.e., the ones without infinite piles. Moreover, every configuration $x \in \mathbb{Z}^{\mathbb{Z}^d}$ made of regular steps (i.e., in dimension 1, for all $i \in \mathbb{Z}$, $x_i - x_{i-1} = x_{i+1} - x_i$) is invariant by the SA rule (possibly composing it with the vertical shift). So it cannot reach nor approach a constant configuration. Thus, the larger reasonable set on which the new stability notion might be defined is the set of bounded configurations $\mathcal{B}$.

Besides, assume F is an SA of radius r. For any $k \in \mathbb{Z}^d$, consider the configuration $x^k \in \mathcal{B}$ defined by $x^k_0 = k$ and $x^k_i = 0$ for any $i \in \mathbb{Z}^d \setminus \{0\}$. Since the pile of height k may decrease at most by r during one step of evolution of the SA, and the other piles may increase at most by r, x^k requires at least $\lceil k/2r \rceil$ steps to reach a constant configuration. Thus, there exists no common integer n such that all configurations x^k reach a constant configuration in time n. This is a major difference with CA, which is essentially due to the unbounded set of states and to the infinity-preserving property.

Thus, we propose to consider flattening SAs, i.e., those which make every pile approach a constant value, but not necessarily reaching it ultimately. This notion, inspired by the equivalence proved for CA in Corollary 4.2 and [20], is formalized as follows for an SA F:

$$\exists c \in \mathbb{Z},\ \forall x \in \mathcal{B}, \quad \lim_{n\to\infty} \mathrm{d}(F^n(x), \underline{c}) = 0.$$

Remark that c shall not be taken in the full state set $\widetilde{\mathbb{Z}}$, because allowing infinite values for c would not correspond to the intuitive idea that a flattening SA "destroys" a configuration (otherwise, the raising map would be flattening). Anyway, this definition is not satisfying because of the vertical commutativity: two configurations which differ by a vertical shift reach two different configurations, and then no flattening SA may exist. A possible way to work around this issue is to make the limit configuration depend on the initial one, which leads to the following formal definition of flatteningness for SA.

Definition 4.2 (*Flattening SA*). A SA F is flattening if and only if

$$\forall x \in \mathcal{B},\ \exists c \in \mathbb{Z}, \quad \lim_{n\to\infty} \mathrm{d}(F^n(x), \underline{c}) = 0.$$

The following proposition shows that the class of flattening SAs is nonempty.

Proposition 4.4. *The SA $\mathcal{N}$ from Example 2.1 is flattening.*

Proof. Let $x \in \mathcal{B}$, let $i \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$, $x_j \geq x_i$. Clearly, after $x_{i+1} - x_i$ steps, $F_{\mathcal{N}}^{x_{i+1}-x_i}(x)_{i+1} = F_{\mathcal{N}}^{x_{i+1}-x_i}(x)_i = x_i$. By immediate induction, we obtain that for all $j \in \mathbb{Z}$ there exists $n_j \in \mathbb{N}$ such that $F_{\mathcal{N}}^{n_j}(x)_j = x_i$, hence $\lim_{n\to\infty} \mathrm{d}(F_{\mathcal{N}}^n(x), \underline{x_i}) = 0$. □

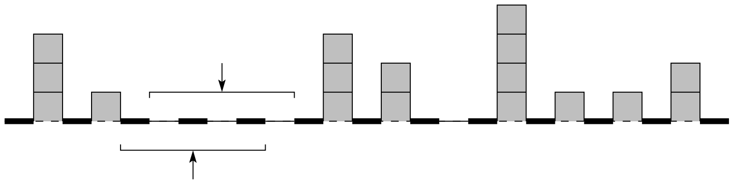


Fig. 5. Illustration of the function ξ used in the simulation of the spreading CA $\mathcal{S}$ by $\mathcal{A}$. The thick segments are the markers used to distinguish the states of the CA, put at height 0. There is an ambiguity for the two piles indicated by the arrows: with a radius 2, the neighborhoods are the same, although one of the piles is a marker and the other the state 0.

Similar flattening SAs can be constructed with any radius and in any dimension.

Remark that this flatteningness notion is different from the nilpotency of the higher-dimensional CA simulating the SA. Indeed, the nilpotent CA would transform a configuration by putting 1 or 0 everywhere. This corresponds to changing each column into $\pm\infty$ after a while, which is not possible in the sense of SA. Besides, CA nilpotency immediately implies equicontinuity. In the context of SA, this is not the case anymore since, for example, $\mathcal{N}$ is not ultimately periodic. An interesting question would be to see whether flattening SAs satisfy other properties, weaker than equicontinuity.

4.3. Undecidability

The main result of this section is that SA flatteningness is undecidable (Theorem 4.5), which is proved by reducing the nilpotency of spreading CAs to it. This emphasizes the fact that the dynamical behavior of SA is very difficult to predict. We think that this result might be used as the reference undecidable problem for further questions on SA.

Problem Nil
INSTANCE: an SA $\mathcal{A} = \langle d, r, \lambda\rangle$;
QUESTION: is $\mathcal{A}$ flattening?

Theorem 4.5. *The problem **Nil** is undecidable.*

Proof. This is proved by reducing the nilpotency of spreading cellular automata to **Nil**. Remark that it is sufficient to show the result in dimension 1. Let $\mathcal{S}$ be a spreading cellular automaton $\mathcal{S} = \langle A, 1, s, g\rangle$ of global rule G, with a finite set of integer states $A \subset \mathbb{N}$ containing the spreading state 0. We simulate $\mathcal{S}$ with the sand automaton $\mathcal{A} = \langle 1, r = \max(2s, \max A), f\rangle$ of global rule F using the following technique, also developed in [9]. Let $\xi : A^{\mathbb{Z}} \to \mathcal{B}$ be a function which inserts markers every two cells in the CA configuration to obtain a bounded SA configuration. These markers allow the local rule of the SA to know the absolute state of each pile and behave as the local rule of the CA. To simplify the proof, the markers are put at height 0 (see Fig. 5):

$$\forall y \in A^{\mathbb{Z}}, \forall i \in \mathbb{Z}, \quad \xi(y)_i = \begin{cases} 0\,(\text{marker}) & \text{if } i \text{ is odd,} \\ y_{i/2} & \text{otherwise.} \end{cases}$$

This can lead to an ambiguity when all the states in the neighborhood of size $4s + 1$ are at state 0, as shown in the picture. However, as in this special case the state 0 is quiescent for g, this is not a problem: the state 0 is preserved, and markers are preserved.

The local rule f is defined as follows, for all ranges $R \in \mathcal{R}_r^1$,

$$f(R) = \begin{cases} 0 \quad \text{if } R_{-2s+1}, R_{-2s+3}, \ldots, R_{-1}, R_1, \ldots, R_{2s-1} \in A, \\ g(R_{-2s} + a, R_{-2s+2} + a, \ldots, R_{-2} + a, a, R_2 + a, \ldots, R_{2s} + a) - a \\ \quad \text{if } R_{-2s+1} = R_{-2s+3} = \cdots = R_{2s-1} = a < 0 \text{ and } -a \in A. \end{cases} \tag{2}$$

The first case is for the markers (and state 0) which remain unchanged, the second case is the simulation of g in the even piles. As proved in [9], for any $y \in A^{\mathbb{Z}}$ it holds that $\xi(G(y)) = F(\xi(y))$. The images by f of the remaining ranges will be defined later on, first a few new notions need to be introduced.

A sequence of consecutive piles $(x_i, \ldots, x_j)$ from a configuration $x \in \mathcal{B}$ is said to be *valid* if it is part of an encoding of a CA configuration, i.e., $x_i = x_{i+2} = \cdots = x_j$ (these piles are markers) and for all $k \in \mathbb{N}$ such that $0 \le k < (j-i)/2$, $x_{i+2k+1} - x_i \in A$ (this is a valid state). We extend this definition to configurations, when $i = -\infty$ and $j = +\infty$, i.e., $x \in \rho^c \circ \xi(A^{\mathbb{Z}})$ for a given $c \in \mathbb{Z}$ ($x \in \mathcal{B}$ is valid if it is the raised image of a CA configuration). A sequence (or a configuration) in *invalid* if it is not valid.

First we show that starting from a valid configuration, the SA $\mathcal{A}$ is flattening if and only if $\mathcal{S}$ is nilpotent. This is due to the fact that we chose to put the markers at height 0, hence for any valid encoding of the CA $x = \rho^c \circ \xi(y)$, with $y \in A^{\mathbb{Z}}$ and $c \in \mathbb{Z}$,

$$\lim_{n\to\infty} \mathrm{d}_T(G^n(y), \underline{0}) = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \mathrm{d}(F^n(x), \underline{c}) = 0.$$

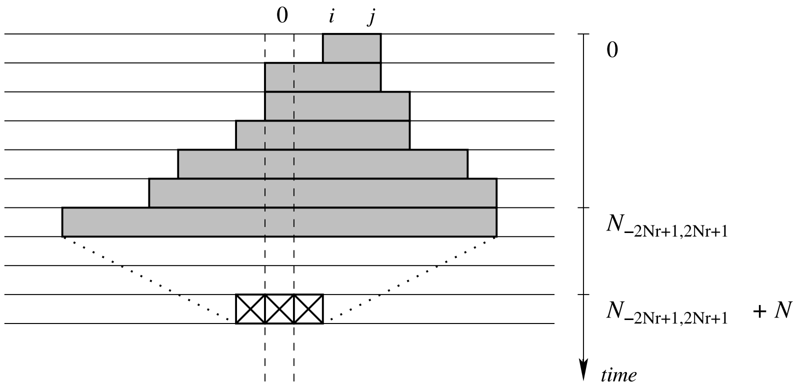


Fig. 6. Destruction of the invalid parts. The lowest valid sequence (in gray) extends until it is large enough. Then after N other steps the 3 central piles (hatched) are destroyed because the rule of the CA is applied correctly.

It remains to prove that for any invalid configuration, $\mathcal{A}$ is also flattening. In order to have this behavior, we add to the local rule f the rules of the flattening automaton $\mathcal{N}$ for every invalid neighborhood of width $4s+1$. For all ranges $R \in \mathcal{R}_r^1$ not considered in Eq. (2),

$$f(R) = \begin{cases} -1 & \text{if } R_{-r} < 0 \text{ or } R_{-r+1} < 0 \text{ or } \cdots \text{ or } R_r < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Let $x \in \mathcal{B}$ be an invalid configuration. Let $k \in \mathbb{Z}$ be any index such that $\forall l \in \mathbb{Z}, x_l \geq x_k$. Let $i, j \in \mathbb{Z}$ be respectively the smallest and largest indices such that $i \leq k \leq j$ and $(x_i, \dots, x_j)$ is valid (i may equal j). Remark that for all $n \in \mathbb{N}$, $(F^n(x)_i, \dots, F^n(x)_j)$ remains valid. Indeed, the markers are by construction the lowest piles and Eqs. (2) and (3) do not modify them. The piles coding for non-zero states can change their state by Eq. (2), or decrease it by 1 by Eq. (3), which in both cases is a valid encoding. Moreover, the piles x_{i-1} and x_{j+1} will reach a valid value after a finite number of steps: as long as they are invalid, they decrease by 1 until they reach a value which encodes for a valid state. Hence, by induction, for any indices $a, b \in \mathbb{Z}$, there exists $N_{a,b}$ such that for all $n \geq N_{a,b}$ the sequence $(F^n(x)_a, \dots, F^n(x)_b)$ is valid.

In particular, after $N_{-2Nr-1,2Nr+1}$ step, there is a valid sequence of length $4Nr+3$ centered on the origin (here, N is the number of steps needed by $\mathcal{S}$ to reach the configuration $\underline{0}$, given by Definition 4.1). Hence, after $N_{-2Nr,2Nr} + N$ steps, the local rule of the CA $\mathcal{S}$ applied on this valid sequence leads to 3 consecutive zeros at positions $-1, 0, 1$. All these steps are illustrated on Fig. 6.

Similarly, we prove that for all integers n greater than $N_{-2Nr-k,2Nr+k} + N$, $(F^n(x)_{-k}, \dots, F^n(x)_k)$ is a constant sequence which does not evolve. Therefore, there exists $c \in \mathbb{Z}$ such that $\lim_{n\to\infty} \mathrm{d}(F^n(x), \underline{c}) = 0$. We just proved that $\mathcal{A}$ is flattening, i.e., $\lim_{n\to\infty} \mathrm{d}(F^n(x), \underline{c}) = 0$ for all $x \in \mathcal{B}$, if and only if $\mathcal{S}$ is nilpotent (because of the equivalence of definitions given by Corollary 4.2), so **Nil** is undecidable (Proposition 4.3). □

5. Conclusion

In this article we have continued the study of sand automata, by introducing a compact topology on the SA configuration space. In this new context of study, the characterization of SA functions of [10,9] still holds. Moreover, a topological conjugacy of any SA with a suitable CA acting on a particular subshift might facilitate future studies about dynamical and topological properties of SA, as for the proof of the equivalence between equicontinuity and ultimate periodicity (Proposition 3.14). Further research could also consider CA acting on more general subshifts (possibly of finite type), in order to see if similar properties are preserved. These studies could be made easier by investigations on the dynamical behavior of CA in any dimension (see [7,14,13,15,1], for some recent results).

We also think that this new context could lead to interesting results concerning SAs classification according to equicontinuity properties. There could be three or four classes, depending on the fact that there may exist non-sensitive SAs without equicontinuity points. Also, studying transitivity, a stronger form of sensitivity for CA, could result in another class replacing expansivity. The authors are currently trying to see whether this is a meaningful property for SA.

Then, we have defined flatteningness as a notion of extreme stability for SA. Although it differs from the nilpotency of CA, it captures the intuitive idea of a "destruction" of configurations. Even though flattening SAs may not completely destroy the initial configuration, they flatten them progressively. Finally, we have proved that SA flatteningness is undecidable (Theorem 4.5). This fact enhances the idea that the behavior of an SA is hard to predict. We also think that this result might be used as a fundamental undecidability result, which could be reduced to other SA properties.

Indeed, very little is known on the decidability of SA properties. The study of global properties such as injectivity and surjectivity and their corresponding dimension-dependent decidability problems could help to understand if d-dimensional SAs look more like d-dimensional or $(d+1)$-dimensional CAs. New evidence for that issue could be found by solving the question of the dichotomy between sensitive SAs and those with equicontinuous configurations. This would give a more precise idea of the dynamical behaviors represented by SA.

References

[1] L. Acerbi, A. Dennunzio, E. Formenti, Conservation of some dynamical properties for operations on cellular automata, Theoretical Computer Science 410 (38–40) (2009) 3685–3693.
[2] E. Akin, The General Topology of Dynamical Systems, in: Graduate Studies in Mathematics, vol. 1, AMS, 1993.
[3] P. Bak, K. Chen, C. Tang, A forest-fire model and some thoughts on turbulence, Physics Letters A 147 (1990) 297–300.
[4] P. Bak, C. Tang, K. Wiesenfeld, Self-organized criticality, Physical Review A 38 (1) (1988) 364–374.
[5] P. Bak, C. Tang, Earthquakes as a self-organized critical phenomenon, Journal of Geophysical Research 94 (1989) 15635–15637.
[6] P. Bak, How Nature Works — The Science of SOC, Oxford University Press, 1997.
[7] J. Cervelle, A. Dennunzio, E. Formenti, Chaotic behavior of cellular automata, in: Mathematical Basis of Cellular Automata, in: Encyclopedia of Complexity and System Science, Springer, 2009.
[8] J. Cervelle, E. Formenti, B. Masson, Basic properties for sand automata, in: 30th International Symposium on Mathematical Foundations of Computer Science, MFCS'05, in: Lecture Notes in Computer Science, vol. 3618, Springer, 2005, pp. 192–211.
[9] J. Cervelle, E. Formenti, B. Masson, From sandpiles to sand automata, Theoretical Computer Science 381 (2007) 1–28.
[10] J. Cervelle, E. Formenti, On sand automata, in: 20th Symposium on Theoretical Aspects of Computer Science, STACS'03, in: Lecture Notes in Computer Science, vol. 2607, Springer, 2003, pp. 642–653.
[11] J. Cervelle, P. Guillon, Towards a Rice theorem on traces of cellular automata, in: 32nd International Symposium on Mathematical Foundations of Computer Science, MFCS'07, in: Lecture Notes in Computer Science, vol. 4708, Springer, 2007, pp. 310–319.
[12] K. Čulik, J. Pachl, S. Yu, On the limit sets of cellular automata, SIAM Journal of Computing 18 (4) (1989) 831–842.
[13] A. Dennunzio, E. Formenti, P. Kůrka, Cellular automata dynamical systems, in: Handbook of Natural Computing: Theory, Experiments, and Applications, Springer, 2009 (in press).
[14] A. Dennunzio, E. Formenti, Decidable properties of 2D cellular automata, in: 33th Conference on Developments in Language Theory, DLT'08, in: Lecture Notes in Computer Science, vol. 5257, Springer, 2008, pp. 264–275.
[15] A. Dennunzio, E. Formenti, 2D cellular automata: New constructions and dynamics, 2009 (Preprint).
[16] A. Dennunzio, P. Guillon, B. Masson, Stable dynamics of sand automata, in: 5th IFIP International Conference on Theoretical Computer Science, TCS'08, vol. 273, Springer, 2008, pp. 157–169.
[17] A. Dennunzio, P. Guillon, B. Masson, Topological properties of sand automata as cellular automata, in: 1st Symposium on Cellular Automata, JAC'08, 2008, pp. 216–227.
[18] E. Goles, M.A. Kiwi, Games on line graphs and sandpile automata, Theoretical Computer Science 115 (1993) 321–349.
[19] E. Goles, M. Morvan, H.D. Phan, Sand piles and order structure of integer partitions, Discrete Applied Mathematics 117 (1–3) (2002) 51–64.
[20] P. Guillon, G. Richard, Nilpotency and limit sets of cellular automata, in: 33rd International Symposium on Mathematical Foundations of Computer Science, MFCS'08, in: Lecture Notes in Computer Science, vol. 5162, Springer, 2008, pp. 375–386.
[21] G.A. Hedlund, Endomorphisms and automorphisms of the shift dynamical system, Mathematical Systems Theory 3 (1969) 320–375.
[22] J. Kari, The nilpotency problem of one-dimensional cellular automata, SIAM Journal of Computing 21 (3) (1992) 571–586.
[23] P. Kůrka, Topological and Symbolic Dynamics, in: Cours spécialisés, vol. 11, Société Mathématique de France, Paris, 2003.
[24] P. Kůrka, Languages, equicontinuity and attractors in cellular automata, Ergodic Theory & Dynamical Systems 17 (1997) 417–433.
[25] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.
[26] M. Sablik, G. Theyssier, Topological dynamics of 2D cellular automata, in: 4th Conference on Computability in Europe, CiE'08, in: Lecture Notes in Computer Science, vol. 5028, Springer, 2008, pp. 523–532.
[27] M.A. Shereshevsky, Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms, Indagationes Mathematicœ 4 (2) (1993) 203–210.
[28] R. Subramanian, I. Scherson, An analysis of diffusive load-balancing, in: ACM Symposium on Parallel Algorithms and Architecture, SPAA'94, 1994, pp. 220–225.